THE THEORY AND PRACTICE OF REINFORCED CONCRETE

BY

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PREFACE TO THE SECOND EDITION

In preparing this Second Edition, special attention has been given to methods for the preliminary and approximate design of beams; bond between concrete and reinforcement has been treated more fully than in the First Edition; a beam-and-girder floor has been worked out as an illustrative problem; the design of rectangular and unsymmetrical footings with direct loads and overturning moments has been shown; and the material on the design of columns, two-way slabs, and flat slabs has been rewritten to agree with the 1940 report of the Joint Committee. Various other smaller additions have been made to improve the text; some minor material has been dropped. Of course, one book cannot illustrate every kind of reinforced-concrete structure in complete detail but the revisions and the new material meet certain existing needs and add greatly to the value of the book as a whole.

The tables and diagrams in the Appendix are useful for many purposes. References to them are made frequently in order to have the student appreciate their utility and to learn when and how to use them. However, the author's chief emphasis is still upon the basic principles of the design of various types of structures and upon the practical features which are so important in planning and building them.

The author is especially indebted to Professors Hardy Cross, Francis M. Baron, and Henry A. Pfisterer of Yale University for helpful suggestions which, as far as is practicable, have been included in the text.

C. W. Dunham.

New Haven, Conn.,
August, 1944.

PREFACE TO THE FIRST EDITION

The purpose of this text is to present in a simple and understandable manner the theories upon which the design of most ordinary types of reinforced-concrete construction is founded and, along with them, to show the practical considerations that so greatly affect the finished product. The first six or seven chapters are intended for use in an elementary course in the design of reinforced concrete; the rest of the text is designed especially for the benefit of the practicing engineer and for use in advanced courses.

The author hopes to teach the reader to visualize how each part of a structure acts; to design these parts so that each one will perform safely the service for which it is intended; and, finally, to plan the operations in the field so that the entire work will be a thing of which he is proud. Sound judgment and practical engineering sense are exceedingly important. These essentials are attained chiefly through hard work and long experience by the individual. However, the author hopes to expedite their attainment by presenting the subject from the viewpoint of the practicing engineer.

The design of all kinds of structures cannot be given in one book. However, certain important types have been selected, and each one is treated as thoroughly as space permits. The author has tried to illustrate many of the hard but practical problems which are likely to be encountered in high-grade construction. If the reader will study the principles and the methods that are illustrated herein until they become almost a part of his subconscious thinking, he will be able to use them as tools in the design of a great variety of structures.

The methods of solving the illustrative problems are those which may be encountered in a large engineering office. The reader will notice that, in many cases, this book does not go into extreme refinements of design and calculation. When one realizes that the assumed loads, their distribution, and the allowable unit stresses are often rather approximate and that they are

based upon judgment and experience, it seems to be inadvisable to carry subsequent computations to a degree of refinement that is not justified by the accuracy of the fundamental data from which the calculations are started. Therefore, the use of the slide rule is sufficient for all work except certain portions of the analysis of indeterminate structures. In many cases, the numerical answers are rounded to the nearest important significant figure. The methods of analysis that are employed are designed to show fundamental principles and their application. They are believed to yield results that are on the side of safety and to be sufficiently accurate for all practical purposes.

Furthermore, this text is not a handbook filled with tables to be used mechanically. Certain useful data of this type are included, but they are not acceptable substitutes for a mastery of the theory of the design of structures. The reader can resort to short-cut methods after he understands the bases of structural action.

Specific recommendations have been given for many theoretical and practical procedures. Where this has been done without reference to other authorities, the author merely attempts to provide the reader with some definite suggestions, but he does not pretend to set up unchangeable specifications.

The author is indebted to many associates and other friends for useful data and for helpful suggestions. He is especially grateful to Mr. W. B. Sinnickson, Engineer of Tests, The Port of New York Authority, who contributed Chap. 1; to Mr. Samuel Potashnick, formerly of The Port of New York Authority, for his cooperation in the development of the data given in Chaps. 11 and 12; and to The Port of New York Authority for the use of its photographs and data. Messrs. A. C. Seaman, Leon Kirsch, Walter Gadkowski, and William J. Delaney have rendered great assistance in the checking of calculations; Professors William S. Lalonde, Jr., Leroy W. Clark, and Bert B. Williams, and Mr. Paul F. Pape have given many useful suggestions.

C. W. DUNHAM.

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THE THEORY AND PRACTICE OF REINFORCED CONCRETE

CHAPTER 1

PROPERTIES AND MANUFACTURE OF CONCRETE[1]

1-1. Introduction. A concrete structure, either plain or reinforced, maintains a unique position among the various systems of modern construction. With few exceptions it is the only type of structure that is completely manufactured from its component raw materials on the site of the work. In most instances, the quality of its essential raw materials is decidedly variable. The compounding of its ingredients, the control of its chemical processes, and the arrangement of its component members are all too often performed by the least skilled of mechanical artisans. The inspection of its fabrication is sometimes delegated to the least experienced member of a supervisory force, and may occasionally be neglected entirely.

The personal element—the care with which work is executed in the field—is of major importance in concrete construction. Structures built of steel, stone masonry, or various other materials are composed of elementary units which are partially or entirely prefabricated in factories or shops by skilled workmen. These other materials are fitted or assembled on the work by skilled mechanics, but concrete is usually manufactured at the site of the structure by unskilled laborers. The designer of reinforced-concrete structures should remember this. He must know the useful properties and practical limitations of the materials with which his plan will be constructed. With this knowledge he should plan the work in such a manner that desirable results are easily and correctly attained in the field.

[1] Contributed by W. B. Sinnickson, Engineer of Tests, The Port of New York Authority.

1-2. Definition and Description of Concrete. Concrete is an artificial stone, cast in place in a plastic condition. Its essential ingredients are cement and water which react with each other chemically to form another material possessing structural strength. A mixture of cement and water is termed *cement paste.* Such a mixture is expensive. In order to increase the volume of artificial stone produced from a definite amount of cement it is customary to add inert filler materials known as *aggregates.* A large amount of cement paste to which has been added a small amount of fine aggregate, to produce a mixture of fluid consistency, is called *grout.* When the amount of fine aggregate is increased to the extent that the mixture loses its fluidity and behaves as a cohesive plastic, the resulting mixture is termed *mortar.* With the further addition of coarse aggregate, the mixture is called *concrete.*

It is a custom of long standing to designate these mixtures in terms of the relative volumes of cement, fine aggregate, and coarse aggregate of which they are composed. The ingredients are always indicated in the same order: cement first, fine aggregate next, and coarse aggregate last. For example, a 1:2:4 concrete is a mixture of 1 cu. ft. of cement, 2 cu. ft. of fine aggregate, and 4 cu. ft. of coarse aggregate plus a nonspecified amount of water sufficient to produce a plastic consistency. A proportion given as 1:3 is intended to mean a mixture of cement and fine aggregate plus an indefinite amount of water but without the addition of coarse aggregate. Such a mixture would be classified as mortar.

This system of specifying concrete proportions by volume is rapidly becoming obsolete on major works but is still used for small projects. The current practice of progressive engineers is to indicate the proportions of materials in the same order but by weight and, frequently, to indicate the amount of water to be used. For instance, the desirable proportions of materials for concrete in the anchorages of the Bronx-Whitestone Bridge in New York City were determined by experiment and are given as 94:184:380 lb. plus 5.6 gal. of water.

Water, cement, and aggregates when mixed together in properly predetermined proportions produce concrete that is a plastic mass capable of being poured or cast into molds. These molds, which are actually called *forms*, must be built of such size and

shape as to restrain the plastic mass until it solidifies. With few exceptions the forms must be constructed in such a manner that the concrete, when poured, will be in its final position in the structure. Besides restraining the plastic mass until solidification occurs, the forms serve a less obvious purpose which should not be overlooked. They support the solidified mass until it has attained sufficient strength to support itself without undue deflection or complete collapse.

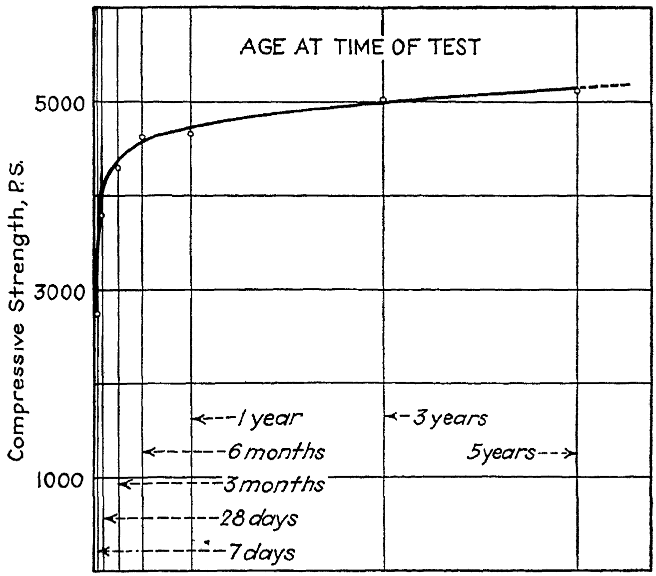


FIG. 1-1.—Relation of compressive strength to age. George Washington Bridge, Riverside Drive connections to the New York approach. Each point is the average unit strength of twenty-five 6- by 12-in. cylinders mixed in the laboratory and cured in moist air at 70°F.

Concrete does not solidify or attain appreciable strength instantaneously. The chemical reaction of cement and water is slow and requires time for its completion. The reaction continues for many years. It is frequently divided, for purposes of description, into three distinct phases. The first, called the initial setting time, requires approximately 45 min. to 8 hr. for completion. During this time, the freshly mixed concrete gradually decreases in plasticity and develops pronounced resistance to flow. Disturbance of the mass, or remixing during this time,

may cause serious damage. The second phase is an interval during which the concrete may be considered as a soft solid without surface hardness. It will support light loads without indentation, but it is easily abraded. Its surface can be scored, roughened, or otherwise marred without appreciable effort. This phase is frequently termed the *interval of final set.* Its duration is very indefinite but may be considered to exist for approximately 5 to 20 hr. after the original mixing operation. The third phase is one of progressive hardening and increase in strength. For concrete of good quality this progressive improvement continues indefinitely. It is rapid during early ages until about one month after mixing, at which time the mass has attained the major portion of its potential hardness and strength. After one month the improvement continues at a greatly reduced rate. This peculiar property of improvement with age will be discussed later in greater detail. It is graphically illustrated in Fig. 1-1.

1-3. Cement. For more than two thousand years man has used various materials of a cementaceous nature in the building of his important structures. These cementing materials can be arbitrarily placed in two distinct categories: those which do and those which do not set and harden in the presence of appreciable amounts of water. The former class is of major structural importance and is said to have hydraulic properties. A general classification of hydraulic cements should include *puzzolanic material,* of volcanic origin, so effectively used by the Romans; *hydraulic lime* frequently used today in France; *natural cement* such as that produced near Louisville, Ky., and Rosendale, N. Y., of the type used in constructing the Brooklyn Bridge; *alumina cement* which is currently popular for sea-water construction in European countries; and *Portland cement.*

Each of these types of hydraulic cement is in current use in some part of the world. The specific use of any particular type is a matter of engineering psychology and economic necessity. Engineering practice, in the United States, is inalterably linked to the rapid construction and early utilization of a structure, for which reason Portland cement is almost universally used. Although aluminous and natural elements are sometimes used in this country for structures of a specialized nature, the further discussion of reinforced-concrete construction will be confined to that in which Portland cement is the binding agent.

As defined in the Standard Specifications for Portland Cement, A.S.T.M. Designation: C9-38 of the American Society for Testing Materials,

Portland cement is the product obtained by pulverizing clinker consisting essentially of calcium silicates, to which no additions have been made subsequent to calcination other than water and/or untreated calcium sulfate except that additions not to exceed 1 per cent of other materials may be added, provided such materials have been shown not to be harmful by tests prescribed and carried out by Committee C-1 on Cement.

In 1904, the A.S.T.M. adopted a standard specification for Portland cement. The specification was definitely regulatory, but the restricting limits were of such range that manufacturers, in all parts of the country, might comply with the requirements with little difficulty. This country-wide inclusive standard was revised at various times as the art and ingenuity of the industry progressed. In 1912, the Federal government adopted a standard specification differing little in its requirements from those of the A.S.T.M. specifications then in use. These standards, although revised at various times, remained similar in essential requirements. Their great latitude made it possible for the cement manufacturers to solve individual problems of raw material composition and plant operation with little attention to restrictive specification requirements.

Contiguous to the year 1920, it was realized that, although most cements met the requirements of the existent specifications, they were often different in accomplishment. Some attained strengths greatly in excess of the requirements; others might attain the specified strength within the required time but show evidence of retrogression at extended ages. Some were durable when used in specific localities, while others were not. Different lots of apparently similar origin and composition were, when mixed with water, decidedly different in plasticity. Many other differences in behavior were observed. It was evident that, although there were only two standard specifications for Portland cement, each similar in its requirements to the other, there were many kinds of cement.

The International Cement Corporation, in 1927, placed on the market a *high-early-strength* Portland cement which was rapidly

duplicated by others. There was little difference in the composition of this material from that of contemporary normal cement, but its behavior was radically different. Concrete made with high-early-strength cement sets in approximately the same time as does that made with normal cement, but it develops strength and hardness at a much earlier age. This special cement, produced from selected raw materials by rigid process control, attains strength at the ages of 3 and 5 days equivalent to that attained by concrete made with normal cement at 7- and 28-day ages. The A.S.T.M. in 1930, and the Federal government in 1936, adopted specifications for this type of Portland cement.

Research during recent years has made it possible for the engineer to select suitable cements for specialized projects. During 1936, the Federal government adopted specifications for *moderate-heat-of-hardening* and *sulfate-resisting* types of Portland cement. The moderate-heat-of-hardening type is desirable in massive structures such as dams, where the heat generated by the reaction of cement and water may cause serious cracking of the structure, especially because of subsequent shrinkage upon cooling. The sulfate-resisting type affords greater assurance against disintegration of a concrete structure from the corrosive action of the water-soluble sulfates of magnesium, sodium, and potassium, which salts exist in sea water and in the soil of the arid regions of many Western states. In September, 1940, the A.S.T.M. adopted a Tentative Specification, designated C150-40T, embodying specific requirements for all the foregoing types as well as for a general-purpose type for use in general concrete construction where special properties are not required.

Special types of Portland cement behave, during mixing and setting, in a manner similar to normal Portland cement, but their rates of hardening and increase in strength are usually slower than in the case of normal cement. This is particularly true of the moderate-heat-of-hardening (A.S.T.M. *low-heat-of-hydration*) type, and designers are cautioned against promiscuous use of these special cements without a thorough knowledge of their properties.

The manufacture of Portland cement is widely distributed throughout the United States, the greatest producing district being the Lehigh Valley in Pennsylvania. The close proximity of hard clayey limestone, hard pure limestone, and coal are

responsible for the advantageous position maintained by the Lehigh Valley. With the advance in the technical application of other raw materials such as blast-furnace slag, shale, clay, and high-grade limestones and chalk, in conjunction with improved plant operation, this one-time advantage has been greatly reduced within recent years. The recent application of a flotation process[1] similar to that used in ore dressing may eventually eliminate any territorial advantage in the manufacture of Portland cement except that of close proximity to fuel supply.

Portland cement is a conglomerate material. It cannot be described as any definite chemical compound. It is a mixture of many compounds in variable proportions. Its composition is dependent on the impurities that are present in its raw materials and upon the treatment during and after the manufacturing processes of calcination and grinding. An ideal Portland cement is arbitrarily described for the purpose of this discussion as a chemical combination of lime and silica to form dicalcium and tricalcium silicates. Actually, such a reaction is economically impractical because of temperature limitations. The resulting product would probably be difficult to handle in construction work, but it serves as an excellent illustration of the ultimate composition of Portland cement. Appreciable amounts of alumina and iron, present as impurities in uncalcined-cement raw mixtures, serve as fluxing agents which accelerate the reaction of lime and silica at commercially practicable temperatures. The probable catalytic and possible inhibitory influence of small amounts of sodium, potassium, titanium, phosphorus, and manganese compounds which are normal constituents of Portland cement has only recently been recognized, and the action of these constituents is as yet unstudied. So also is it recognized that heat-treatment during manufacture, as well as the addition of abnormal modifying agents, may decidedly change the properties of cement. Magnesia in excess of 5 per cent is generally considered deleterious throughout the United States, but Portland cement containing this constituent in an amount as high as 8 per cent has given satisfactory service in Brazilian highway construction.[2] The amount of sulfur trioxide present in Portland

[1] Breerwood patent 1931921, Oct. 24, 1933.

[2] H. H. Vaughan, President, Canadian Foreign Investment Corp., Ltd., Montreal, Canada.

cement is an indication of the amount of gypsum used to retard the set. A large amount of this constituent generally indicates that the raw mixture was of improper composition or was improperly calcined, and the resulting cement has a *flash set* with a normal amount of retarder.

With the exception of soda and potash which are largely volatile at calcination temperatures, and sulfur trioxide which is added as gypsum during finish grinding, all the impurities contained in the raw mix plus those added by the fuel ash are constituents of the calcined-cement clinker. Iron and alumina combine with lime to form compounds having little cementing value. They thus fix an appreciable amount of lime which might otherwise combine with silica to form a compound of great cementing value. The lime-alumina constituent of poor cementing value is generally accepted as the primary cause of concrete disintegration from the chemical action of sulfate waters.[1]

The chemical reaction of cement and water is a combination of hydrolysis and hydration. Hydrolysis is most readily described as the disintegration of a material into other materials influenced by the chemical action of water. Hydration is the direct combination of a material with water. Some of the constituents of Portland cement disintegrate in the presence of water and form other compounds which subsequently combine with water. Other constituents combine directly with water to form compounds containing water of crystallization as part of the reaction product. Briefly, the tricalcium silicate constituent disintegrates slowly by hydrolysis, and the disintegration products then hydrate; dicalcium silicate apparently reacts directly with water to form a hydration product, but the reaction takes place slowly. The lime-alumina constituent and the lime-alumina-iron constituent combine with water to form hydrates. The lime-alumina reaction with water takes place much more rapidly than do the other combinations, and because of this and other considerations the alumina is credited with having the greatest influence on the initial setting properties. The slow combination of dicalcium silicates and also one of the disintegration products of tricalcium silicates with water to form calcium-silica hydrates

[1] P.C.A. Fellowship, *Paper* 28, Bogue, Lerch and Taylor.

is responsible for the progressive increase in strength of Portland-cement concrete.[1]

The foregoing description of the composition and chemical reactions of Portland cement is exceedingly meager as to detail, but so is our present knowledge of this common engineering material. The material is exceedingly complicated, and its chemical behavior and composition are highly controversial. The chemical terms used are intended to be descriptive rather than scientifically correct. The designing engineer should conservatively apply the generally accepted standard specifications for this material and leave improvements and departures from standard practice to the specialized field of the cement technologist and materials engineer.

1-4. Aggregates. Aggregates, either fine or coarse, are inert filler materials added to cement paste to increase its bulk. A proper appreciation of this single function of aggregate will make the entire subject of concrete proportioning much easier to understand. It is customary to consider fine aggregate as any structurally sound filler material which will pass through a sieve having $\frac{1}{4}$-in. square openings. It follows that those particles which are larger than $\frac{1}{4}$ in. in average dimensions are classed as coarse aggregate. Filler materials may be either natural or artificial in origin. Because of widespread distribution, natural sand and gravel and mechanically crushed rock are the most abundantly used materials. For the production of special types of concrete, artificial aggregates such as calcined clay, blast-furnace slag, steam cinders, sawdust, excelsior, etc., are sometimes used.

It is of major importance that aggregate be nonreactive with cement and water and that it shall be structurally sound, strong, and durable. Hardness is also a desirable property, especially in highway construction, where resistance to abrasion is a factor in the ultimate utility of the structure. In general, igneous and metamorphic rocks and siliceous sands and gravel are relatively much harder than cementing material and are usually of excellent quality for use as concrete aggregate. Sedimentary rocks in general should be considered unsuitable as concrete aggregates unless thorough and complete tests or extensive experience in

[1] P.C.A. Fellowship, *Paper* 27, Bogue and Lerch.

their use have proved their satisfactory quality. Natural materials having pronounced planes of weakness or cleavage such as slate, shale, and micaceous materials should, in general, be considered as undesirable. An aggregate material having uniform shearing strength along all possible planes throughout its volume is ideal from a structural standpoint for use in concrete.

Aggregates should be clean. Particles of aggregate coated with dust, clay, silt, or organic material will not develop maximum bond with the surrounding cement paste. The interface between an aggregate particle and its cementing medium is the critical plane through which tensile or shearing failures are most likely to occur. This low strength plane or area of contact should not be further weakened by interposing an infinitesimal layer of dirt between the aggregate particle and its cementing medium. Natural sands are particularly subject to surface coatings, especially when obtained from near the surface of a deposit. Materials from sources of supply containing clay, silt, and natural overburden of disintegrated organic matter should be thoroughly washed before use. Even after washing they should be tested in comparison with a material of known dependability. The contamination of concrete aggregate with topsoil, humus, or other earthy material containing products of organic origin even in small percentages is practically certain to result in early disintegration or complete collapse of a structure. Aggregate obtained from subaqueous sources is found, in some instances, to be coated with algae and contaminated with marine vegetation and marine animals. Such substances of marine origin are usually serious in their destructive action. Many rocks are of such a nature that, during crushing, a detrimental amount of dust is formed. Such materials should be washed free of crusher dust before incorporation in a concrete mixture.

Aggregate should be reasonably impermeable to water. Materials that possess pronounced tendencies to absorb water are more susceptible to disintegration by repeated freezing and thawing while in the saturated condition. The density of a material is a fair indication of its absorbent quality. Materials of lower density usually absorb greater amounts of water and are usually less desirable as concrete aggregates.

The size and shape of aggregate particles as well as the relative number of particles of various size are of decided importance with

regard to utility as a filler material. With the thought in mind that the function of an aggregate is to act as the bulky filler in an expensive water-cement paste it is readily apparent that an aggregate of maximum-bulk volume and maximum mobility or minimum resistance to displacement while the concrete is in the plastic state is highly desirable.

A consideration of shape indicates that aggregate particles may be generally classified as spherical, cubical, or prismoidal in character. Particles of spherical shape possess maximum mobility; those of prismoidal shape have maximum resistance to relative change in position. Round particles roll against each other, and their relative position in a group is easily changed. Angular particles slide against each other, and more effort is required to accomplish rearrangement. Flat, elongated, or prismoidal particles tend to interlock with each other and behave in a manner similar to a log jam. The bridging and keying action of this latter type of particle imparts harshness and relatively poor mobility to a plastic concrete mixture. A particle of spherical shape also possesses maximum volume per unit of surface, whereas that of prismoidal shape is of the inverse nature. Because of these differences in shape and surface area, aggregate of the spherical type produces a minimum stiffening effect on a particular consistency of cement paste. Since for a fixed plasticity of concrete more spherical material may be added to the expensive paste than can be done with cubical or prismoidal particles, natural gravel is generally more economical than is crushed stone as a filler material for use in massive concrete construction.

Consideration of particle size indicates that this factor also influences the economy of a concrete mixture. A single aggregate particle, cubical in shape and each side having a dimension of 1 in., has a surface area of 6 sq. in., and its volume is 1 cu. in. If the particle is physically divided by planes through the center of each face, the volume of material will still be 1 cu. in., but the number of individual particles will be increased to eight cubes, each of which measures $\frac{1}{2}$ in. on each side. The total surface area of the material is increased to 12 sq. in. Further subdivision of these particles produces no increase in the spacial volume of the material, but the number of particles and the sum of their individual surface areas soon reach extremely large values. The

entire surface of each aggregate particle in a plastic concrete mixture must be coated and wet by cement paste, and for this reason a specific absolute volume of coarse particles would have less stiffening effect than an equal absolute volume of finer particles, because the coarse particles would possess less surface area. Since a greater absolute volume of large particles can be used as filler than can an equal spacial volume of smaller particles with less stiffening effect on the paste, the economy of a concrete mixture is greater as the maximum size of the aggregate is increased.

The relative number of particles of various sizes of aggregate is a further factor influencing the economy of plastic concrete mixtures. For example, consider a cubical box of exactly 12 in. internal dimensions. If this box were filled with spherical particles, each 1 in. in diameter and each tangent to the other, there would be contained in the box 12 horizontal layers each consisting of 144 spheres. On first thought, it might be inferred that the box was completely filled, but further consideration indicates that there is air space between adjacent spheres. The sum of the individual absolute volumes of 1,728 spherical particles each 1 in. in diameter is 0.52 cu. ft. The volume of the interstitial space is 0.48 cu. ft. If the same number of spheres were more closely arranged in a system of hexagonal rather than cubical packing, the interstitial volume would be 44 per cent of the total volume that they apparently occupy. This interstitial space between aggregate particles is commonly termed *voids*.

The foregoing uniformly sized particles, in their most compact arrangement, would require 0.26 cu. ft. of cement paste to fill the voids and cement them into a solid mass. If smaller aggregate particles were placed in each interstice, and the resulting voids further filled with even smaller particles, the amount of cement paste necessary to fill the remaining interstices would be greatly reduced. By using fine and coarse aggregates which are graded and consist of various sizes of individual particles, it is often possible to decrease the volume of the interstitial space to less than this 26 per cent of the apparent volume of the aggregate. To produce good concrete it is necessary to use sufficient cement paste to fill the aggregate voids, plus an additional amount of paste necessary to separate the particles at the points

of tangency or planes of contact and thus produce plasticity. Since graded aggregate possesses fewer voids, it requires less cement paste to produce a mixture of predetermined plasticity and is usually economically desirable.

Specifications for fine and coarse aggregates should require that the material be "clean, hard, strong, durable, and sound material free from injurious amounts of shale, loam, alkali, organic matter, or soft, friable, thin, elongated, or laminated pieces." In addition, the size of the aggregate should be specified in terms of permissible percentages of material greater or lesser in size than certain testing sieves. Where economy of cement or uniformity of grading is desirable, it is frequently specified that aggregates conform with particle-size distribution limits as measured by testing sieves. For examples of typical specification limits as to size of particle and particle-size distribution, the recommendations of the Joint Committee on Standard Specifications for Concrete and Reinforced Concrete of the American Society of Civil Engineers, The American Society for Testing Materials, the American Railway Engineering Association, the Portland Cement Association, the American Concrete Institute, and the American Institute of Architects should be consulted.

A specific example of somewhat rigid limitations of size and grading of fine and coarse aggregate is given in Table 1-1, the specification requirements of The Port of New York Authority. These limits require the production of aggregates of excellent grading and minimum voids. The New York metropolitan area is fortunate in that its major sources of aggregate supply are tremendous deposits of glacial sand and gravel and vast outcroppings of igneous and metamorphic rock. Most producers of commercial aggregates supplying the New York metropolitan area are generally able to furnish material in conformity with the requirements of The Port of New York Authority without undue hardship or expense.

The student is cautioned against the unrestricted adoption of a specification as rigid in its limits as that in Table 1-1. Many sections of the country are less fortunate than is New York City in that their available deposits of material suitable for concrete aggregate may not be susceptible to such close limitation without a prohibitive increase in cost. Local sources must be investigated in order to arrive at a reasonably restrictive specification. In

any case, specification restrictions should be commensurate with the importance of the work.

Table 1-1.—Grading of Concrete Aggregate, The Port of New York Authority

Sieve size (square openings)	Per cent passing by weight		
	Fine aggregate	¾-in. coarse aggregate	1½-in. coarse aggregate
2 in.	....	....	100
1½ in.	....	100	95 to 100
1 in.	....	95 to 100	55 to 75
¾ in.	....	60 to 80	35 to 55
⅜ in.	....	15 to 40	5 to 20
No. 4	90 to 100	0 to 5	0 to 5
No. 8	80 to 95	(Used for thin sections)	(Used for heavy sections)
No. 16	45 to 80		
No. 30	25 to 55		
No. 50	13 to 25		
No. 100	0 to 5		

Many systems of sieves for measuring the size and grading of aggregate particles are in use throughout the United States.[1] Some state highway departments and other consumers of crushed stone use arbitrary sizes of round-opening screens for separating the various sizes of material for testing purposes. A more logical system of mechanical analysis widely used is that of sieves having square openings each related to the next larger opening in the ratio of $1:\sqrt{2}$. This system of square-mesh sieve sizes was first produced in 1910[2] and is described in the 1913 A.S.T.M. *Proceedings*. A practical application of the square-root series was first suggested by Prof. Abrams[3] in 1918. The Abrams system of classification expresses the grading of a particular aggregate as an abstract number termed the *fineness modulus* which is one-hundredth of the sum of the percentages of material coarser than each of the following U.S. Standard sieves: Nos. 100, 50, 30, 16, 8, 4, ⅜ in., ¾ in., 1½ in., 3 in.[4] Other

[1] Edmund Shaw, Sieve Testing of Aggregates, *Rock Products*, May 9, 1931.

[2] W. S. Tyler Company, Cleveland, Ohio.

[3] Lewis Institute, Structural Materials Research Laboratory, *Bull.* 1.

[4] A.S.T.M. Tentative Specifications for Concrete Aggregates (C33-36T).

sieves in addition to the foregoing series may be used to determine maximum size and grading more accurately, but those given are the only ones used to determine the fineness modulus. The most modern system[1] is one which is so arranged that successive sieve openings are related in size as $1:\sqrt[4]{2}$.

The abstract number termed the fineness modulus is essentially a summation of the volumes of material coarser than certain prescribed sieves. It is an indication of the average surface area of the aggregate. Many combinations of various percentages of different-sized particles may possess similar fineness moduli, and, for any particular type of aggregate, similarity of fineness moduli is reasonable assurance of similar plastic properties when incorporated in a concrete mixture. For this reason, the fineness modulus of an aggregate is an excellent aid to inspection and readily discloses a change in concrete-making quality of material from a single source.

It is a general practice to specify the maximum amount of deleterious substances such as clay, silt, shale, and organic matter that will be acceptable in a concrete aggregate. The methods of testing to determine the quantity of clay, silt, and shale present in an aggregate are of a flotation or elutriation nature. A dry sample of aggregate is weighed and washed with water (in the test for shale, with a high specific gravity liquid) until free of the undesirable materials. The amount present is then determined by difference in weight of the washed material as compared with the weight of the original sample.

The presence of organic impurities may be detected by soaking aggregate for 24 hr. in a 3 per cent solution of sodium hydroxide. Harmful amounts of organic impurities may be estimated by using fixed amounts of aggregate and solution and comparing with a prepared color standard. Aggregate producing a deeper color than that of the prepared colorimetric standard should be used only after exhaustive laboratory tests have proved its satisfactory quality.

Aggregate is tested for soundness by saturating the material with a concentrated solution of sodium or magnesium sulfate and permitting the saturated material to dry. These salts crystallize and during the process fix water of crystallization as a physical

[1] A.S.T.M. Standard Specifications for Sieves for Testing Purposes (E11-39).

part of the sulfate molecule. The fixation of this water of crystallization, and the subsequent growth of additional crystals from solution absorbed within the mass of an aggregate particle, exert a disruptive force tending to burst the particle. Repeated cycles of saturation and crystallization simulate the action of freezing and thawing of absorbed water, and the number of such cycles withstood by a material is an indication of its soundness.

Standard and tentative standard methods of performing these various tests to determine the quality of aggregates are included in the literature of the American Society for Testing Materials, the American Concrete Institute, and the Portland Cement Association. The Association of State Highway Officials and many individual state highway departments sponsor and use somewhat different methods of tests which are peculiarly adaptable to concrete highway-construction materials.

1-5. Compressive Strength of Concrete. Concrete is an artificial stone, and, as such, it has most of the principal assets of natural stone. The excellent resistance of hardened concrete to compression and its ready conformation during the plastic state to irregular rock and earth surfaces and to prearranged contours make it an ideal material for foundations and other structural members subject to compressive forces. Its quality, particularly with regard to compressive strength, may to a certain extent be predetermined by design, and as a consequence its cost need be no greater than is necessary for the service that it is required to perform. The compressive strength of concrete is customarily indicated in terms of its unit strength at the age of 28 days unless specific reference is made to some other time interval. The ultimate strength of concrete in compression f'_c may range from approximately 1,000 to more than 7,000 lb. per sq. in. and is susceptible to approximate control during manufacture.

Although there is no definite relationship between compressive strength and other physical properties of concrete, the strength in compression is, for concrete containing similar ingredients, a reasonable index of the quality of other physical properties. It is generally true that, for concrete made from similar materials, a greater compressive strength is accompanied by a greater tensile strength, a greater transverse strength, higher modulus of elasticity, greater density, lesser permeability, and greater

durability. It is further indicated that greater compressive strength has been attained at a greater materials cost per cubic yard of concrete.

The ultimate compressive strength of concrete is influenced primarily by the richness in cement of the water-cement paste used to glue the aggregate into a solid mass. Factors such as aggregate type and quality, temperature and duration of water-cement reaction, treatment during the early hardening period, workmanship during mixing and placing, and age are of secondary but definitely important influence. Those secondary factors which influence strength approach primary importance as the quantity of cement is reduced. In other words, a high-strength concrete, containing a large amount of cement per unit quantity of water, will tolerate a greater number and degree of unfavorable factors of secondary influence with less evident harm than will a low-strength concrete lean in cement content. These various factors influencing the quality of concrete will be discussed later in greater detail.

Until the year 1918, the attainment of a predetermined compressive strength of concrete was a matter of chance. Empirical combinations of cement and aggregate were expected to attain certain strengths. For example, materials measured volumetrically in the proportions of 1:2:4 were generally expected to produce concrete having a compressive strength of 2,000 lb. per sq. in. at the age of 28 days. Mixtures proportioned 1:2½:5 were expected to attain a strength of 1,500 lb. per sq. in. at the same age. In December, 1918, Duff A. Abrams, professor in charge of the Structural Materials Research Laboratory, Lewis Institute, Chicago, published a theory which has since revolutionized the control of concrete quality, particularly with respect to compressive strength. Professor Abrams' theory was derived from the results of many tests made under his supervision, and his original premise has been fully confirmed in practice. The basic concept of the theory as expressed by Abrams is that "With given concrete materials and conditions of test the quantity of mixing water used determines the strength of concrete, so long as the mix is of a workable plasticity."[1]

The research of Abrams was performed during a period when concrete ingredients were generally measured by volume. As a

[1] P.C.A. minutes of annual meeting, December, 1918.

consequence, Abrams' original theory and empirical equations expressed the strength of concrete as a function of the volumes of water and cement contained in the paste. A recent accidental discovery of Inge Lyse,[1] then research assistant professor of engineering materials, Fritz Engineering Laboratory, Lehigh University, has simplified the application of the water:strength relation. Abrams expressed the compressive strength of concrete as an equation wherein the ratio of water to cement by volume was expressed as an exponential independent variable.

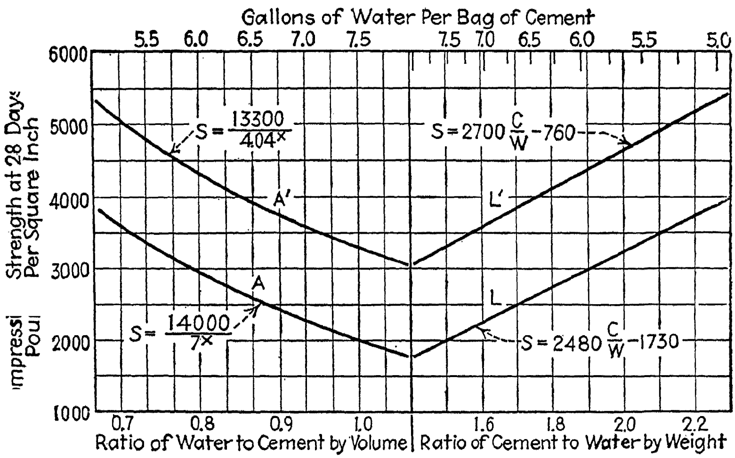


FIG. 1-2.—Relation of compressive strength to water content.

Lyse, using weights of cement and water, discovered the relationship to be linear in the first degree. Calculations made in the laboratory of The Port of New York Authority, by both methods, indicate that the sum of the squares of the residual errors is a minimum for the weight relationship, within the range of practical concrete mixtures.

Figure 1-2 illustrates graphically the relationship of strength to water-cement content by both Abrams' and Lyse's methods. Curve A conforms with Abrams' original equation, and curve L is the same plotted by Lyse's method. Curves A' and L' indicate the strengths obtained with various cements and aggregates of recent production. It should be noted that the compressive strength attained by concrete made with modern Portland cement

[1] *Eng. News-Record,* 1932.

is much greater than that obtained by Abrams using cements manufactured during the period 1915–1918.

The indicated improvement in compressive strength is the result of modified chemical composition, improved manufacturing processes, and greater fineness of grinding, voluntarily adopted by the Portland cement industry within recent years in order to compete with the contemporary production of high-early-strength cement. As a result of this improvement in the quality of cements, the engineer must modify his concept of concrete quality as measured by compressive strength. Concrete, made with cement of recent manufacture, attains a compressive strength approximately one and one-half times greater than that of concrete made prior to about the year 1928. Other physical properties of concrete such as density and impermeability have been benefited to a certain extent by the improvement in cement quality, but it is overoptimistic to infer that these other properties of concrete have been improved to the same degree as has the compressive strength. The engineer should remember this when comparing modern concrete with that produced prior to about the year 1928.

1-6. Modulus of Elasticity of Concrete. The modulus of elasticity of concrete in compression E_c is the ratio of the unit stress in pounds per square inch to the deformation in inches per inch of gauge length. A typical stress-strain diagram is illustrated in Fig. 1-3. It should be noted that the locus is a curve and that E_c is of variable value depending on the magnitude of the stress. Many contradictory values may be assigned to the modulus of elasticity as is shown in the illustration. Of these, the secant modulus at the value of the unit stress used for purposes of design is the proper one to be used for E_c in the design of reinforced-concrete structures.

The 1936 Building Regulations for Reinforced Concrete, A.C.I. 501-36T, prepared by the American Concrete Institute, indicate that

$$E_c = 1{,}000f'_c.$$

This value is an approximation but is reasonably correct. Various factors influence the elastic behavior of concrete, the primary one being the relative amounts of cement and water used to bind the aggregates together. The age and condition of curing

affect E_c in that both factors are of direct influence on the completeness of combination of cement and water. The density and degree of saturation of the hardened concrete are also influential factors.

In general, greater cement content, greater compressive strength, better curing conditions, greater age, and greater density of concrete tend to increase the value of the secant modulus E_c, which approaches the value of the initial tangent

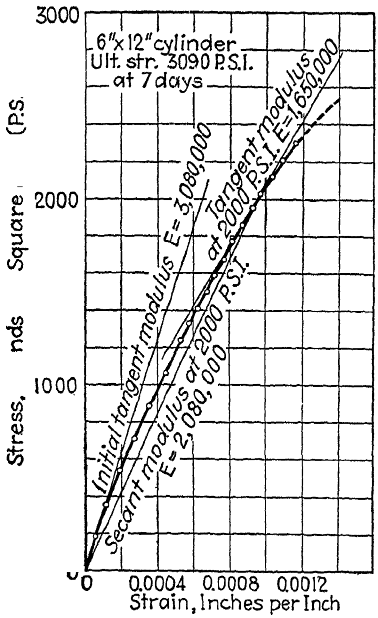


FIG. 1-3.—Modulus of elasticity of concrete.

modulus as a limit. E_c varies slightly with the degree of water saturation of the hardened concrete mass, completely saturated concrete having a slightly higher modulus than the same concrete in a partially saturated condition. Specimens containing Portland cement of current manufacture and having water contents of approximately 5 to $5\frac{1}{2}$ gal. per bag have been tested and found to have secant moduli closely approximating their tangent moduli over a major portion of their ultimate strength. Another factor having a pronounced influence on the value of E_c is the type and quality of aggregate contained in the mass. As a

general rule, the value of E_c is greater for concrete made with crushed stone aggregate consisting of prismoidal-shaped particles as compared with concrete made with gravel largely composed of rounded particles.

In the design of reinforced-concrete structures, it is assumed that concrete is an elastic material. For concrete of good quality the assumption is safe and practical, but it is fundamentally incorrect. In actual fact, concrete must be recognized as an extremely rigid plastic which is subject to progressive and

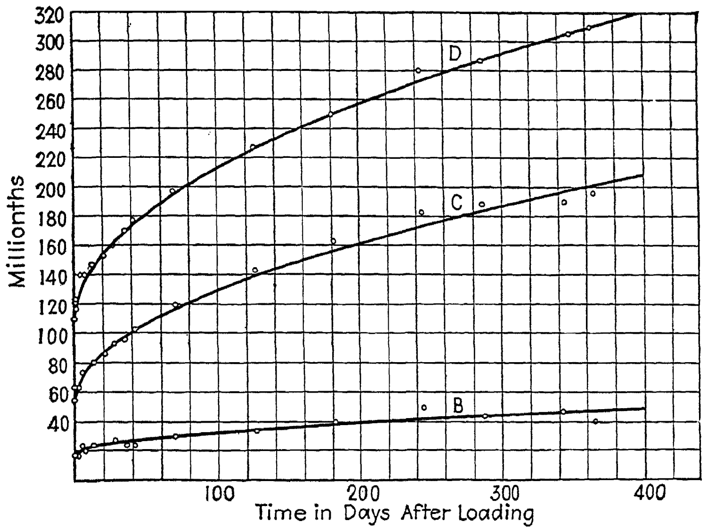


FIG. 1-4.—Plastic flow due to sustained load.

cumulative deformation under load. This property, termed "plastic flow," is negligible in most instances, but its existence should be recognized and considered in the design of unusual structures. Figure 1-4 shows the results of tests made by The Port of New York Authority to determine the magnitude of the permanent set or plastic flow under sustained load in connection with a study of the design of a reinforced-concrete arch crossing Riverside Drive and connecting with the George Washington Bridge.

The tests illustrated in Fig. 1-4 were made on concrete columns 10 by 10 by 48 in. containing 1 per cent reinforcement. Curves B, C, and D indicate the permanent deformation over a period

of one year of specimens loaded with sustained loads of 100, 300, and 500 lb. per sq. in., respectively. It is readily seen that the permanent deformation is measurable and is proportional to the amount and duration of loading.

1-7. Tensile Strength of Concrete. Concrete is poorly resistant to tensile loads, and, as a consequence, in the design of reinforced-concrete structures its tensile strength is assumed to be negligible. Tests of concrete having a relatively rich cement content and an ultimate compressive strength of 4,840 lb. per sq. in. at the age of 28 days indicated an ultimate tensile strength of 317 lb. per sq. in.[1]

1-8. Shearing Strength of Concrete. Pure, or "punching," shear is the sliding of one plane upon an adjacent plane. It is difficult to determine the resistance of concrete to this condition of failure. Concrete is not a homogeneous material, and the interlocking and bridging action of the coarse aggregate particles within the mass is a complicating factor which tends to distribute a shearing load as a combination of tension, compression, and shear throughout the mass. This subject is discussed at greater length in Chap. 4.

1-9. Miscellaneous Properties. The weight of ordinary concrete ranges between 140 and 160 lb. per cu. ft. and is dependent on the proportions, the degree of consolidation, and the relative densities of its ingredients and, also, upon the amount of water physically absorbed by the mass. Concrete weighing as little as 90 lb. per cu. ft. may be made with special lightweight aggregates. It has been customary for many years to assume the average weight of plain concrete to be 145 lb. and that of reinforced concrete 150 lb. per cu. ft.

Concrete shrinks when stored in air and expands in water. The amount of volume change is a function of the chemical composition of the cement, the water-cement ratio of the paste, and the richness of the mixture. In general, the amount and condition of the magnesia and the amounts of uncombined lime, sulfur trioxide, alumina, and iron oxide constituents of the cement influence the permanent volumetric change. Temporary or reversible changes in volume are caused by colloidal swelling and contraction of the cementaceous calcium silicates as influ-

[1] Gumensky and Whitman, Steel Stresses in Concrete Pipe, *Eng. News-Record,* Oct. 7, 1937.

enced by the degree of water saturation of the concrete mass. It is evident that volumetric variation of a concrete mass is the algebraic sum of permanent and temporary variation. It is exceedingly difficult, if not impossible, to predict or control the magnitude of these volumetric changes.

The actual shrinkage of concrete of good quality, stored in air, ranges approximately between 0.0002 and 0.0005 in. per in. Concrete stored in water may expand as much as 0.002 in. per in. within a period of one year. These volumetric changes must be considered in the design of long structural members such as pavement slabs, retaining walls, and dams, and provision must be made for a sufficient number of properly located expansion and contraction joints. In addition to the foregoing deformations, the effect of temperature must also be considered. The coefficient of thermal expansion of concrete may, for all practical purposes, be assumed to be 0.000006 in. per in. per °F.

1-10. Proportioning of Concrete. Briefly, the formulation of proportions of materials for concrete of good quality is the process of determining by experience or by trial what quantities of fine and coarse aggregate to mix with a cement paste of a specific dilution. The following essential facts should be remembered:

1. Dilution of cement paste with water influences the compressive strength and density of concrete. Less dilution results in greater ultimate strength and greater density, hence greater durability.
2. The dilution of cement paste influences the cost of the mixture. Greater dilution permits the incorporation of a greater amount of aggregate filler to produce a predetermined consistency of mixture. The introduction of more aggregate into a mixture produces a greater volumetric yield of concrete per unit quantity of cement used.
3. The minimum volume of cement paste included in a mixture should be the amount necessary to fill completely the voids between aggregate particles, completely coat each aggregate particle, and separate each particle with a film of paste sufficient to produce the desired workability.
4. The fine aggregate content of a mixture should be the minimum amount consistent with good plasticity. Because of greater surface area, less fine than coarse aggregate may be incorporated in a mixture, and as a result, for a given consistency, the economy is reduced as the ratio of fine to coarse aggregate is increased.
5. Shrinkage during setting and volume changes after hardening are functions directly proportional to the paste content of a mix. Since greater fine-to-coarse aggregate ratios require more paste for a given plasticity,

oversanded mixtures are more susceptible to volume change and possible shrinkage cracking in service than are those of minimum sand content.

6. Because more plastic mixtures require more paste to produce the greater workability, wet mixtures are more susceptible to shrinkage and to volume changes than are drier mixtures.

7. Although more coarse aggregate can be incorporated in a mix of given plasticity than can fine aggregate, the substitution of coarse for fine as a measure of economy is limited in scope. The minimum fine-to-coarse aggregate ratio is that with sufficient fine material to produce a cohesive plastic mixture free from segregation and honeycomb.

The object of fixing concrete proportions is to produce concrete of a quality suitable for the required structural service and, at the same time, limit the cost of the concrete in a manner reasonably commensurable with the importance of the work. The first consideration in the use of any particular mixture should be its ability adequately to withstand the forces imposed upon it in service. Of practically equal importance is the necessity that the hardened concrete shall be durable. Three methods of insuring adequate structural strength and durability are in general use. The first, most widely used, is the specification of a minimum cement content such as five, six, or seven bags of cement per cubic yard of concrete. The second, preferable from a technical view, is the specification of a maximum permissible water content per bag of cement. The latter method is based on Abrams' fundamental concept that the strength of concrete is a function of the relative amounts of cement and water used in the mixture. A third method, used on minor works and rapidly becoming obsolete, is the specification of arbitrary quantities of cement, fine aggregate, and coarse aggregate to be used in the mixture.

A secondary factor which should be considered is the attainment of proper plasticity without sacrifice of either strength or durability. Modern cement is of such quality that excellent strength may be attained with less cement than is required for dense and durable concrete. During recent years, great stress was placed by many engineering organizations on cement economy. This practice should be followed cautiously but never to the extent of sacrificing durability. The Joint Committee on Standard Specifications for Concrete and Reinforced Concrete, of the American Society of Civil Engineers and other affiliated organizations, have recommended in their 1940 report that the

water content, in gallons per bag of cement, be limited depending on the exposure condition. These recommendations with reference to reinforced concrete are indicated in Table 1-2.

TABLE 1-2.—WATER CONTENTS SUITABLE FOR VARIOUS CONDITIONS OF EXPOSURE
(Gallons per sack of cement)

Type and location of structure	Severe or moderate climate, wide range of temperatures, rain and long freezing spells or frequent freezing and thawing		Mild climate, rain or semiarid, rarely snow or frost	
	Thin sections reinforced	Moderate sections reinforced	Thin sections reinforced	Moderate sections reinforced
1. At the water line in water-front structures where complete or intermittent saturation is possible but not where the structure is continuously submerged:				
In sea water	5	5½	5	5½
In fresh water	5½	6	5½	6
2. Portions of water-front structures some distance from water line but subject to frequent wetting:				
By sea water	5½	6	5½	6½
By fresh water	6	6½	6	7
3. Ordinary exposed structures, buildings, and portions of bridges not covered above	6	6½	6	7
4. Complete continuous submergence:				
In sea water	6	6½	6	6½
In fresh water	6½	7	6½	7
5. Concrete deposited through water	..	5½	..	5½
6. Pavement slabs directly on ground:				
Wearing slabs	5½	..	6	
Base slabs	6½	..	7	

7. Special cases:
   a. For concrete exposed to strong sulfate ground water, or other corrosive liquids or salts, the maximum water content shall not exceed 5 gal. per sack.
   b. For concrete not exposed to weather, such as the interior of buildings and portions of structures entirely below ground, no exposure hazard is involved, and the water content should be selected on the basis of the strength and workability requirements.

It is customary, on work of limited cost or minor importance, to specify arbitrarily selected volumes of materials such as 1:2:3 or 1:2½:5 for the manufacture of concrete. Batch quantities of aggregates are sometimes measured volumetrically in wheelbarrows, boxes, or hoppers adjusted to contain the desired amount of material. It is a peculiar property of granular materials, and particularly so of small particles, that moisture influences the apparent volume of the material.[1] If a container having a capacity of 1 cu. ft. is filled with dry sand, it will be found to hold approximately 100 lb. of material. If this sand is dumped from its container and is moistened with water, all of it cannot be replaced in the container. An amount that may be as great as 25 lb. will remain. This bulking, or increase in apparent volume, is influenced in amount by the moisture content of the material. The apparent volume of a specific weight of sand may increase 30 per cent or more as its moisture content increases from zero to approximately 8 per cent. Moisture in excess of about 8 per cent results in a reduction in the amount of bulking until, at the condition of complete saturation, the apparent volume is approximately equivalent to the volume of the sand measured dry. The same effect is evident in the volumetric measurement of coarse aggregate except that the maximum increase in apparent volume due to moisture seldom exceeds about 5 per cent.

Small changes in moisture condition are accompanied by great changes in bulking, especially in the case of fine aggregate. When aggregates are measured by volume, it is necessary, in order to obtain a specific amount of material for the batch, to determine the bulked volume of the material and to adjust the measuring device accordingly. Such correction is seldom applied on small jobs, and even when made with extreme care it is of doubtful reliability, since the moisture condition of an aggregate supply may be quite variable over a small time interval. For this reason, concrete proportioned by volumetric measurement is usually variable in consistency, composition, and quality. When volumetric proportions are specified, and uniform quality of concrete is desired, it is preferable that the weight per unit volume of each aggregate be determined and used to convert

[1] A. T. Goldbeck, National Crushed Stone Association, *Bull.* 1, 1927.

the proportions into weight units and that the aggregates be so measured.

In comparison with the foregoing method, it is desirable, even on unimportant work, to specify arbitrary proportions by weight such as 94:200:350. Such a requirement eliminates any possible misunderstanding with regard to method of measurement. For example, about 98 lb. of well-graded gravel will fill a cubic-foot measure. If, during filling, the gravel is tamped and consolidated, as much as 110 lb. of material may be required to fill the measure. Whether a cubic foot of the material should be measured in a loosely filled or in a partially or tightly consolidated condition is a matter of opinion, and, unless defined, the intent of the specification may be misinterpreted. This is not the case if proportions are specified by weight. A specific weight of material is not subject to misinterpretation. When 200 lb. of sand is required, it means just that and not 190 lb. of sand and 10 lb. of water. Because of its unquestionable precision of meaning, a weight specification is exceedingly advantageous.

As a guide to the proportioning of concrete for minor work on the basis of damp weights, Table 1-3 has been calculated for fine and coarse aggregate having a saturated surface-dry specific gravity of 2.65 and average moisture conditions of 5 and 1 per cent free surface moisture, respectively. This tabulation may be used in conjunction with Table 1-2 (Water Contents Suitable for Various Conditions of Exposure). It is based on the recommendations of the Joint Committee of the A.S.C.E. and affiliated committees in their progress report of 1937 and is intended to produce concrete having a consistency of 4 in. as measured by the slump.[1] Table 1-3 may be used for other aggregates of known specific gravity, provided the indicated weights are multiplied by the ratio of actual specific gravity to the assumed value of 2.65. The designer is cautioned that this tabulation represents approximations of correct weights and that slight changes in weight of fine or coarse aggregate, or volume of added mixing water, may be necessary to produce concrete of optimum quality.

Since the volumetric measurement of aggregates is very unsat-

[1] The consistency as measured by the slump is the subsidence of a plastic concrete specimen, the original height of which was 12 in.; A.S.T.M. Designation: C138-32T.

isfactory, the practice is rapidly becoming obsolete.[1] Current practice on work of any importance is the measurement of aggregates by weight, at the same time making allowance for the moisture carried by the damp aggregate. For example, if 1,000 lb. of surface-dry sand are desired in a batch, and the available material contains 5 per cent of moisture by weight, then 1,053 lb. of sand weighed into the batch would contain 0.95 × 1,053, or 1,000 lb., of dry sand and 53 lb. of water. Similar correction would be made for the amount of surface moisture carried by the coarse aggregate. The 53 lb. or 6.4 gal. of water weighed with the sand plus the water carried into the batch by the coarse aggregate should be deducted from the desired total water content of the batch, and the difference used as mixing water.

TABLE 1-3.—WEIGHTS OF MATERIALS FOR CONCRETE

		¾- to 1-in. coarse aggregate					1½- to 2-in. coarse aggregate				
Total water, gal. per bag	Probable strength, lb. per sq. in., 28 days	Cement, lb.	Fine aggregate, lb.	Coarse aggregate, lb.	Mixing water, gal.	Cement factor, bags per cu. yd.	Cement, lb.	Fine aggregate, lb.	Coarse aggregate, lb.	Mixing water, gal.	Cement factor, bags per cu. yd.
5	4,250	94	147	233	3.8	8.0	94	158	273	3.7	7.4
5½	3,700	94	169	263	4.1	7.2	94	179	303	4.1	6.7
6	3,300	94	200	273	4.4	6.5	94	210	334	4.3	6.0
6½	3,000	94	221	314	4.6	6.0	94	242	364	4.6	5.5
7	2,750	94	253	334	5.0	5.6	94	253	394	5.0	5.1
7½	2,500	94	274	359	5.3	5.2	94	290	420	5.2	4.8
8	2,250	94	295	384	5.7	4.9	94	316	445	5.5	4.5

[1] When one is faced with the necessity of using volumetric measurements on a small, rush job, Fuller's rule for estimating quantities required per cubic yard of concrete may be of some assistance in getting approximate volumes. It is the following:

$$C = \frac{11}{c + s + g}, \qquad S = C \times s \times \frac{3.8}{27}, \qquad G = C \times g \times \frac{3.8}{27}$$

where C = bbl. of cement per cu. yd. of concrete, S = cu. yd. of fine aggregate per cu. yd. of concrete, G = cu. yd. of coarse aggregate per cu. yd. of concrete, c = volumetric ratio of cement, s = volumetric ratio of fine aggregate, g = volumetric ratio of coarse aggregate.

For example, a 1:2:4 mix is represented by $c{:}s{:}g$.

See *Plain Concrete* by Edward E. Bauer.

Accurate measurement of water is necessary for the satisfactory control of concrete quality. Reference to Table 1-3 indicates that, for average moisture conditions, between 20 and 30 per cent of the total water included in a concrete batch is introduced with the aggregates. So as not to exceed a predetermined water content it is necessary to determine and compensate for the amount of water introduced with the aggregates. Water added in addition to that carried by the aggregates is termed "mixing water." For concrete of uniform quality, the total water and the cement contents of various batches must be maintained constant. This is best accomplished by regulating the mixing water with a metering device or with a calibrated measuring tank. It is current practice among progressive engineers who require concrete of good quality and uniformity to require that a water-measuring device be arranged so that it will automatically contain or deliver various amounts of water without the personal attention of an operator except to charge or discharge the device.

The proportioning of concrete, for major structures of considerable size, is most economical when done on an experimental basis. Large structures, such as the Triborough Bridge in New York City, require such volumes of concrete as to warrant special investigation of concrete proportions by laboratory methods. The A.S.T.M. in its Standard Method of Test for Compressive Strength of Concrete, A.S.T.M. Designation: C39-42, outlines an excellent program for such investigation. Such studies usually determine the desirable combinations of available aggregate necessary to produce maximum plasticity and maximum density of concrete. It is customary, at the same time, to determine the amount of water which will insure the desired structural strength.

Two rules are of material assistance to the study of experimental concrete data. The first is that of Abrams as modified by Lyse: "The compressive strength of concrete is a function of the ratio *by weight* of cement to water used in the mix so long as the mixture is plastic." This relationship is readily demonstrated provided a sufficient number of observations are available. The New York City Building Code indicates that not less than four observations at each of four different cement-water ratios should be used to determine this relationship. This number is without doubt a safe minimum, but a greater number

of observations should be used wherever possible. The second rule of material assistance in the interpretation of test results was first disclosed, in principle, by Lyse. In its present modified form, as applied in the laboratory of The Port of New York Authority, it is stated that "The consistency of concrete remains constant, regardless of the richness of the mix, if the general size and type of aggregate and the water content per unit of freshly mixed concrete remain constant." The problem of proportioning concrete mixtures on an experimental basis is much too involved for complete discussion in this chapter. For further information on the subject the reader is referred to the Recommended Practices and Standard Specifications for Concrete and Reinforced Concrete by the Joint Committee of the A.S.C.E. and affiliated organizations (herein referred to as the "Code") published in June, 1940, by the American Concrete Institute, Detroit, Mich., and to current publications of the Portland Cement Association.

1-11. Concrete Yield Computations. The computations peculiar to concrete proportioning consist of the arithmetical application of certain conversion factors of constant value and others whose value must be determined by test. The values assigned to cement and water are assumed to be constants. Those assigned to fine and coarse aggregate must be determined by test of the particular material in the condition being considered.

The weights of one bag or one barrel of cement are standardized by specification and are 94 and 376 lb., respectively. One bag of cement weighing 94 lb. is generally assumed to have an apparent volume of 1 cu. ft. The specific gravity of cement may vary over a very limited range, but it is safe to assume that its value is 3.15. Water is generally assumed to weigh 62.5 lb. per cubic foot and 8.33 lb. per gal. One cubic foot of water is equivalent to $7\frac{1}{2}$ gal. These assumed values are sufficiently correct to be within the limit of accuracy of measurement of batch quantities.

The *yield* of a concrete batch and the amount of cement required for 1 cu. yd. of concrete are factors of great interest to the engineer. The most accurate method of determining the yield is to compute the absolute volume of each ingredient in a batch, including the water. The sum of the absolute volumes of all the ingredients is the theoretical yield and differs very little from the actual yield. From the yield of a specific quantity

of cement it is easy to determine the amount of cement required to produce 27 cu. ft., or 1 cu. yd. of concrete.

For example, the cement required for the first mixture in the $\frac{3}{4}$-in. aggregate column of Table 1-3 is calculated as shown in the following table:

	Cement	Fine aggregate	Coarse aggregate	Water
Batch quantities	94 lb.	147 lb.	233 lb.	3.8 gal.
Moisture content, %		5	1	
Amount of water in aggregate		7.4 lb.	2.3 lb.	1.2 gal.
Weight of dry material, lb	94	139.6	230.7	
Specific gravity	3.15	2.65	2.65	1.00
Absolute volume of material, cu. ft.	0.48	0.84	1.39	0.67

The sum of the individual absolute volumes of the various ingredients is 3.38 cubic feet of mixed material. If one bag (94 lb.) of cement yields 3.38 cu. ft. of concrete, then 7.99 bags, or practically 2 bbl., of cement will be required to produce 1 cu. yd. of concrete. (x bags: 1 bag : : 27 cu. ft.:3.38 cu. ft.)

In the foregoing computation, a step that is sometimes misunderstood is the conversion of weight of dry materials to absolute volume. In order to make this conversion, the specific gravity of each material must be known or determined by test. By multiplying 62.5 (the weight of a cubic foot of water) by the specific gravity of a material, the weight of a solid cubic foot of that material is derived. This absolute or solid weight per cubic foot is the basis of conversion from weight to absolute volume.

The quantities of materials required for a 1-cu. yd. batch of concrete are obtained by multiplying the quantities given for a one-bag batch by the cement factor per cubic yard. In the foregoing example, a 1-cu. yd. batch would require:

Cement	Damp fine aggregate	Damp coarse aggregate	Mixing water
752 lb.	1,176 lb.	1,864 lb.	30.4 gal.

It is beyond the scope of this work to illustrate all of the computations involved in concrete proportioning. The calculation of proportions of materials in terms of dry-rodded volume requires the determination by test of the apparent weight per cubic foot of the aggregates in the dry-rodded condition. So also does the computation of proportions by damp-loose volume require the actual determination of the damp-loose weights per cubic foot of the respective aggregates. For a more complete discussion of these and other computations, the student is referred to "Design and Control of Concrete Mixtures," published by the Portland Cement Association.

1-12. Mixing Concrete. Except under very unusual circumstances, concrete is mixed in a power-driven drum equipped with blades so arranged as to agitate, stir, and interfold the plastic mass. The size and shape of the drum, its speed of rotation, the arrangement and condition of the blades, and the duration of mixing all influence the quality of concrete. Of these factors it is customary for the engineer to control the rate and duration of mixing. From experience it has been determined that a concrete mixer should revolve at an approximate peripheral speed of 200 ft. per min. Speeds greater than about 225 and less than 100 ft. per min. are usually found to be unsatisfactory. As a general rule, the peripheral speed for greatest mixing efficiency is inversely proportional to the diameter of the drum.

The quality of concrete is improved to a certain extent by longer and more thorough mixing, but, beyond an indefinite optimum time, further mixing may produce a reduction in quality. Experience indicates that in no instance should the duration of mechanical mixing be less than 1 min. after all the materials including the water are in the drum. Current practice on projects requiring concrete of uniformly good quality is to require a minimum mixing time of $1\frac{1}{2}$ min. for a mixer of 1 cu. yd. capacity and a greater time for larger mixers. A safe rule which may generally be used to indicate the desirable duration of mixing is to require $1\frac{1}{2}$ min. for the first cubic yard of mixer capacity and an additional $\frac{1}{2}$ min. for each additional cubic yard thereafter.

It cannot be too strongly emphasized that the duration of mixing should be counted from the time when all the materials, including the water, have been introduced into the drum. A

batch of materials mixed with less than the designed amount of mixing water is prone to roll and ball up in the mixer and, except with continued mixing of abnormally long duration, will be nonuniform in composition when discharged from the drum. This condition is particularly evident in large mixers of the truck-mix type.

Modern job mixers differ mainly in detail and not in type or principle. Their main abuse is one of overcharging. Mixers cannot efficiently or satisfactorily function when they are charged with a volume of material greater than one-half the gross volume of the mixer drum. They are designed by their manufacturers efficiently and properly to mix a specific volume of concrete. This design capacity is usually indicated by their catalogue numbers. Thus a No. 54S mixer is designed to mix 54 cu. ft. of concrete, whereas a No. 27E mixer is intended to mix 27 cu. ft. of concrete.

Within recent years, the use of mixers mounted on truck bodies and used to deliver concrete or concrete materials from a remote central batcher plant to the site of the work has rapidly increased. These mixers range in capacity from 2 to 6 cu. yd. Early models of truck mixers were prone to produce concrete of variable quality throughout a single batch, but most truck mixers manufactured since the year 1935 have been properly designed to produce uniform mixtures. The general trend in design has been to lower the ratio of drum length to diameter and to improve the interfolding action of properly arranged mixing blades. Improvement has also been made in water-transfer systems, and many truck mixers are equipped with mechanical water pumps. Rapid transfer of water into the drum materially reduces the time required for satisfactory mixing and improves the uniformity of the concrete throughout the batch.

Portland cement of recent manufacture is usually well proportioned and well calcined, but even today some underburned cement may occasionally be produced. Material of this nature is liable to form greater than normal amounts of laitance upon continued mixing. For this reason, the mixing of concrete for periods greater than approximately 20 min. should be discouraged.

1-13. Handling and Placing of Concrete. A most thorough and careful design may be completely defeated by improper construction practices in the handling and placing of concrete as

well as of concrete ingredients. It is obvious that cement should be properly stored and protected from moisture, but it is less obvious that care must be observed in the storage and handling of aggregates. Aggregate storage should be arranged to protect the material from contamination with earth, coal dust, plaster, scrap lumber, and similar materials which are undesirable in concrete. In addition, fine and coarse aggregates should not be permitted to intermingle with, or contaminate, each other prior to batch measurement.

Aggregates should be transported and stored in such a manner as to preclude segregation of the material. This is particularly true of coarse aggregate which is especially susceptible to separation of sizes if dropped through the air from a crane bucket or other conveyor. As a general rule, aggregates should be stored in properly constructed bins or in low stock piles, the slopes of which are less than the angle of repose of the material. Every precaution should be taken to minimize the dropping or rolling of aggregate through any appreciable distance. The unrestrained dropping of aggregate on the apex of a pile or on the slope of natural repose will result in the coarser particles segregating and concentrating at the toe of the slope.

These same general considerations should be observed in the handling and placing of mixed concrete. Unrestrained dropping, steep chuting, and horizontal flow of concrete should not be permitted. Concrete should be transferred from the mixer to the forms with a minimum of handling, and persistent precautions should be observed to prevent segregation of the mixture. Concrete is analogous to an emulsion of materials of different specific gravities. Improper handling may result in separation of certain constituents. When concrete must be dropped from a height or slid down a chute, its fall or flow should be restrained by frequent deflecting baffles in such a manner that it continues to resemble a cohesive mass rather than a goblike shower.

Concrete should be placed vertically in the form in its final resting place. Chuting into the form at an angle and flow of concrete in the form should not be tolerated. The movement of concrete in any direction except the vertical is a differential movement of its component ingredients. This tendency of coarse aggregate and mortar to come to rest at different places when moved horizontally or on an incline must be minimized by

rigid observation of good construction practices. The placing of concrete in horizontal layers should be required, and any practice in handling or placing that tends to produce sloping planes of contact between successive lifts should be firmly disapproved.

After placing, concrete should be thoroughly consolidated by tamping, rodding, and spading. During recent years, the introduction of mechanical vibration has greatly reduced the amount of tamping and spading of concrete. Vibration should be considered only as an auxiliary method and not as a substitute for these earlier forms of consolidation. Spading, in particular, is necessary for the production of good form finishes and smooth exposed surfaces of optimum durability. The spading of concrete should be done in such a manner as to work the coarse aggregate *away* from the form and not toward it. This insures a maximum of mortar against the form where it is needed to minimize possible honeycomb or air voids. The coarse aggregate worked away from a form during proper spading will tend to return to the face because of semifluid pressure of other coarse particles within the plastic mass.

The *proper* use of mechanical vibration is beneficial to the density, compressive strength, bond strength between concrete and steel, and economy of the mixture. The *improper* use of vibrators or use on the wrong type of concrete may result in greater harm than would occur if the concrete were placed without any special effort toward consolidation. The most serious abuse of vibration is its application to concrete of too great workability. The consistency of concrete to be vibrated, as measured by the slump test, should be not greater than 3 in., and drier mixtures are desirable. In addition, the mixture should definitely possess a minimum sand content to the extent of being harsh. Another serious abuse of vibration is the prevalent tendency to overvibrate. Workmen soon observe that concrete can be guided as a stream by leading the mass with a vibrator. This requires much less effort than does the use of a shovel, so they usually move the concrete horizontally with the machine, thus causing decided segregation.

Continued vibration moves the ingredients of lower specific gravity vertically toward the surface of the mass. This produces differential water gain in the forms. The cement-water ratio

of the concrete in the upper section of a lift is adversely affected, and a plane of low durability is usually produced at the top of the lift. In addition, the vertical movement of water upward and cement downward produces pockets reasonably devoid of cement on the undersurface of horizontal reinforcement and on the underface of many aggregate particles. These pockets frequently can be seen with the naked eye in sawn sections of overvibrated concrete. Only with competent and rigid inspection is the use of mechanical vibration a desirable aid in the consolidation of concrete.

Concrete, during the period of its plastic life before initial set, shrinks appreciably in volume. A well-proportioned mixture of reasonably stiff consistency, made with aggregates of favorable texture, ordinarily shrinks much less than 1 per cent. In some instances, particularly in the case of oversanded mixtures of wet consistency containing less desirable types of fine aggregate, the volumetric shrinkage prior to initial set may be greater than 2 per cent. Plastic concrete is a suspension of solids in water. The component ingredients have different relative densities. Because of the difference in relative density, the heavier constituents tend to settle and thus consolidate themselves to a greater extent than can be done with any known method of mechanical tamping or vibration. Because of this tendency toward slow vertical segregation, the upper surfaces of flat slabs and the upper sections of deep lifts are usually weaker and less durable than are the lower portions. This tendency toward volumetric consolidation at the expense of the vertical dimension frequently requires that surface finishing be delayed until the shrinkage has considerably progressed.

In order to minimize the undesirable effects of wet shrinkage, the workability of concrete should be maintained at the minimum plasticity consistent with the sectional size of forms and desired form finishes. All concrete, and especially that in pavement slabs, should be tamped, spaded, vibrated, and worked only in an amount sufficient to produce satisfactory consolidation and finish. This is particularly true of concrete of wet consistency, a type that should be protected against all unnecessary working in the forms.

1-14. Curing. The reaction of Portland cement and water is one requiring time and favorable temperature conditions for its

completion. It may be considered essentially as a progressive reaction of a liquid phase (water) and a reasonably insoluble phase (cement). It is true that cement constituents are partly soluble in water, but a condition of complete saturation of the solute is soon reached without much diminution of the amount of cement available for solution. The reaction actually occurs in this saturated or possibly supersaturated solution. As in practically all chemical reactions, the process approaches equilibrium unless the products of reaction are removed from the field of action. Water must be available in order that the successive solution, reaction, and deposition of reaction products by crystallization or precipitation may repeatedly occur.

As in other chemical processes, the reaction of cement and water is influenced by the prevailing temperature. It is logical to infer that, like most other soluble or partially soluble substances, the solubility of cement in water is less at lower temperatures. Lower solubility is productive of lesser concentration of material available for reaction. It is a well-established fact that the velocity of a chemical reaction becomes less as the concentration of the reactive materials is decreased. In all probability, the effect of lower solubility of cementaceous material and the resulting lesser concentration of reactive agents on the velocity of reaction at reduced temperatures are the most logical explanation of the fact that concrete cured entirely at low temperature is inferior in quality. This probable explanation is further confirmed by the fact that, after prolonged curing at low temperature and further storage at normal temperatures, in the presence of favorable moisture conditions, the reaction continues at a normal rate, and the concrete improves in compressive strength and other qualities.

Freshly placed concrete should be protected from reduced temperatures and from loss of water by evaporation. An amount varying between 35 and 75 lb. of water per 94-lb. bag of cement is included in a concrete batch. From tests reported by F. R. McMillan,[1] director of research of the Portland Cement Association, it is indicated that approximately 20 lb. of this water is chemically combined with the cement in hardened concrete. Water in excess of this amount is the medium within

[1] "Basic Principles of Concrete Making," McGraw-Hill Book Company, Inc., 1929.

which the chemical reaction occurs. Since the reaction is very slow and requires much time for its completion, it is necessary to maintain the presence of this reacting medium within the mass in order that the reaction may continue. It is further desirable that the temperature of the mass be maintained above a minimum limit favorable to chemical action.

The maintenance of the foregoing condition is called "curing." Loss of water (the reaction medium) from the mass may be minimized by leaving the forms in place, by interposing wet burlap or earth between the hardened mass and the atmosphere, by frequently wetting the mass to replace evaporation loss, or by sealing the surface of the mass with a waterproofing film. Curing, or the conservation of the reaction medium within the mass, should be observed for a period of approximately 10 days in order to insure that the reaction progresses sufficiently. If conservation of moisture within the mass is not observed, the concrete may be weak, porous, and of poor durability.

Conservation of moisture is easily attained with determined supervision, but the maintenance of proper temperatures is more difficult. Concrete may be safely mixed and placed during winter temperatures provided the aggregates and water are heated to a temperature of between 50 and 100°F. The temperature of the freshly placed concrete should be maintained above 50°F. for at least 7 days in order that chemical reaction may progress. This is especially true of thin sections usually existent in reinforced structures. The reaction of cement and water generates some heat, and in massive construction the liberated energy may be sufficient to maintain a favorable temperature within the mass, but the radiation from thin sections is greater in amount than the heat energy supplied by chemical reaction. For this reason heat must be conserved or supplied from some external source. The insulating value of salt hay covering is sometimes sufficient to conserve heat energy, but, in most instances, artificially heated tarpaulin enclosures are used to protect the work properly.

The reaction of cement and water ceases, for all practical purposes, at a temperature of about 35°F. Concrete maintained at that temperature will possess little strength but is not necessarily damaged by such exposure. If its temperature is increased

to 50°F., or more, the reaction will progress at a greater rate, and the concrete will gain in strength. The greatest danger of cold-weather concreting to be avoided is the possibility of partially or recently set concrete being frozen. The freezing of the water contained in a recently hardened mass produces disruptive forces at a time when the hardened paste is too weak to resist the condition, and as a result tensile failures of aggregate bond and possibly complete fractures of interstitial paste will occur. Although these fractures may partially recover at later dates by what is termed *autogenous healing*, they cannot recover completely, and the existent structure is irreparably weakened. In most instances, concrete that has been frozen shortly after final set must be replaced with satisfactory material.

For a more complete description of the effect of curing on the physical properties of concrete, the studies of H. F. Gonnerman, published in the A.C.I. *Proceedings*, Vol. XXVI (1930), p. 359, and the observations of H. J. Gilkey on the Moist Curing of Concrete reported in *Engineering News-Record*, Oct. 14, 1937, should be consulted. Gilkey has found that the compressive strength of concrete at the age of one year may vary between 45 and 135 per cent of the value obtained on concrete cured at 70°F. and completely saturated, the variation depending on the degree and duration of moist curing and the moisture content at the time of test.

1-15. Forms. Forms are intended to define the contour and locate the position of individual members with reference to the structure as a whole. In order for them to limit satisfactorily the size, shape, and position of these members, it is necessary that they be built to resist the forces imposed upon them. Freshly mixed concrete is a fluid and is generally assumed to exert a horizontal pressure equal to the hydrostatic head of a liquid weighing 145 lb. per cu. ft.

Concrete that is subjected to high-frequency vibration acts as a fluid throughout its depth, and, as a consequence, the full hydrostatic head must be considered. Concrete placed without the aid of vibration exerts a fluid pressure for a depth depending on the rate and temperature of placing. The consolidation and interlocking of aggregate and the initial setting of cement tend to neutralize fluidity, and, as a consequence, the horizontal pres-

sures are somewhat less than the full hydrostatic head. From literature of the Universal Form Clamp Company[1] it is indicated that concrete having a temperature of 50°F. and placed at the rate of 6 vertical ft. per hour exerts a maximum horizontal thrust of 1,030 lb. per sq. ft. at a 9-ft. depth of head. At greater depths the horizontal pressure is reported to be less until, at a head of 12 ft., it is given as 870 lb. per sq. ft. The maximum pressure indicated above is approximately 80 per cent of the theoretical fluid pressure.

Construction practices that are favorable to the early setting of the concrete mass result in smaller maximum fluid pressures and lesser effective hydraulic head. For example, it is indicated in the literature referred to in the preceding paragraph that concrete poured at a temperature of 70°F., and at a similar rate to that given above, produces a maximum fluid pressure of 740 lb. per sq. ft. at a depth of 7 ft. This value is approximately 73 per cent of the theoretical fluid pressure, but its depth of maximum action is 22 per cent less than that of concrete having a temperature of 50°F.

The foregoing is sufficient indication of the problems involved in the design and construction of adequate forms. A discussion of other data regarding forms will be found in Chaps. 15 and 16.

1-16. Reinforcing Steel. Concrete cannot be relied upon to withstand appreciable tensile stress. This deficiency is overcome by embedding steel rods in those parts of a section that are subjected to tension. The two materials act in conjunction with each other, each doing the work for which it is best suited. The combination acting as a unit, concrete resisting compression and steel resisting tension, is called "reinforced concrete." It is a strong, durable, and economical system of construction, and it has proved satisfactory for a large variety of structures.

Much has been said about concrete, and little about steel. The latter is of equal interest and importance. Because of greater knowledge and better control during manufacture, reinforcing steel is a standardized material of dependable uniformity. It is ordinarily manufactured in the form of plain, deformed, or twisted rods or bars of various cross-sectional areas, ranging from $\frac{1}{4}$ in. round to $1\frac{1}{4}$ in. square. It is regularly available as wire and wire mesh in various sizes and combinations of weave.

[1] P.C.A., Forms for Architectural Concrete.

Reinforcing bars are hot rolled from car axles, tee rails of standard section, and new steel billets. Because of greater dependability from the standpoint of uniformity, it is usual, in the case of important structures, to require the use of bars manufactured from new billets. Billet-steel and axle-steel bars are produced in what are termed *structural*, *intermediate*, and *hard* grades. Rail-steel bars are similar in properties to the billet and axle types of hard grade. The essential differences in properties of the structural-, intermediate-, and hard-grade materials are, respectively, greater ultimate strengths and higher yield-point values accompanied by progressively lower ductility. Higher yield-point values are desirable because of greater permissible working stresses but are accompanied by lower ductility and result in greater cracking of concrete in regions of tension. The desirable balance between these properties is a matter of personal opinion and professional judgment.

Wire and wire mesh are ordinarily available in sizes ranging from No. 4-0 to No. 14 gauge and are manufactured from cold-drawn steel wire. It is fabricated as mesh by welding in a large variety and combinations of weave. The ultimate strength of cold-drawn reinforcing wire is approximately equal to that of hard grades of hot-rolled bars, but the yield point of this material is appreciably greater than that of rolled bar steel. Table 1-4 indicates the comparative physical properties of these various grades of steel as specified by the A.S.T.M.

The modulus of elasticity in tension of plain carbon steel may vary between 28,000,000 and 31,000,000 lb. per sq. in. It is generally assumed to be a constant having a value of 30,000,000 lb. per sq. in. At normal atmospheric temperatures, steel is considered to be a truly elastic material throughout a loading range below a certain value termed the *proportional*, or *elastic*, limit. The proportional limit is not coincident with the yield point as determined by ordinary acceptance tests but is sufficiently close for all practical purposes.

Reinforcing steel should be well embedded within the concrete mass in order to protect it from corrosion and from damage by fire. General experience has proved that a minimum cover of 2 in. is sufficient protection except for structures that are continuously wet, in which case a 3- or 4-in. cover is desirable. Embedment of steel within the basic concrete medium serves to

TABLE 1-4.—A.S.T.M. PHYSICAL REQUIREMENTS FOR REINFORCING STEEL

Properties	Type	Plain bars, grade			Deformed bars, grade			Twisted[3] bars
		Structural	Intermediate	Hard	Structural	Intermediate	Hard	
Tensile strength, lb. per sq. in.	New billet	55,000 to 70,000	70,000 to 90,000	80,000 min.	55,000 to 70,000	70,000 to 90,000	80,000 min.	Record only
	Car axle	55,000 to 70,000	70,000 to 90,000	80,000 min.	55,000 to 70,000	70,000 to 90,000	80,000 min.	
	Rail steel	....	....	80,000 min.	....	....	80,000 min.	80,000 min.
	C-D wire	....	....	80,000 min.				
Yield point, min., lb. per sq. in.	New billet	33,000	40,000	50,000	33,000	40,000	50,000	55,000
	Car axle	33,000	40,000	50,000	33,000	40,000	50,000	
	Rail steel	....	....	50,000	....	....	50,000	50,000
	C-D wire	....	....	0.8 tens. str.				
Elongation in 8 in., min., %	New billet	1,400,000[1] / Tens. str.	1,300,000[1] / Tens. str.	1,200,000[1] / Tens. str.	1,250,000[1] / Tens. str.	1,125,000[1] / Tens. str.	1,000,000[1] / Tens. str.	5
	Car axle	1,400,000[1] / Tens. str.	1,300,000[1] / Tens. str.	1,200,000[1] / Tens. str.	1,250,000[1] / Tens. str.	1,125,000[1] / Tens. str.	1,000,000[1] / Tens. str.	
	Rail steel	....	....	1,200,000[1] / Tens. str.	....	....	1,000,000[1] / Tens. str.	1,000,000[1] / Tens. str.
	C-D wire	Reduction of area 30 per cent min. (greater than 100,000 tens. str., 25 per cent min.)						
Bend test[2]								
Under ¾ in.	New billet	180° $d = t$	180° $d = 2t$	180° $d = 3t$	180° $d = t$	180° $d = 3t$	180° $d = 4t$	180° $d = 2t$
	Car axle	180° $d = t$	180° $d = 2t$	180° $d = 3t$	180° $d = t$	180° $d = 3t$	180° $d = 4t$	
	Rail steel	....	....	180° $d = 3t$	....	....	180° $d = 4t$	
0.3 in. and under	C-D wire	....	....	180° $d = t$				180° $d = 4t$
¾ in. and over	New billet	180° $d = t$	90° $d = 2t$	90° $d = 3t$	180° $d = 2t$	90° $d = 3t$	90° $d = 4t$	180° $d = 3t$
	Car axle	180° $d = t$	90° $d = 2t$	90° $d = 3t$	180° $d = 2t$	90° $d = 3t$	90° $d = 4t$	
	Rail steel	....	....	90° $d = 3t$	....	....	90° $d = 4t$	
Over 0.3 in.	C-D wire	....	....	180° $d = 2t$				90° $d = 4t$

[1] Specific modifications depending on size.
[2] d = diameter of pin around which specimen is bent; t = thickness or diameter of specimen.
[3] Billet-steel bars twisted cold; rail-steel bars twisted hot.

inhibit oxidation and corrosion. In many instances, the demolition of old structures has shown that the steel was well protected and in excellent condition.

1-17. Allowable Unit Stress and Safety Factor. The allowable unit stress and the safety factor are interdependent and are based on experience. The safety factor is the proportional amount by which the ultimate strength exceeds the permissible unit stress used in the design. The values shown in Table 1-5 are the permissible unit stresses based upon the 1940 report of the Joint Committee on Standard Specifications for Concrete and Reinforced Concrete—the Code.

The problems given in the text are for the purpose of demonstrating the design of reinforced concrete as encountered in practice. For that reason, a wide range of allowable unit stresses is used in order to illustrate that the quality of the materials, the uncertainties of construction, and the nature of the work are factors that must all be considered.

The permissible unit stress in steel reinforcement for design purposes varies with differences of the applicable codes, the quality of steel, wartime or other necessity for economizing on steel tonnages used, as well as with the individual opinion and professional judgment of the designer. As examples of variations due to individual conditions and illustrating the need of good judgment, the following may be cited:

1. Culverts or tunnels under deep embankments, bins designed for maximum filling, foundations supporting massive superstructures, and, in general, most structures carrying relatively large determinable loads compared with any possible future increases—these may take advantage of higher allowable design unit stresses provided the accompanying cracking of the concrete in the regions of tension is not objectionable.

2. Bridges, crane girders, some floors and other structures or members that may be subjected in the future to large increases of loads, impacts, earthquake shocks, and other things causing forces of unknown magnitude—these should be designed conservatively. This should also be done in the case of important structures whose failure may cause great loss of life or of economic values.

In this connection, all should realize that the saving of steel due to an increase in the unit stress permissible in design is not

TABLE 1-5.—RECOMMENDED PERMISSIBLE UNIT STRESSES

Reinforcing steel		Concrete	
		Description	f'_c / n
		Modular ratio: n as follows	2,000 to 2,400 lb. = 15 2,500 to 2,900 lb. = 12 3,000 to 3,900 lb. = 10 4,000 to 4,900 lb. = 8 Above 5,000 lb. = 6
Tension			
Structural-grade bars,	f_s = 18,000 lb. per sq. in.	Flexure f_c	
		Extreme fiber stress in compression	$f_c = 0.45f'_c$
Structural shapes,	f_s = 18,000 lb. per sq. in.	Shear v	
		Beams with no web reinforcement and without special anchorage of longitudinal steel	$v_c = 0.02f'_c$
Intermediate- and hard-grade bars,	f_s = 20,000 lb. per sq. in.	Beams with no web reinforcement but with special anchorage of longitudinal steel	$v_c = 0.03f'_c$
For one-way solid slabs—wire mesh or bars not exceeding ½ in.—50 per cent minimum yield point but not to exceed	f_s = 25,000 lb. per sq. in.	Beams with properly designed web reinforcement but without special anchorage of longitudinal steel	$v = 0.06f'_c$
		Beams with properly designed web reinforcement and with special anchorage of longitudinal steel	$v = 0.12f'_c$
Web reinforcement[1] (all grades)	f_s = 16,000 lb. per sq. in.	Flat slabs at distance d from edge of column capital or dropped panel	$v_c = 0.03f'_c$

Column reinforcement
Verticals:
Intermediate grade, $f_s = 16{,}000$ lb. per sq. in.
Hard grade, $f_s = 20{,}000$ lb. per sq. in.
Spirals, useful limit:
Intermediate grade, $f'_s = 40{,}000$ lb. per sq. in.
Cold-drawn wire, $f'_s = 60{,}000$ lb. per sq. in.

Footings where longitudinal bars have no special anchorage $v_c = 0.02f'_c$
Footings where longitudinal bars have special anchorage $v_c = 0.03f'_c$

Bond u
In beams and slabs and one-way footings:
Plain bars *or structural shapes* $u = 0.04f'_c$ but not exceeding 160 lb. per sq. in.
Deformed bars $u = 0.05f'_c$ but not exceeding 200 lb. per sq. in.
In two-way footings:
Plain bars *or structural shapes* $u = 0.03f'_c$
Deformed bars $u = 0.0375f'_c$
(Where special anchorage is provided, *one and one-half times* these values in bond may be used, but in no case to exceed 200 lb. per sq. in. for plain bars or structural shapes and 250 lb. per sq. in. for deformed bars)

Bearing f_c
Full area loaded $f_c = 0.25f'_c$
One-third area[2] loaded $f_c = 0.375f'_c$ (approx.)

Axial compression f_c
In pedestals $f_c = 0.25f'_c$
In columns with lateral ties (max.)[3] $f_c = 0.18f'_c$
In columns with continuous spirals (max.)[3] $f_c = 0.225f'_c$

[1] For illustration, varying values are used in problems herein.
[2] The allowable bearing stress on an area between one-third and the full area shall be interpolated between the values given.
[3] See Chap. 6.

proportional to that increase. In fact, in most large structures properly designed, the saving is likely to be in the neighborhood of 5 to 10 per cent of that relative increase. Obviously, this is a small part of the cost of the reinforcement which, in turn, is usually a rather small percentage of the cost of the job. However, the safety of the structure may really decrease as the permissible unit stress is raised, hence the need for good judgment in determining these matters.

These comments are generally applicable to the concrete also, although the strength of the concrete is not so likely to be critical and the percentage of saving in the cost due to increasing permissible unit stresses in the concrete is usually far less than in the case of the steel.

1-18. Importance of Workmanship. The need for honest and intelligent workmanship in the field during the building of a concrete structure has been emphasized herein. It should be emphasized again and again. The best of designs can be ruined if the intent of the plans is not carried out in the field. Proper reinforced-concrete construction depends upon men—men who understand the action of structures, men who know the characteristics and the limitations of the material that they are handling, and men who are conscientious and determined to conduct their work with honor to themselves and with credit to their profession.

CHAPTER 2

STRESSES IN BEAMS DUE TO BENDING

2-1. Introduction. In general, a beam is a member which carries loads that act transversely with respect to its longitudinal axis so as to cause the member to bend. The most simple reinforced-concrete beams are those whose cross sections are rectangular in shape. In fact, an ordinary floor slab, like that shown in Fig. 2-1, may be thought of as a series of such beams of unit width b (usually 12 in.), represented by the piece $ABCD$ having a depth equal to t and a span equal to L.

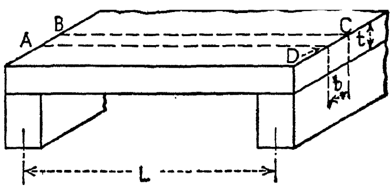


FIG. 2-1.

The effective span of a beam should be taken equal to the clear distance between supporting members plus some allowance for end bearing but not more than the distance between the centers of the supports. Some engineers assume the span to be the clear distance plus about 4 in. for light members or plus 12 in. for heavy construction. When a beam rests upon a heavy wall, the end of the span may be assumed to be near the inner edge of the wall; when it is supported by an edge beam or in a manner that restrains it only slightly, assume the end to be at the center of the support.

In dealing with such a beam there are two types of problem to consider. One is the analyzing or testing of existing and assumed beams of given dimensions to compute what forces they can withstand or what stresses they will have when subjected to specific loads; the other is the designing or proportioning of beams to support certain given forces or loads. The fundamental theories

by which these problems may be solved will now be explained, and formulas for use in connection therewith will be developed.

A good training in theory is essential for any designer. However, when one enters practical engineering work he will find that the design of reinforced-concrete structures is influenced to a large extent by general specifications, codes, and customary practices. Such codes may change—and they should do so as the art progresses and our knowledge increases. Each engineer should have a copy of the latest report issued by the Joint Committee. References to their report of June, 1940, herein called the Code, will appear when applicable to the subject matter in the text. Of course, there are many other specifications, building codes, and regulations in use: e.g., the Building Code of the City of New York; and United States Navy Department, Bureau of Yards and Docks, Standards of Design for Concrete, No. 3*Yb*. The designer should be careful to comply with whatever ones govern his work, bearing in mind that they are general rules to be followed unless he has important reasons for making the construction even better and safer.

2-2. Table of Symbols and Their Meanings. The symbols which are used in the 1936 Specifications of the American Concrete Institute have been adopted for this text as far as it has been practicable to do so. An explanation of those that are used herein is given generally wherever they are first encountered. However, those which are utilized in this chapter are grouped for convenience. They are fundamental symbols for use throughout the work. Their meanings should be memorized.

The list is as follows:

A_s = area of steel in tension, square inches.
A'_s = area of steel in compression, square inches.
b = width of rectangular beam or flange of T-beam, inches.
b' = width of stem of T-beam, inches.
C or C_c = total force of compression in concrete, pounds.
C' or C_s = total force of compression in steel, pounds.
d = depth from compression face of beam or slab to the center of gravity of the longitudinal tensile reinforcement, or to the center of the longitudinal tensile reinforcement which is farthest from the neutral axis, inches (called "effective depth").
d' = depth from compression face of beam or slab to center of longitudinal compressive reinforcement which is farthest from the neutral axis, inches.

D = total over-all depth of a beam, inches.
E_c = modulus of elasticity of concrete in compression, pounds per square inch.
E_s = modulus of elasticity of steel in tension or compression, pounds per square inch.
f_c = compressive unit stress in extreme fiber of concrete, pounds per square inch.
f'_c = ultimate compressive strength of concrete—usually at age of 28 days—pounds per square inch.
f_s = tensile unit stress in longitudinal reinforcement, pounds per square inch.
f'_s = compressive unit stress in longitudinal reinforcement, pounds per square inch.
I_c = moment of inertia of transformed section in terms of concrete, in.^{4}
j = ratio of distance between centroid of compression and center of gravity of tensile reinforcement or the extreme row of tensile reinforcement to the depth d.
k = ratio of distance between the compression face of the beam and the neutral axis to the depth d.
kip = 1,000 lb. (sometimes abbreviated as "k.").
L = span of beam or slab, usually in feet.
M_c = internal resisting moment in terms of the strength of the concrete, inch-pounds.
M_s = internal resisting moment in terms of the strength of the steel, inch-pounds.
n = ratio of modulus of elasticity of steel to that of concrete.
p = ratio of area of tensile reinforcement to the effective area of concrete in beams and slabs.
S_c = section modulus of transformed section in terms of concrete, in.^{3}
S_s = section modulus of transformed section in terms of steel, in.^{3}
t = thickness of slab or flange of T-beam, inches.
T or T_s = total force of tension in steel, pounds.
bd = effective area of a beam or slab, square inches.

2-3. Distribution of Stresses in a Reinforced-concrete Beam. A beam must curve if it is subjected to a bending moment; the fibers that are in compression must shorten; those which are subjected to tension must elongate. The typical method which is used herein to enable one to visualize these actions is shown in Fig. 2-2(a). The short, irregular lines represent cracks. They are drawn in the regions where tension exists—the weak portions of the concrete which must be *reinforced* with steel. No cracks are shown in the portions that are subjected to compression because these are the regions in which the concrete is effective by

itself. Such pictures do not mean that concrete beams always crack so excessively; they are to indicate the condition that the beams may reach if they are loaded sufficiently.

When a load is applied gradually to a reinforced-concrete beam, the entire cross section of the member resists the bending moment until the tensile strength of the concrete is exceeded. During this initial stage, the deformation of the longitudinal reinforcement is the same as that of the concrete beside it; but as soon as the concrete cracks, the entire situation changes because the tensile forces are then mostly dependent upon the reinforcement which must act as a chain or tie that keeps the beam from being ruptured. If the reinforcement is not parallel to the tensile force, it is satisfactory to assume that the effective area of a rod

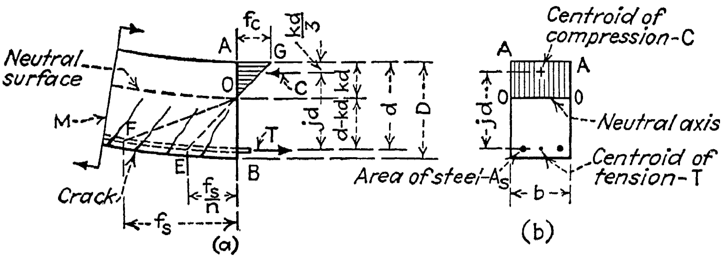


FIG. 2-2.

is its normal cross-sectional area times the cosine of the angle between the longitudinal axis of the rod and the direction of the force.

To illustrate the distribution of the stresses in the beam at this stage, let Fig. 2-2(*a*) represent a small portion of a reinforced-concrete beam as seen in side elevation. Let Fig. 2-2(*b*) show the cross section *AB* of this beam. Remember also that the member is bent because of the application of the external loads. Let the couple *M* represent the magnitude and the direction of the bending moment which is caused by these forces. Inasmuch as the beam is in equilibrium, this portion of it may be considered as a free body in space which is acted upon by the external couple *M* and by a second couple of equal moment but with opposite direction. Obviously, this second couple must be provided by the strength of the materials composing the beam itself.

As previously explained, the tensile strength of concrete is rather low and unreliable. Although its effect is considerable in

many instances, it is advisable to assume that the tensile strength of the concrete is zero. Therefore, the tensile force T [Fig. 2-2(*a*)] is assumed to be provided by the steel alone.

Although the intensity of the stress in a rod may vary slightly across its area, increasing with the distance from the neutral axis, it is sufficiently accurate to assume that it has an average intensity of stress f_s which is equal to the unit stress at its center. This value, multiplied by the area of the steel, gives the tensile force

$$T = A_s f_s. \tag{2-1}$$

On the other hand, the concrete of the upper portion of the beam resists the compression, but this force is spread over the affected area with varying intensity. Although this distribution may not vary uniformly, it is generally satisfactory to consider that it does so under ordinary working conditions, for simplicity of calculation. Therefore, the maximum compression occurs at the top fibers, and the stress decreases to zero at the line *O-O*, the neutral axis. The height of this area which is in compression is called kd. Since this pressure diagram of the compressive forces is a triangular wedge, the resultant force C must equal the volume of the wedge, and it must be located at the center of gravity of this imaginary solid, which is at a distance $kd/3$ from the top fibers. It is therefore clear that

$$C = \tfrac{1}{2}f_c(kd)(b) = \tfrac{1}{2}f_c kbd. \tag{2-2}$$

It is also apparent that the magnitudes of these forces C and T must be equal in order to have equilibrium, for which $\Sigma H = 0$. Therefore,

$$\tfrac{1}{2}f_c kbd = A_s f_s. \tag{2-3}$$

From Fig. 2-2 it is easily seen that the lever arm of the internal resisting couple with forces C and T is the distance between the points of application of these forces, jd. Thus,

$$jd = d - \frac{kd}{3}, \qquad \text{or} \qquad j = 1 - \frac{k}{3}. \tag{2-4}$$

Inasmuch as the moment of a couple equals the magnitude of either of the equal, opposite forces times the lever arm between them,

$$M_c = \tfrac{1}{2}f_ckbd(jd) = \tfrac{1}{2}f_ckjbd^2 = Kbd^2* \qquad (2\text{-}5)$$

and

$$M_s = A_sf_sjd. \qquad (2\text{-}6)$$

Formulas (2-5) and (2-6) are fundamental in the analysis and design of rectangular beams with steel for resisting tension only. Both formulas are needed because reinforced concrete is not a homogeneous material.

If the beam of Fig. 2-2 is progressively loaded to failure, the concrete may yield under compression before the steel gives way under tension. If so, the beam is said to be *overreinforced* because it has more than the necessary amount of steel. In case the reverse is true, and the steel fails first, the beam is *under-reinforced.* From the standpoint of the efficient use of materials, the best design is one that results in a beam in which the maximum safe working strength of the concrete and that of the steel are reached simultaneously—a *balanced design.* However, cost and other practical matters affect one's designs. Sometimes it is wise to use more than the theoretical amount of reinforcement in order to simplify the details by making the rods in many different beams alike; sometimes it is advisable to use more concrete than needed for strength alone because of mass or general dimensions desired; often it is best to avoid an excessive variety of sizes which increases the formwork because the cost of a reinforced-concrete structure does not vary directly with the volume of concrete used in it.

By definition,

$$p = \frac{A_s}{bd}, \qquad \text{or} \qquad A_s = pbd.$$

Substituting this value of A_s in Eq. (2-6) gives

$$M_s = pf_sjbd^2 = Kbd^2.* \qquad (2\text{-}7)$$

Equations (2-5) and (2-7) may be restated as

$$bd^2 = \frac{2M_c}{f_ckj} = \frac{M_c}{K}. \qquad (2\text{-}8)$$

$$bd^2 = \frac{M_s}{pf_sj} = \frac{M_s}{K}. \qquad (2\text{-}9)$$

* K is computed as a coefficient which can be tabulated for balanced designs. For such data, see Tables 5 and 6 in the Appendix.

Another convenient form for Eq. (2-6) is

$$A_s = \frac{M}{f_s jd} \tag{2-10}$$

Equations (2-5), (2-6), and (2-7) are in convenient form for analyzing beams, whereas Eqs. (2-8), (2-9), and (2-10) are handier for use in designing them.

2-4. Compressive and Tensile Stresses and Location of Neutral Axis. Let line AB (Fig. 2-3) represent a plane cross section through a beam before the external loads are applied. If this section is considered to remain a plane after bending has taken place, it will move to $A'B'$. The shortening due to compression at the top can be represented by l_c; the elongation of the rods caused by tension can be represented by l_s.

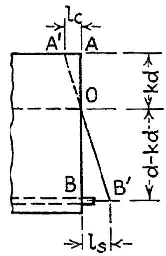


FIG. 2-3.

For any material that is not stressed beyond its elastic limit, the modulus of elasticity equals the stress per square inch divided by the corresponding deformation in inches per inch of length, or $E = f/l$. Therefore,

$$E_c = \frac{f_c}{l_{c1}}, \qquad \text{and} \qquad E_s = \frac{f_s}{l_{s1}}$$

where l_{c1} and l_{s1} represent the strains per unit of length. From Fig. 2-3, by the use of similar triangles, it is found that

$$\frac{l_c}{l_s} = \frac{kd}{d - kd} = \frac{k}{1 - k}. \tag{2-11}$$

The ratio of the modulus of elasticity of steel to that of concrete is

$$n = \frac{E_s}{\ } \tag{2-12}$$

$$n = \frac{f_s}{\ } \qquad \frac{f_s l_{c1}}{\ } \tag{2-13}$$

Referring to Eq. (2-13), it is clear that, if the deformations of the steel and of the concrete are equal, then

$$n = \frac{f_s}{f_c}, \qquad \text{or} \qquad f_s = nf_c. \tag{2-14}$$

These equations show that, once the qualities of the steel and of the concrete to be used are determined by tests, E_s, E_c, and n will have definite magnitudes, and maximum permissible values may be specified for f_s and f_c. The usual values for the foregoing quantities are

$$
\begin{aligned}
E_s &= 29{,}000{,}000 \text{ to } 30{,}000{,}000 \text{ lb. per sq. in.} \\
E_c &= 2{,}000{,}000 \text{ to } 3{,}500{,}000 \text{ lb. per sq. in.} \\
f_s &= 16{,}000 \text{ to } 20{,}000 \text{ lb. per sq. in.} \\
f_c &= 800 \text{ to } 1{,}200 \text{ lb. per sq. in.} \\
n &= 8 \text{ to } 15.
\end{aligned}
$$

Equation (2-14) explains why EB of Fig. 2-2(a) is labeled $\frac{f_s}{n}$. To some scale, AG and EB represent stresses in the concrete for a straight-line variation of stress. The real stress in the steel would have to be $n(EB)$, which is plotted as FB.

In order to assist in determining a value for n, the Code gives it the following approximate magnitude:

$$n = \frac{30{,}000}{f'_c}.$$

Obviously, the qualities of the steel and the concrete should be known or decided upon before making the calculations.

Several other formulas that are useful in developing office standards and tables or diagrams for analysis and design can be derived from Fig. 2-2 and the formulas already given. They are the following:

$$n = \frac{f_s k}{f_c(1 - k)}. \tag{2-15}$$

$$k = \frac{nf_c}{f_s + nf_c}, \text{ or } k = \frac{n}{n + \frac{f_s}{f_c}}, \text{ or } k = \frac{1}{1 + \frac{f_s}{nf_c}}. \tag{2-16}$$

$$f_s = \frac{nf_c(1 - k)}{k}. \tag{2-17}$$

$$f_c = \frac{f_s k}{n(1 - k)}. \tag{2-18}$$

$$k = \frac{2A_s f_s}{f_c bd} = \frac{2pf_s}{f_c}. \tag{2-19}$$

$$p = \frac{\frac{1}{2}}{\frac{f_s}{f_c}\left(\frac{f_s}{nf_c} + 1\right)}. \tag{2-20}$$

$$k = \sqrt{2pn + (pn)^2} - pn. \tag{2-21}$$

$$f_c = \frac{2A_sf_s}{kbd} = \frac{2f_sp}{k}.$$

$$p = \frac{k^2}{2n(1-k)}. \tag{2-22a}$$

The assumptions upon which these formulas are based should be studied carefully.[1] It must be remembered that they are for beams with tensile reinforcement only, additional methods of calculation for other cases being derived in subsequent articles. They also assume that the concrete cracks so that it cannot withstand the tensile forces. This is equivalent to saying that, before the member will fail, these conditions will occur, and the beam will be safe in spite of them but that any resistance to tension that the concrete may provide will merely add to the safety of the structure. Then, finally, the value of n for concrete is considered to be the same for tension as it is for compression—an assumption that will be made use of later in Art. 2-8.

2-5. Problems in the Analysis and Design of Beams. Many of the problems to be illustrated in this text are solved by the use of the slide rule. Its accuracy will be sufficient for practical purposes, although differences may appear in the third significant figure. The student will find that, in many cases, the calculations are not carried out to unnecessary extremes but are limited in accordance with good judgment.

Generally, E_c is assumed to be equal to $1{,}000f'_c$ and this value is used in the determination of n. However, the allowable working stress in the concrete f_c is much less than the ultimate strength f'_c. The ratio of f'_c to f_c is called the *safety factor*. For beams,

[1] A different basic assumption regarding the distribution of the compressive stresses in reinforced-concrete beams is presented by Charles S. Whitney in Plastic Theory of Reinforced Concrete Design, A.S.C.E., *Trans.*, Vol. 68, 1942. In reality, the differences presented therein are not of great importance in practical design at ordinary allowable stresses because the steel is the critical part of most reinforced-concrete beams. Varying assumptions of the distribution of the concrete stresses have a minor effect upon the lever arm of the steel, as the reader will realize after thinking over the matter carefully.

the Code gives it a magnitude of about 2.5. For purposes of illustration in problems, different allowable unit stresses will be utilized for the concrete.

Furthermore, an examination of Fig. 2-2(a) will remind one that a high working stress in the longitudinal reinforcement will be accompanied by correspondingly severe cracking of the concrete in the vicinity of the steel. Therefore, the designated values of f_s are not constant but are varied in order to remind one that the specifications and good judgment may limit its magnitude to suit any special case.

For use in the solution of problems, the weight of concrete will be taken at 150 lb. per cu. ft. The areas and perimeters of reinforcing rods of the ordinary sizes are given in Table 1 in the Appendix.

Analysis of Beams.

Problem 2-1. If the beam shown in Fig. 2-4 is subjected to a bending moment of 300,000 in.-lb., and $n = 15$, find f_s and f_c.

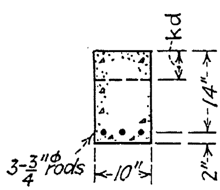


FIG. 2-4.

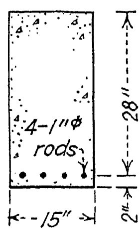


FIG. 2-5.

Incidentally, the designer is not so much interested in the magnitudes of f_c and f_s for any given case for their own sakes as he is in comparing them with the allowable unit stresses to see whether or not the member is safe on the one hand and economical on the other.

$$0.44 = 1.32 \text{ sq. in.} \quad \text{(See Table 3, Appendix.)}$$

$$\frac{A_s}{bd} = 1.32 \div 10 \times 14 = 0.0094$$

$$k = \sqrt{2pn + (pn)^2} - pn = \sqrt{2 \times 0.0094 \times 15 + (0.0094 \times 15)^2} - 0.0094 \times 15 = 0.408$$

(See Fig. 10, Appendix.)

$$j = 1 - \frac{k}{3} = 1 - \frac{0.408}{3} = 0.864$$

$$f_c = \frac{2M}{kjbd^2} = \frac{2 \times 300{,}000}{0.408 \times 0.864 \times 10 \times 14^2} = 868 \text{ lb. per sq. in.}$$

$$f_s = \frac{M}{A_s jd} = \frac{300{,}000}{1.32 \times 0.864 \times 14} = 18{,}800 \text{ lb. per sq. in.}$$

Problem 2-2. Assume E_s and E_c = 30,000,000 and 2,500,000 lb. per sq. in., respectively; also, the allowable f_s and f_c = 18,000 and 900 lb. per sq. in., respectively. Find the safe resisting moment of the beam shown in Fig. 2-5.

$$n = \frac{E_s}{E_c} = \frac{30{,}000{,}000}{2{,}500{,}000} = 12$$

$$A_s = 4 \times 0.79 = 3.16 \text{ sq. in.}$$

$$p = \frac{A_s}{bd} = 3.16 \div 15 \times 28 = 0.0075$$

$$k = \sqrt{2pn + (pn)^2} - pn = \sqrt{2 \times 0.0075 \times 12 + (0.0075 \times 12)^2} - 0.0075 \times 12 = 0.344$$

$$j = 1 - \frac{k}{3} = 1 - \frac{0.344}{3} = 0.885$$

$$M_s = A_s f_s jd = 3.16 \times 18{,}000 \times 0.885 \times 28 = 1{,}410{,}000 \text{ in.-lb.}$$

$$M_c = \tfrac{1}{2} f_c kjbd^2 = \tfrac{1}{2} \times 900 \times 0.344 \times 0.885 \times 15 \times 28^2 = 1{,}610{,}000 \text{ in.-lb.}$$

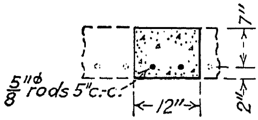


FIG. 2-6.

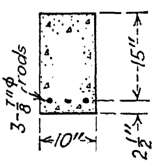


FIG. 2-7.

From the fact that the foregoing figures show the safe value of M_s to be less than M_c, it is apparent that this beam is somewhat underreinforced. Using the magnitude of M_s and solving for the simultaneous value of f_c gives

$$1{,}410{,}000 = \tfrac{1}{2} \times f_c \times 0.344 \times 0.885 \times 15 \times 28^2.$$
$$f_c = 788 \text{ lb. per sq. in.}$$

It can also be said that f_c varies as the magnitude of the resisting moment,

$$f_c : 900 :: 1{,}410{,}000 : 1{,}610{,}000.$$

Problem 2-3. Assume a simply supported slab 9 in. thick with $\frac{5}{8}$-in. round rods 5 in. c.c. (center to center) located 2 in. above the bottom as shown in Fig. 2-6. Find the safe uniform live load for this beam if the span = 9 ft., n = 12, and the allowable f_s and f_c = 20,000 and 900 lb. per sq. in., respectively.

First, imagine a slice 12 in. wide to be cut out of the slab from one support to the other, parallel to the reinforcement. Each such piece will be a rectangular beam. It will contain an equivalent of $\frac{12}{5}$ rods.[1] Thus

$$A_s = \frac{12 \times 0.31}{5} = 0.74 \text{ sq. in.,} \qquad d = 7 \text{ in.,} \qquad b = 12 \text{ in.}$$

$$p = \frac{A_s}{bd} = \frac{0.74}{12 \times 7} = 0.0088$$

$$k = \sqrt{2pn + (pn)^2} - pn = \sqrt{2 \times 0.0088 \times 12 + (0.0088 \times 12)^2} - 0.0088 \times 12 = 0.37$$

$$j = 1 - \frac{k}{3} = 1 - \frac{0.37}{2} = 0.88$$

$$M_s = A_s f_s jd = 0.74 \times 20{,}000 \times 0.88 \times 7 = 91{,}200 \text{ in.-lb.}$$
$$M_c = \tfrac{1}{2} f_c kjbd^2 = \tfrac{1}{2} \times 900 \times 0.37 \times 0.88 \times 12 \times 7^2 = 86{,}200 \text{ in.-lb.}$$

The strength of the concrete, in this case, limits the safe resisting moment, and the slab is slightly overreinforced. The actual magnitude of f_s is

$$f_s = 20{,}000 \times \frac{86{,}200}{91{,}200} = 18{,}900 \text{ lb. per sq. in.}$$

Considering that the weight of concrete is usually assumed to be 150 lb. per cu. ft., the dead load of the slab is $150 \times \frac{9}{12} = 112$ lb. per sq. ft. of horizontal area. Assuming that $M = wL^2/8$ for a simply supported beam and that the safe live load is the reserve supporting capacity of the slab over and above that required for the dead load, it is apparent that

$$M = 86{,}200 = \frac{w \times 9^2 \times 12}{8}, \text{ or } w = 710 \text{ lb. per sq. ft.}$$

Therefore, the live load permissible in this case is $710 - 112 = 598$ lb. per sq. ft.

Design of Beams. The term "design of a beam" is used to denote the determination of the size and the materials that are required to constitute a beam that can safely support specified loads under certain definite conditions of span, stresses, etc. Ordinarily, there are many beams of varying proportions which can safely serve the same purpose. However, it is generally reasonable and economical to proportion a rectangular beam so that its depth equals about twice its width unless these dimensions are controlled by other conditions. Sufficient lateral stiffness, economy and efficiency in the use of materials, strength in shear

[1] See Table 2, Appendix.

as well as in bending, space for proper placing of rods—these are some of the practical reasons for using such proportions.

The Code specifies the following relationship between the allowable unit stress f_c, the distance between lateral supports L, and the least width of the compression face of a reinforced-concrete beam: If $L/b = 24$, $f_c =$ full allowable value. Max. $L/b = 36$, whereat $f_c = 50$ per cent of full allowable value. For intermediate values of L/b, f_c is reduced proportionately.

Problem 2-4. Design a beam to carry a bending moment of 400,000 in.-lb. if $n = 10$ and the allowable f_s and $f_c = 18{,}000$ and 1,100 lb. per sq. in., respectively.

From Eq. (2-16) the theoretical value of k should be

$$k = \frac{n}{n + \dfrac{f_s}{f_c}} = \frac{10}{10 + \dfrac{18{,}000}{1{,}100}} = 0.38.$$

$$j = 1 - \frac{k}{3} = 1 - \frac{0.38}{3} = 0.873.$$

Using Eq. (2-8),

$$bd^2 = \frac{2 \times M_c}{f_c kj} = \frac{2 \times 400{,}000}{1{,}100 \times 0.38 \times 0.873} = 2{,}190.$$

The problem now resolves itself into a case of "cut and try." There are a multitude of possible values for the width and the depth of the beam. However, one way is to assume d and test for b, changing the assumptions until proper and reasonable dimensions are found. Assuming $d = 18$ in. gives

$$b = \frac{2{,}190}{18^2} = 6.75 \text{ in.}$$

Experience will soon show that this value of b is so small that it will be difficult or impossible to place the reinforcing rods properly. However, assuming $d = 15$ in. gives

$$b = \frac{2{,}190}{15^2} = 9.75 \text{ in., or, say, } 10 \text{ in.}$$

Taking this value of $d = 15$ in. and substituting it in Eq. (2-10) yields

$$A_s = \frac{M_s}{f_s jd} = \frac{400{,}000}{18{,}000 \times 0.873 \times 15} = 1.7 \text{ sq. in.}$$

If three $\frac{7}{8}$-in. round rods are used, $A_s = 3 \times 0.6 = 1.8$ sq. in. Placing these rods in the assumed beam gives a section as pictured in Fig. 2-7. The $2\frac{1}{2}$ in. of concrete below the steel provides slightly more than the minimum required cover of 2 in.

Ordinarily it would be unnecessary to test this beam further because the width of the member and the area of the steel used are slightly greater than the minimum required by the calculations. However, the actual values of f_s and f_c will be found by analyzing the beam as follows:

$$p = \frac{A_s}{bd} = 1.8 \div (10 \times 15) = 0.012$$

$$k = \sqrt{2pn + (pn)^2} - pn = \sqrt{2 \times 0.012 \times 10 + (0.012 \times 10)^2} - 0.012 \times 10 = 0.383$$

$$1 - \frac{}{3} = 1 - \frac{0.383}{3} = 0.872$$

$$\frac{M_s}{A_s jd} = \frac{400{,}000}{1.8 \times 0.872 \times 15} = 17{,}000 \text{ lb. per sq. in.}$$

$$f_c = \frac{2M_c}{kjbd^2} - \frac{2 \times 400{,}000}{0.383 \times 0.872 \times 10 \times 15^2} \quad 1{,}066 \text{ lb. per sq. in.}$$

Solving Eq. (2-20) for the theoretically best percentage of steel gives

$$\frac{18{,}000}{1{,}100}\left(\frac{18{,}000}{10 \times 1{,}100} + 1\right.$$

instead of the value 0.012 actually used. Since the latter exceeds the percentage for a balanced design (0.0116 in this case), the beam is slightly over-reinforced, a fact which is also shown by the relative magnitudes of f_s and f_c given above.

In assuming depths of beams for designing ordinary structures, the following may be of some service as a general guide or starting point, d being in inches and the span L in ft.:

1. For slabs for roofs and floors, assume $d = 0.5L$ in. (if $L = 8$ ft., $d = 4$ in.).
2. For light beams and heavy slabs, assume $d = 0.8L$ in.
3. For heavy beams, and headers or girders supporting cross-beams, assume $d = 1.0L$ to $1.25L$ in., depending upon the severity of loads and lengths of spans.
4. For ordinary continuous beams and girders, assume d somewhat less than given above.

Many times it is desirable to use approximate formulas when making a first try at the design of a beam in order to minimize

time and labor. This tentative design can be checked later by more theoretically correct methods. A casual inspection of Tables 5 and 6 in the Appendix shows that the value of k generally lies somewhere between 0.3 and 0.45, hence j is somewhere from 0.9 to 0.85. Therefore, assume $k = 0.38$ and $j = 0.88$, as average values. Then Eq. (2-5) gives

$$M = \frac{1}{2}f_c \times 0.38 \times 0.88bd^2 = \frac{1}{6}f_cbd^2 \qquad \text{or} \qquad f_c = \frac{6M}{bd^2} \qquad (2\text{-}5a)$$

and Eq. (2-6) becomes

$$M = 0.88A_sf_sd \qquad \text{or} \qquad A_s = \frac{M}{0.88f_sd}. \qquad (2\text{-}6a)$$

These formulas will be labeled as shown to denote that they are approximations of the original ones. They are easily remembered and are useful if one wishes to get a scale on sizes required, especially when he has no books to which he can refer.

The values of K shown in Tables 5 and 6 in the Appendix are also very useful in expediting the design of rectangular beams. To illustrate this, assume the following problem:

Design a beam along the edge of a large hatch in a floor. It is simply supported and has a span of 22 ft. It carries a uniformly distributed live load of 1,500 lb. per lin. ft., as well as its own weight. An intersecting beam at its center delivers to it a reaction of 20,000 lb. Assume $n = 8$ and the allowable f_c and f_s = 1,200 and 20,000 lb. per sq. in., respectively.

This is a girder with a very heavy load. From the ratios of depths to spans previously given, assume $d = 1.2 \times 22 = 26.4$ in., or call $d = 27$ in., $D = 30$ in. and $b = 18$ in. The dead load of the beam is

$$w = \left(\frac{18 \times 30}{144}\right)150 = 560 \text{ lb. per lin. ft.}$$

$$M = (1{,}500 + 560)\frac{22^2}{8} + 20{,}000 \times \frac{22}{4} = 235{,}000 \text{ ft.-lb.}$$

From Table 6, $K = 173$, then

$$bd^2 = \frac{M}{K} \text{ or, using the assumed } d,\ b \quad \frac{235{,}000 \times 12}{27^2 \times 173} = 22.4 \text{ in.}$$

This is greater than the assumed b of 18 in. It seems wise to deepen the beam; hence assume $d = 30$ in., $D = 34$ in., and $b = 20$ in. The new dead load is 710 lb. per lin. ft., and $M = 244{,}000$ ft.-lb. Testing for b again,

$$\frac{M}{d^2K} = \frac{244{,}000 \times 12}{30^2 \times 173} = 18.8 \text{ in.}$$

This shows that b might be 19 in., but the 20-in. dimension is conservative, satisfactory, and simple for use in formwork.

Technically, k and j should be computed for these conditions, but the magnitude of j would change but slightly, being a little more than the 0.892 given in Table 6. Therefore it is conservative and close enough for practical purposes. Using $j = 0.892$,

$$A_s = \frac{M}{f_s j d} = \frac{244{,}000 \times 12}{20{,}000 \times 0.892 \times 30} = 5.47 \text{ sq. in.}$$

Table 3 in the Appendix shows that nine $\frac{7}{8}$-in. round, seven 1-in. round, six 1-in. square, five $1\frac{1}{8}$-in. square, or four $1\frac{1}{4}$-in. square rods may be used. Next, considering Table 8 in the Appendix, and assuming $\frac{3}{4}$-in. aggregate, seven 1-in. round rods with five in the bottom row and two in a second row 3 in. above it gives a satisfactory spacing and arrangement of rods.

Checking this beam, if considered necessary, and assuming that the rods are concentrated at their center of gravity ($\frac{7}{8}$ in. above the bottom row), the following are found:

Minimum depth $= 30 + \frac{7}{8} + \frac{1}{2} +$ say $2\frac{1}{2}$ for cover $= 33\frac{7}{8}$ in. The assumed 34-in. figure is satisfactory.

$$p = \frac{5.53}{20 \times 30} = 0.0092$$

$$k = \sqrt{2 \times 0.0092 \times 8 + (0.0092 \times 8)^2} - 0.0092 \times 8 = 0.309.$$

(Check this with Fig. 10 of the Appendix.)

$$j = 1 - 0.103 = 0.897$$

$$A_s = \frac{244{,}000 \times 12}{20{,}000 \times 0.897 \times 30} = 5.43 \text{ sq. in.}$$

$$f_c = \frac{2 \times 244{,}000 \times 12}{0.309 \times 0.897 \times 20 \times 30^2} = 1{,}170 \text{ lb. per sq. in.}$$

2-6. The Transformed-section Method. The method of analysis and design that has just been explained has treated the concrete and steel as two separate materials which act together in the beam. It has also been stated that if concrete and steel are deformed equally, the unit stress in the steel is n times as great as that in the concrete. This fact can be utilized to good advantage because 1 sq. in. of steel may be considered equivalent to n sq. in. of concrete, as far as its resistance to deformation is concerned. Therefore, if all of the rods in a beam are assumed to be replaced in the cross section by the equivalent square inches of concrete in the same location with regard to the neutral axis, an imaginary beam is obtained in which the steel is said to be

"transformed" into concrete. Thus, for any beam of given shape, dimensions, and make-up, there is a definite substitute beam of homogeneous material which can be used in its stead for the purpose of calculation. Such a substitute beam will have a definite location for its neutral axis, which is at the center of gravity of the section; it will also have a definite moment of inertia. Therefore, these values can be substituted in the equation $M = sI/c$ in order to find the resisting moment of the beam.

Figure 2-8(*a*) shows the cross section of a reinforced-concrete beam. The substitute, or transformed, beam is pictured in Fig. 2-8(*b*). The neutral axis of this "transformed-concrete" beam is represented by the line *O-O*.

The location of this axis is found by utilizing the fact that, for a beam of homogeneous material but of any shape, the static moment of the area of the cross section about the neutral axis is zero. Then the area above the neutral axis times the lever arm to its own center of gravity must equal the area below the neutral axis times the lever arm to its particular center of gravity.

Furthermore, to find the moment of inertia of the transformed-concrete beam about this neutral axis—called I_c—it is merely necessary to make the usual calculation for ΣAx^2 about the line *O-O*. Then the section modulus of this transformed beam will naturally have two values, viz., $\frac{I_c}{kd}$ for the top and $\frac{I_c}{d - kd}$ for the bottom. With these values, the compressive stress in the concrete is simply

$$f_c = M \div \frac{I_c}{kd} \tag{2-23}$$

whereas the tensile stress in the steel is

$$f_s = n\left(M \div \frac{I_c}{d - kd}\right) \tag{2-24}$$

Although the method of finding f_c and f_s that is indicated above is satisfactory, it is sometimes hard to visualize what is going on and to avoid errors resulting from improper use of n or of the distances to the extreme fibers. On this account it is an advantage to have the section modulus for use in determining the compressive stress in the concrete differentiated from that utilized

in finding the tensile stress in the steel. Therefore, call the former S_c and the latter S_s. Then

$$S_c = \frac{I_c}{kd} \tag{2-25}$$

and

$$S_s = \frac{I_c}{n(d - kd)}. \tag{2-26}$$

Equation (2-26) is found from Eq. (2-24) as follows:

$$f_s = \frac{nM}{\dfrac{I_c}{d - kd}} = \frac{M}{\dfrac{I_c}{n(d - kd)}} = \frac{M}{S_s}.$$

Therefore, by finding S_c and S_s immediately after calculating I_c, the designer can realize thereafter that the one with the subscript c goes with the concrete and the one having the subscript s is for the steel-stress calculations. However, Eqs. (2-23) and (2-24) will also be used in later problems. Curves for use in practical design are given in Figs. 4 and 5 of the Appendix. When working to any given specification as to f'_c, and therefore n, tables or curves giving section moduli can be prepared for a variety of beams so that these properties can be used in designing without repeated computations.

In general, the transformed-section method is so simple and understandable that it will be used extensively in subsequent calculations. The general procedure is to assume a section for the beam and then to analyze it. The transformed-section method will be explained further by direct application to a problem.

Problem 2-5. Let Fig. 2-8(a) represent the cross section of a beam with dimensions and make-up as shown. Assume that this beam carries a bending moment of 300,000 in.-lb. and that the materials are such that n equals 15. Find f_c and f_s.

The area of the rods is $3 \times 0.44 = 1.32$ sq. in. They carry a total tensile force of A_sf_s which, with the compressive forces, is needed to hold the beam in equilibrium. Since the steel is n times as effective as the same area of concrete, the rods are replaced by an area of concrete that will be $nA_s = 15 \times 1.32 = 19.8$ sq. in. If this area is arbitrarily assumed to have a depth equal to the diameter of the rods, its length will be $19.8 \div 0.75 = 26.4$ in., as shown in Fig. 2-8(b), which pictures the transformed section of the beam of Fig. 2-8(a) in terms of concrete.

On the other hand, replacing the concrete of the area in compression above the neutral axis *O-O* by steel will produce an equivalent area of steel in compression equal to $bkd \div n = 10 \times kd \div 15 = 0.667kd$. Since the real neutral axis *O-O* does not shift because of this juggling of figures, the distance kd must remain unchanged, and the width of the substituted steel area must become 0.667 in. as shown in Fig. 2-8(*c*). This last figure may be called a "transformed section" of the beam of Fig. 2-8(*a*) in terms of steel. This procedure is not necessary but is given here merely to show the student that it is possible to substitute an equivalent steel beam instead of the concrete one. Hereafter, the substitute steel beam will be discarded in order to adhere to one standard system—the transformed section in terms of concrete.

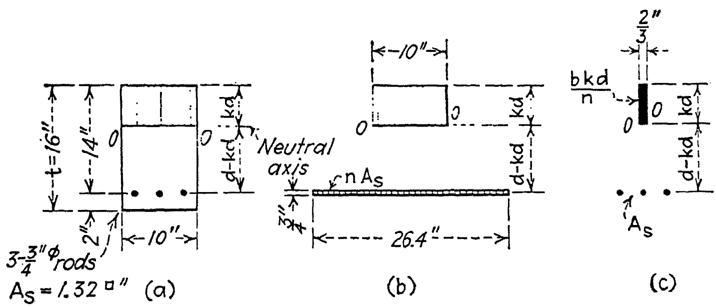


FIG. 2-8.

The value of kd is found by taking moments of the equivalent areas of Fig. 2-8(*b*) about the neutral axis *O-O*, where the sum of these moments is zero.

Solving for kd gives

$$bkd(kd) \quad nA_s(d - kd)$$

$$\frac{10(kd)^2}{2} = 15 \times 1.32(14 - kd)$$

$$kd = 5.72 \text{ in., and } d - kd = 14 - 5.72 = 8.28 \text{ in.}$$

$$I_c = b\frac{(kd)^3}{3} + nA_s(d - kd)^2 = \frac{10(kd)^3}{3} + (15 \times 1.32)(14 - kd)^2 = 1{,}984 \text{ in.}^4$$

Notice that I of the transformed section of the rods about their own axes is neglected.

From Eq. (2-25),

$$S_c = 1{,}984 \div 5.72 = 347 \text{ in.}^3$$

From the ordinary flexure formula, $M = sI/c$, with $s = f_c$, the value of f_c is found. Thus,

$$f_c = \frac{M}{S_c} = \frac{300{,}000}{347} = 865 \text{ lb. per sq. in.}$$

The section modulus to use in finding the stress in the rods, using Eq. (2-26), is

$$S_s = \frac{I_c}{n(d - kd)} = \frac{1{,}984}{15 \times 8.28} = 15.9 \text{ in.}^3$$

$$f_s = \frac{M}{S_s} = \frac{300{,}000}{15.9} = 18{,}900 \text{ lb. per sq. in.}$$

2-7. Effect of n and Tension in Concrete. In order to show the effect of a change in the value of n upon the location of the neutral axis and upon the magnitude of the calculated unit stresses, the previous problem should be computed again, using $n = 10$. If this is done, it is found that $kd = 4.9$ in., $d - kd = 9.1$ in., $I_c = 1{,}484$ in.^{4}, $S_c = 303$ in.^{3}, $S_s = 16.3$ in.^{3}, $f_c = 990$ lb. per sq. in., and $f_s = 18{,}400$ lb. per sq. in. Thus, the neutral axis rises, the stress in the concrete increases, and the stress in the steel declines. This is exactly what Eq. (2-21) indicates, but it shows more clearly that the strength of the concrete—of which n is a measure—affects the location of the neutral axis and the relative distribution of the stresses.

It is important to notice that, if a structure is designed upon the basis of a concrete with a certain strength—and hence with a certain value of n—and if a stronger concrete is used later on in the real building of the structure, the computed stresses in the concrete will exceed the calculated values of f_c. However, this excess will not be greater than the proportionate increase in the ultimate strength of the concrete. On the other hand, this increase in concrete stress and this decrease in k relieve the steel of a small part of its calculated stress. Therefore, no structural harm will result from the substitution.

There is considerable uncertainty as to the value of n, and as to the use of different magnitudes for it, especially at the ordinary working conditions and unit stresses. However, with the present state of knowledge of the actions of the materials concerned, and combined as they are, it seems desirable to continue its use when one considers that the service and safety records of almost count-

less structures indicate that the results have been satisfactory. However, further research along these lines is desirable.

There is still another important matter which should be considered. It concerns the case of the reinforced-concrete beam which supposedly does not crack. For instance, take the beam of Fig. 2-8, and analyze it on the basis that the concrete can withstand tension. Assume $n = 15$.

The principles of the transformed-section method can again be used in solving this last problem. The concrete above the neutral axis will be in compression as usual, but all of the concrete below that axis—and the rods, too—will be in tension. Of course, the neutral axis will be in a new position which is found as follows:

$$\frac{b(kd)^2}{2} = \frac{b(t - kd)^2}{2} + (n - 1)A_s(d - kd).$$

$$\frac{10(kd)^2}{2} = \frac{10(16 - kd)^2}{2} + 14 \times 1.32(14 - kd)$$

$$kd = 8.6 \text{ in.,} \qquad \text{and} \qquad d + 2 - kd = 16 - kd = 7.4 \text{ in.}$$

Continuing the solution, using the same principles as those of the previous problems,

$$I_c = \frac{b(kd)^3}{3} + \frac{b(d + 2 - kd)^3}{3} + (n - 1)A_s(d - kd)^2.$$

$$I_c = \frac{10 \times 8.6^3}{3} + \frac{10 \times 7.4^3}{3} + 14 \times 1.32 \times 5.4^2 = 4{,}010 \text{ in.}^4$$

$$S_c \text{ for top} = \frac{I_c}{kd} = \frac{4{,}010}{8.6} = 466 \text{ in.}^3$$

$$S_c' \text{ for bottom} = \frac{I_c}{d + 2 - kd} = \frac{4{,}010}{7.4} = 542 \text{ in.}^3$$

$$f_c \text{ at top} = \frac{M}{S_c} = \frac{300{,}000}{466} = 643 \text{ lb. per sq. in.}$$

$$f_c \text{ at bottom} = \frac{M}{S_c'} = \frac{300{,}000}{542} = 553 \text{ lb. per sq. in.}$$

Of course, the steel has the same deformation as the concrete which is at the same distance from the neutral axis, but its unit stress is n times that of the concrete. The stress in the latter at the location of the rods is

$$f_x : f_c \text{ at bottom} :: (d - kd) : (t - kd)$$
$$f_x : 553 :: 5.4 : 7.4, \text{ or } f_x = 404 \text{ lb. per sq. in.}$$
$$f_s = nf_x = 15 \times 404 = 6{,}050 \text{ lb. per sq. in.}$$

An analysis of the results of this last problem shows that if the concrete does not crack, the compressive stress in it will be 643 lb. per sq. in. instead of 865 lb. per sq. in., but the tensile stress in the concrete will be 553 lb. per sq. in. This tension is very high because, using a concrete with an ultimate compressive strength of 2,000 lb. per sq. in., the limit of its tensile strength will probably be about $0.10 \times f'_c = 0.10 \times 2{,}000 = 200$ lb. per sq. in. Therefore, the concrete must surely crack under a stress that is nearly three times this value. The action cannot be that of a homogeneous solid beam, but, before the member will fail, it will behave in the general manner that has been assumed for such beams; that is, it will crack; the concrete will resist the compression; and the steel will withstand all, or nearly all, of the tension.

No doubt there are cases in which reinforced-concrete beams have not cracked—not even having the "hair cracks" which are the usual kind. This is generally due to a partial arch action of the member, the continuity of the structure beyond the extent that was assumed in the design, the absence of the anticipated loads, or the use of a concrete with greater tensile and compressive strengths than those which were expected. However, the tensile strength of the concrete is too unreliable to permit the designer to depend upon it, even when the calculations may appear to warrant it. Furthermore, a construction joint or an unintentional delay in placing adjacent concrete until after the initial set has occurred may result in a member that cannot withstand tension effectively. Therefore, because life and property are frequently at stake, reliance upon the tensile strength of the concrete should be avoided.

2-8. Beams with Compressive Reinforcement. In the case of continuous rectangular beams and T-beams, it is general practice to have some of the steel run through the bottom of the beam at the support where the lower portion of the beam is in compression. Sometimes it is necessary or advisable to place rods in a beam so as to reinforce its compressive strength, as well as to have other rods which carry tension. For cases like these, it is necessary to realize that both the steel and the concrete will work together in withstanding the compression. The fundamental principles set forth in Arts. 2-3, 2-4, and 2-5 apply here, but the presence of the compressive steel modifies the formulas. In fact, they

become rather cumbersome. Therefore, the transformed-section method will be used in solving such problems because it is a widely adaptable method of analysis.

Let Figs. 2-9(*a*) and (*b*) represent a piece of a beam that has compressive reinforcement and is subjected to bending.

Let A'_s = the area of the compressive reinforcement in square inches and f'_s = its unit stress. It is apparent that the internal resisting couple is made up of the tension T and the resultant of

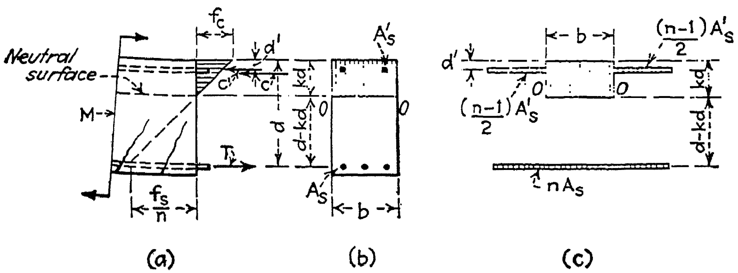


FIG. 2-9.

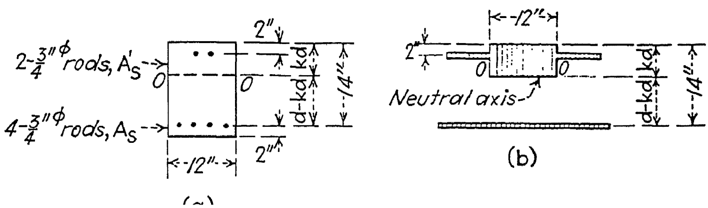


FIG. 2-10.

the two compressive forces C and C', with the proper lever arm. Figure 2-9(*c*) shows the imaginary transformed cross section of this beam with the steel replaced by its equivalent amount of concrete. The area of the concrete that is displaced by the rods should be deducted from the cross section of the concrete. This is equivalent to saying that the area that is substituted for the compressive steel outside of the solid rectangle bkd should be $(n - 1)A'_s$.

The values of kd, I_c, S_c, and S_s can be found as before and as illustrated in the following problems. Figures 6 to 9 in the Appendix are also useful for design and checking purposes.

Problem 2-6. Find the safe resisting moment of the beam shown in Fig. 2-10, assuming that n equals 10 and that the allowable f_s and f_c equal 18,000 and 1,000 lb. per sq. in., respectively.

$$A_s = 4 \times 0.44 = 1.76 \text{ sq. in.}$$
$$A'_s = 2 \times 0.44 = 0.88 \text{ sq. in.}$$
$$nA_s = 17.6 \text{ and } (n - 1)A'_s = 7.92.$$

Taking moments about the neutral axis,

$$\frac{12(kd)^2}{2} + 7.92(kd - 2) = 17.6(14 - kd).$$
$$kd = 4.8 \text{ in.,} \qquad \text{and} \qquad (d - kd) = 9.2 \text{ in.}$$

Taking the moments of inertia about the neutral axis,

$$I_c = 12 \times \frac{4.8^3}{3} + 7.92(4.8 - 2)^2 + 17.6(14 - 4.8)^2 = 1{,}994 \text{ in.}^4$$
$$S_c = \frac{I_c}{kd} = \frac{1{,}994}{4.8} = 415 \text{ in.}^3$$
$$S_s = \frac{I_c}{n(d - kd)} = \frac{1{,}994}{10 \times 9.2} = 21.7 \text{ in.}$$

Since the moment equals the allowable unit stress times the section modulus,

$$M_c = 1{,}000 \times 415 = 415{,}000 \text{ in.-lb.}$$
$$M_s = 18{,}000 \times 21.7 = 391{,}000 \text{ in.-lb.}$$

Therefore, M_s controls the design. The compressive stress in the top rods is

$$f'_s = 18{,}000 \times \frac{2.8}{9.2} = 5{,}480 \text{ lb. per sq. in.}$$

This last equation for f'_s is based upon the fundamental assumption that the stresses vary directly as their distances from the neutral axis. The distance of the compressive reinforcement above *O-O* is 2.8 in.

The magnitude of j for this beam is not quite equal to $1 - \frac{k}{3}$ because the stress in the top rods shifts the centroid of compression slightly away from the center of gravity of the triangular wedge that would apply for the concrete alone. However, it is safe to assume $j = 1 - \frac{k}{3}$ in most cases.

The preceding problem shows the analysis of an existing or assumed beam. In practical design work it is important to be able to determine tentative sections by approximate methods which yield results that will serve as fairly good trial sections. Obviously, it is tedious to make a wild guess and then to have the analysis of the beam show that the guess was not even approximate. Therefore, the following procedures are recommended, being based upon the fundamental action of a beam with compressive reinforcement:

A. For design:

1. Assume $k = 0.38$ and $j = 0.87$ or 0.88.

2. Solve for A_s as in the case of beams with tensile steel only. Then find f_c for the same case.

3. The value of f_c just found minus the allowable value gives the approximate amount of overload of stress in the concrete. One-half of this overload times the area under compression bkd gives the total excess force that the steel may be assumed to withstand.

4. The unit stress in the compressive steel f'_s can be approximated by dividing the allowable f_s by $(d - kd)$ and multiplying by $(kd - d')$. The total excess force divided by this value of f'_s gives the trial value of A'_s. (Then add perhaps 25 per cent to it.)

B. For analysis:

1. Assume $k = 0.38$ and $j = 0.88$.
2. Solve for $f_s = M \div 0.88A_sd$.
3. The total compressive force $C =$ the tensile force $T = f_sA_s$.
4. The compressive force in the steel $C_s = A'_sf_s(kd - d') \div (d - kd)$.
5. $C - C_s =$ total compressive force in the concrete $=$ approximately

kd

$_c$. Solve for f_c.

These trial procedures are based upon an assumed value of kd.

Another useful approximate formula for trial design and analysis of such beams may be derived by assuming $k = 0.36$, $j = 0.88$, and $n = 10$. Also assume that the compressive reinforcement is $\frac{2}{3}kd$ from the neutral axis. Then the total compressive force

$$C = \frac{M}{0.88d} = \frac{bkdf_c}{2} + (n - 1)A'_s\frac{2}{3}f_c.$$

$$\frac{M}{0.88d} = f_c(0.18bd + 6A'_s). \tag{2-27}$$

The first trial method will now be applied to a particular problem.

Problem 2-7. Assume that a simply supported beam must be limited to a width of 12 in. and a total depth of $18\frac{1}{2}$ in. It must resist a bending moment of 600,000 in.-lb. Assume also that n equals 12, the cover over the rods equals $2\frac{1}{2}$ in., and the allowable f_s and f_c equal 18,000 and 900 lb. per sq. in., respectively. Design the beam.

Using the trial method for design given under *A*,

$$kd = 0.38(18.5 - 2.5) \quad 6.1 \text{ in.} \qquad \text{and} \qquad (d - kd) = (16 - 6.1) = 9.9 \text{ in.}$$

$$A_s = \frac{M}{f_sjd} = 600{,}000 \div (18{,}000 \times 0.88 \times 16) = 2.4 \text{ sq. in.}$$

$$f_c = \frac{2M}{kjbd^2} = 2 \times 600{,}000 \div (0.38 \times 0.88 \times 12 \times 16^2) =$$
$$1{,}170 \text{ lb. per sq. in.}$$

Then the excess force $= \frac{1}{2}(1{,}170 - 900) \times 12 \times 6.1 = 9{,}900$ lb.

$$\text{Approx. } f'_s = 18{,}000\ (6.1 - 2.5) \div 9.9 = 6{,}500 \text{ lb. per sq. in.}$$

$$\text{Approx. } A'_s = \frac{9{,}900}{6{,}500} = 1.52 \text{ sq. in.}$$

Try two 1-in. round rods in the top and three of the same size in the bottom. Using these, the student can analyze this trial section, finding

$$\begin{aligned} kd &= 6 \text{ in., and } d - kd = 10 \text{ in.} \\ I_c &= 3{,}920 \text{ in.}^4 \\ S_c &= 653 \text{ in.}^3 \\ S_s &= 32.7 \text{ in.}^3 \\ f_c &= 920 \text{ lb. per sq. in.} \\ f_s &= 18{,}400 \text{ lb. per sq. in.} \\ f_s' &= 6{,}450 \text{ lb. per sq. in.} \end{aligned}$$

These show that the assumed rods are sufficient for the purpose.

As far as the relief of the concrete in compression is concerned, it is sometimes possible that the use of the same weight of steel added to the tensile reinforcement may shift the neutral axis and increase kd so that the concrete alone may be able to resist the compressions safely without the use of compressive reinforcement. However, such reinforcement is often used for practical reasons rather than for the purpose of adding compressive strength to beams. It is probably most useful over the supports of continuous beams. Figure 3-7(*c*) pictures the rods in such a beam. Note the presence of one pair of rods running full length through the top whereas another pair does likewise in the bottom. Incidentally, if the lap of the bottom rods above the column were made long enough, four rods could be counted in the section as compressive reinforcement.

2-9. T-beams. The use of simple concrete slabs of moderate depth and weight is generally limited to spans of 10 to 15 ft. Where it is desired to use concrete for long spans without excessive weight and material, a common type of construction is that shown in Fig. 2-11(*a*). It consists of a relatively thin slab with deep, haunched portions or stems at intervals. Figure 2-11(*b*) gives an exaggerated picture of the action of the slab under vertical loads, whereas Fig. 2-11(*c*) shows the action of the stem if it is simply supported. Instead of considering the stem to be a rectangular beam which carries the load by itself, it is better to realize that all parts of the structure must act simultaneously. In general, the stem and the slab near it can be assumed to act as a unit, forming a "T-beam" as shown in Fig. 2-12.

In this type of construction, there are two general cases to consider. In the first, the neutral axis is located in the slab or flange section as shown in Fig. 2-12(*a*). The problem is then the same as for a rectangular beam of the size shown by the dotted lines, since direct tensile stress in the concrete below the neutral axis is neglected anyway. The second case is one in which the neutral axis lies in the stem as pictured in Fig. 2-12(*b*). The

diagram representing the compressive unit stresses in the flange is a trapezoidal wedge instead of a triangular one. This is shown in Fig. 2-12(*c*). The small portion of the top of the stem $ABB'A'$ which is subjected to compression is generally neglected.

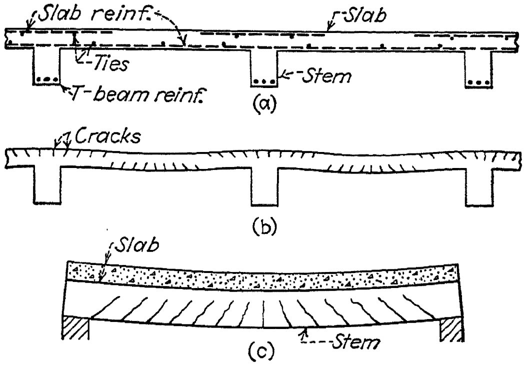


FIG. 2-11.

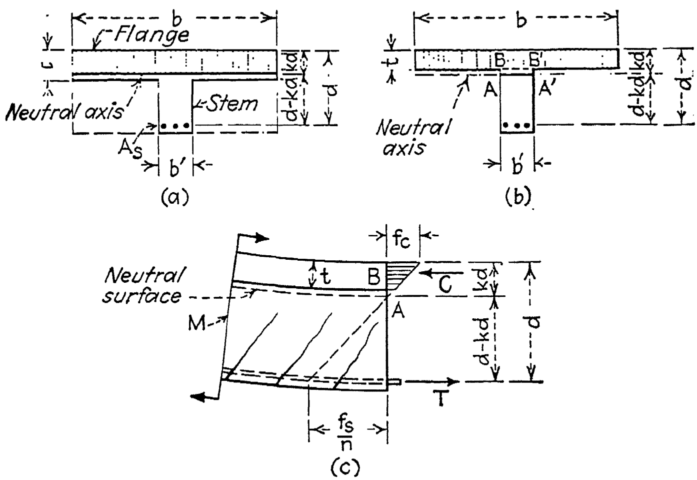


FIG. 2-12.

When a T-beam carries a negative bending moment (as at a support), it is treated as a rectangular beam the width of which is b' because the flange concrete should not be relied upon to withstand tension.

It is possible to derive formulas for k, j, M_c, M_s, etc., for T-beams, but they are rather complicated. It is therefore con-

sidered advisable to use the transformed-section method in this text in the same manner as for more simple beams in order to illustrate the action more clearly.

The Code states that the maximum effective width of the flange on each side of the stem can be assumed to equal eight times the thickness of the flange or one-half the clear distance between adjacent stems. The total flange width must not exceed one-fourth the span of the beam.

The magnitude of j for T-beams is not exactly $1 - \frac{k}{3}$ when the neutral axis lies below the flange. However, it is sufficient for one to assume this value for j in most cases. Figure 11 in the Appendix is sometimes useful in finding k and j.

The tensile steel is almost always the critical part of such beams when subjected to positive bending moments. It is generally sufficient to assume $j =$ from 0.87 to 0.9, then test or design the tensile reinforcement only. However, the analysis of such beams by means of the transformed-section method will be given for purposes of illustration.

Problem 2-8. The T-beam shown in Fig. 2-13 is to carry a bending moment of 1,300,000 in.-lb. If n equals 12, find f_s and f_c.

$$nA_s = 12 \times 1.32 = 15.8 \text{ for each set of rods.}$$
$$48 \times 4(kd - 2) = 15.8(30 - kd + 27 - kd).$$
$$kd = 5.7 \text{ in.} \quad \text{and} \quad d - kd = 24.3 \text{ in.}$$

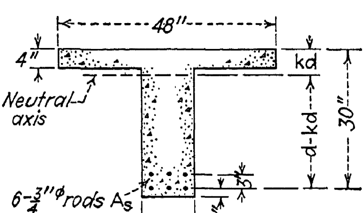


FIG. 2-13.

It should be noted that d is measured to the bottom row of rods rather than to the center of gravity of the group of rods. Although this differs from the usual custom of the past, it is more accurate, and, when the transformed-section method is used, it is not particularly troublesome.

The moment of inertia, computed about the neutral axis, is

$$I_c = 48 \times \frac{4^3}{12} + 48 \times 4 \times 3.7^2 + 15.8(24.3^2 + 21.3^2) = 19{,}390 \text{ in.}^4$$

In computing the moment of inertia of the flange, it must be remembered that the moment of inertia of a section about any axis equals the moment of inertia about its own center of gravity plus the area times the square of

the distance between the two axes. However, I for a set of rods about its own center of gravity is so small that it can be neglected.

$$S_c = \frac{I_c}{kd} = \frac{19{,}390}{5.7} = 3{,}400 \text{ in.}^3$$

$$S_s = \frac{I_c}{n(d - kd)} = 19{,}390 \div (12 \times 24.3) = 66 \text{ in.}^3$$

$$f_c = \frac{M}{S_c} = \frac{1{,}300{,}000}{3{,}400} = 382 \text{ lb. per sq. in.}$$

$$f_s = \frac{M}{S_s} = \frac{1{,}300{,}000}{66} = 19{,}700 \text{ lb. per sq. in.}$$

Problem 2-9. Design a simply supported T-beam to span 25 ft. and to carry a uniformly distributed live load of 300 lb. per sq. ft. plus the dead load. Assume that the framing is such that the slab is $4\frac{1}{2}$ in. thick and the stems are 8 ft. c.c. Take $n = 10$, the allowable f_s and $f_c = 20{,}000$ and 1,200 lb. per sq. in., respectively, and the maximum width of flange $= b' + 12t$.

This problem is another one that requires the making of assumptions and then the testing of them. The procedure is as follows:

1. Assume b', and, from the specifications already set up, find b.
2. Assume d; then call j equal to 0.87,[1] and solve for A_s as though the beam were a rectangular one with a width of b.
3. If A_s permits the use of rods that can be arranged properly, then assume the size and position of the rods, and analyze the trial beam, neglecting shear for the present.

Following the foregoing outline, assume b' equals 16 in. Then b equals $b' + 12t = 16 + 12 \times 4.5 = 70$ in.

Assume $d = 20$ in., $j = 0.87$, and the average dead load $= 100$ lb. per sq. ft. Then

$$M = \frac{wL^2}{8} \times 12 = 8(300 + 100) \times 25^2 \times \frac{12}{8} = 3{,}000{,}000 \text{ in.-lb.}$$

$$A_s = 3{,}000{,}000 \div (20{,}000 \times 0.87 \times 20) = 8.6 \text{ sq. in.}$$

This would require nine rods 1 in. square, but good judgment shows that this is not a practical arrangement. Therefore, assume d equals 24 in. Find a new steel area, which is $A_s = 7.2$ sq. in. Use 8 rods 1 in. square, in two rows of four each, spaced 3 in. c.c. between rows.

Before going farther, check up the assumed dead load. It is found to be 105 lb. per sq. ft. Therefore, $M = 3{,}040{,}000$ in.-lb.

The student can now analyze this trial beam. The value of f_c will be found to be 718 lb. per sq. in.; that of f_s, to be 20,000 lb. per sq. in. This beam is so heavily reinforced that f_c is unusually high for a T-beam. In

[1] One can assume a higher value for j, but that shown gives safe results.

general, one soon learns that the critical part of such a beam is the steel. It is unnecessary to compute f_c, except near the supports when the beam is continuous.

2-10. Irregular Beams. By using the transformed-section method and the principles illustrated in Arts. 2-8 and 2-9, it is generally possible to analyze and design reinforced-concrete beams of irregular, unusual, and unsymmetrical shapes. Practice in the solution of such problems is the best way in which to fix the methods in one's mind. In the problem that follows, an unsymmetrical section is analyzed for the purpose of illustration.

Problem 2-10. Find M_c and M_s for the hexagonal section shown in Fig. 2-14 if $n = 10$ and the allowable f_s and $f_c = 18{,}000$ and $1{,}000$ lb. per sq. in., respectively.

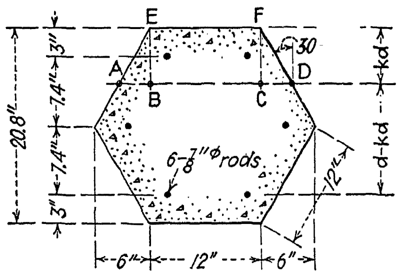


FIG. 2-14.

The neutral axis lies somewhere in the upper half of the beam. The rectangle $EFCB$ has a width of 12 in. and a height equal to kd. The triangles ABE and FCD have altitudes equal to kd, but the bases AB and $CD = kd \times \tan 30°$, or $0.577kd$. Therefore, the equation for kd can be expressed as follows:

$$+ \frac{2kd(0.577kd)kd}{2 \times 3} + (n-1)A_s'(kd-3) = nA_s(10.4 - kd) + nA_s(17.8 - kd)$$

where $nA_s = 10 \times 0.6 \times 2 = 12$, and $(n-1)A_s' = 9 \times 0.6 \times 2 = 10.8$.

Solving this equation gives

$kd = 5.2$ in., and $d - kd = 12.6$ in.

$$I_c = \frac{EF \times EB^3}{3} + \frac{2 \times AB \times EB^3}{12} + (n-1)A_s'(kd-3)^2 + nA_s(10.4 - kd)^2 + nA_s(17.8 - kd)^2$$

$$I_c = \frac{12 \times 5.2^3}{} \quad \frac{2(0.577 \times 5.2)5.2^3}{12} \quad 10.8 \times 2.2^2 + 12 \times 5.2^2 + 12 \times 12\ 6^2$$

$$I_c = 2{,}920 \text{ in.}^4$$

$$S_c = \frac{I_c}{kd} = \frac{2{,}920}{5.2} = 562 \text{ in.}^3$$

$$S_s = \frac{I_c}{n(d - kd)} = \frac{2{,}920}{10 \times 12.6} = 23.2 \text{ in.}^3$$

$$M_c = f_cS_c = 1{,}000 \times 562 = 562{,}000 \text{ in.-lb.}$$

$$M_s = f_sS_s = 18{,}000 \times 23.2 = 418{,}000 \text{ in.-lb.}$$

Practice Problems

The following problems are to be solved by the principles of Arts. 2-3 to 2-6, inclusive. The weight of concrete, including reinforcement, is to be assumed equal to 150 lb. per cu. ft. for all problems.

Problem 2-11. Assume a beam of the cross section shown in Fig. 2-15. It is subjected to a total bending moment of 1,300,000 in.-lb. If $n = 12$, find the total forces C and T, also the unit stresses f_s and f_c.

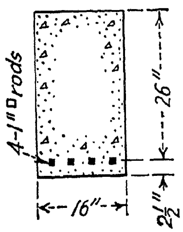


FIG. 2-15.

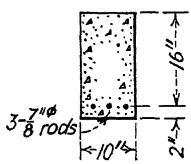


FIG. 2-16.

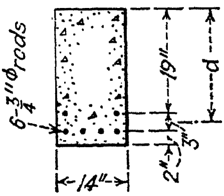


FIG. 2-17.

Problem 2-12. Assume that the beam shown in Fig. 2-16 is simply supported and has a span of 14 ft. If $n = 12$ and the allowable values of f_s and $f_c = 18{,}000$ and 1,000 lb. per sq. in., respectively, find the uniformly distributed live load that this beam will safely support in excess of its own dead load.

Problem 2-13. Find the safe bending moment for the beam shown in Fig. 2-17 if $n = 12$ and the allowable f_s and $f_c = 18{,}000$ and 900 lb. per sq. in., respectively.

Discussion. In this case, assume that the tensile force T is to be considered concentrated at the center of gravity of the group of rods.

Ans. $M_s = 73{,}000$ ft.-lb.; $M_c = 75{,}000$ ft.-lb.

Problem 2-14. The beam pictured in Fig. 2-18 is a cantilever supporting its own weight and a concentrated load of 6,000 lb. at its outer end. If $n = 15$ and the allowable f_s and $f_c = 18{,}000$ and 750 lb. per sq. in., respectively, find the safe length for which this cantilever can be used.

Discussion. Since the compression is at the bottom of this beam, the problem is similar to the previous ones except that the section is reversed. Equate the controlling resisting moment to $wL^2 \times \frac{12}{2} + 12PL$, and solve for L, remembering that $w = \left(10 \times \frac{10}{144}\right)150$ and $P = 6{,}000$ lb.

Problem 2-15. If $n = 12$, $b = 13$ in., $d = 20$ in., and the allowable values of f_s and $f_c = 18{,}000$ and 1,000 lb. per sq. in., respectively, find the required area of the rods for a balanced design.

Discussion. Find p from Eq. (2-20).

Problem 2-16. Design a simply supported slab to carry a uniformly distributed live load of 300 lb. per sq. ft. over a span of 8 ft. if n equals 12 and the allowable values of f_s and f_c equal 18,000 and 900 lb. per sq. in., respectively.

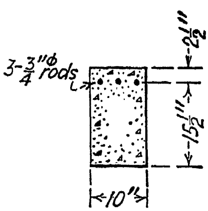


FIG. 2-18.

Discussion. Find k from Eq. (2-16) and p from Eq. (2-20). Use $b = 12$ in. Assume a dead load per square foot for the slab itself, and add it to the live load before computing M. Using Eq. (2-8), solve for d, and add $1\frac{1}{2}$ in. to get the total thickness. Then find $A_s = pbd$, and choose rods that will fit into the 12-in. width of slab at not less than 3 in. c.c.

The following problems are to be solved by the transformed-section method.

Problem 2-17. The beam shown in Fig. 2-19 is subjected to a bending moment of 800,000 in.-lb. Assume $n = 10$, and solve for f_s and f_c.

Discussion. It is advisable to consider one strip of concrete to replace the upper set of two rods and another strip to take the place of the lower set of four rods. This is more accurate than considering the six rods to be grouped at their center of gravity.

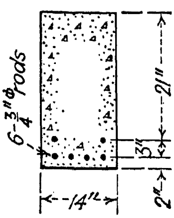


FIG. 2-19.

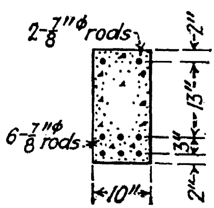


FIG. 2-20.

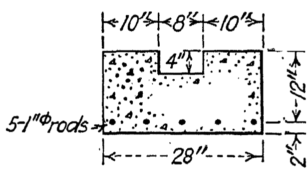


FIG. 2-21.

Problem 2-18. Find the resisting moments M_s and M_c for the beam illustrated in Fig. 2-20 if $n = 15$ and the allowable f_s and $f_c = 18{,}000$ and 900 lb. per sq. in., respectively.

Ans. $M_c = 690{,}000$ in.-lb., $M_s = 780{,}000$ in.-lb.

Problem 2-19. A beam is $12\frac{1}{2}$ in. wide and 24 in. in total depth. It runs across a support where it has a negative bending moment of 950,000 in.-lb. There are three $\frac{7}{8}$-in. round rods $2\frac{1}{2}$ in. above the bottom. Find the number of $\frac{7}{8}$-in. round rods needed in the top of the beam if $n = 12$ and the allowable f_s and $f_c = 18{,}000$ and 900 lb. per sq. in., respectively. Find also the values of f_c, f_s, and f'_s.

Problem 2-20. If the beam shown in Fig. 2-21 carries a bending moment of 400,000 in.-lb., find f_s and f_c, assuming $n = 12$.

Discussion. The 8-in. slot at the center removes all or most of the concrete in the region of compression. Therefore, assume the beam to be 20 in. wide.

Problem 2-21. Assume that the T-beam shown in Fig. 2-22 is carried continuously across its support and that the negative bending moment is 2,000,000 in.-lb. The bottom layer of rods is in compression. Find the

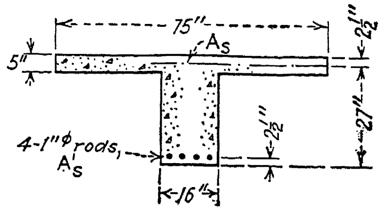


FIG. 2-22.

rods needed to withstand the tension in the top if they are $2\frac{1}{2}$ in. from the top of the slab. The second row (if any) is 3 in. below the upper one, and all rods are 1 in. round. Assume $n = 10$ and the allowable f_s and f_c = 20,000 and 1,200 lb. per sq. in., respectively.

Discussion. The beam is to be analyzed like a rectangular one 16 in. wide. Make a trial calculation for A_s, using $j = 0.87$ and $d = 27$ in.

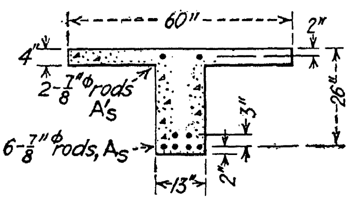


FIG. 2-23.

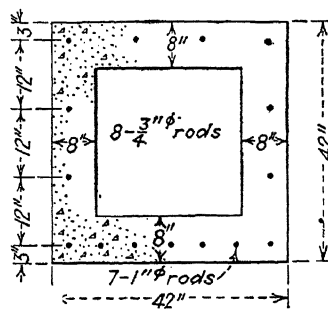


FIG. 2-24.

Thus, A_s is found to be 4.26 sq. in. This requires over five rods, but it will be best to use two rows with four in the top layer and two in the bottom. This is done for symmetry. The trial beam can now be analyzed.

Ans. f_c = 940 lb. per sq. in.
f_s = 19,200 lb. per sq. in.

Problem 2-22. If the beam shown in Fig. 2-23 carries a bending moment of 1,500,000 in.-lb., find f_s and f_c, assuming $n = 12$.

Problem 2-23. Find the safe resisting moment of the box section shown in Fig. 2-24 if $n = 12$ and the allowable f_s and f_c = 20,000 and 900 lb. per sq. in., respectively. As usual, neglect the compression on the stem portions below the flange (top slab).

Problem 2-24. Find the safe resisting moment of the beam pictured in Fig. 2-25 if $n = 15$ and the allowable f_s and f_c = 18,000 and 800 lb. per sq. in., respectively.

Discussion. Solve for kd as usual, remembering that the trapezoidal area in compression may be considered as a rectangle minus two triangles. The altitude of each triangle is kd; its base is $kd/3$.

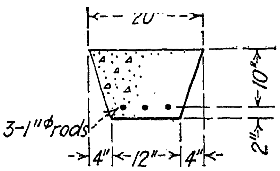


FIG. 2-25.

Problem 2-25. Figure 2-26 shows a cantilevered rectangular beam which it is proposed to use to support the outside stringer or facia of a viaduct. The allowable f_s and f_c = 20,000 and 1,000 lb. per sq. in., respectively; n = 10. Using the loads and dimensions given, check the stresses in the steel and concrete by using the transformed-section method with d = the distance to the center of gravity of the six tensile rods. Neglect the two $\frac{1}{2}$-in. round ties. Compute the bending moment at the center of the column, including the weight of the concrete. Is the member safe? If not, what should be done about it?

Ans. f_s = 18,900 lb. per sq. in.—safe.
f_c = 1,320 lb. per sq. in.—overloaded.

Make the section wider or deeper rather than add compressive reinforcement; or perhaps decrease the length of the cantilever if possible.

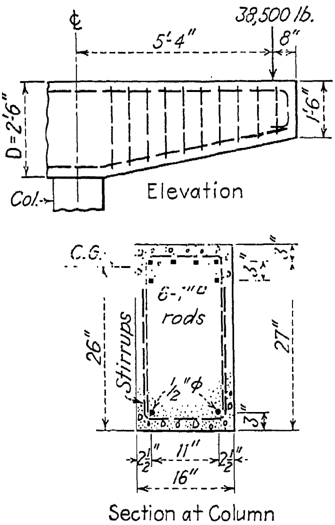


FIG. 2-26.

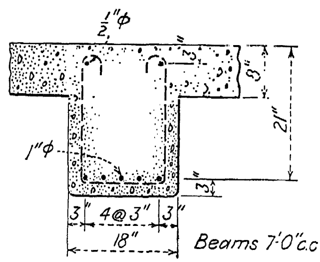


FIG. 2-27.

Problem 2-26. Figure 2-27 shows the center of a T-beam used in the Primary Crushing Plant of the Cananea Consolidated Copper Co., Cananea, Sonora, Mexico. The span = 20 ft. Assuming the beam to be partially restrained so that the bending moment at the center is $wL^2/12$, find the safe uniformly distributed live load which the beam can support. Assume the allowable f_s and f_c = 22,000 and 900 lb. per sq. in., respectively; n = 10; j = 0.88.

Discussion. Since this is a T-beam, f_c will be low and the resisting moment will be determined by the steel.

Ans. M = 134,000 ft.-lb.
LL = 430 lb. per sq. ft.

CHAPTER 3

BOND

3-1. Nature and Magnitude of Bond Stress. When a reinforcing rod is embedded in concrete, the latter adheres to its surface, resisting any force that tends to pull or push out the rod. This is called the "bond" between the concrete and the steel. The intensity of this adhesive force is called the "bond stress," or "bond unit stress." In reality, this bond stress is a resistance to shearing between the surfaces of the steel and the concrete. The action is that of resistance to forces that try to break the concrete away from the surface of the steel in a direction parallel to that surface.

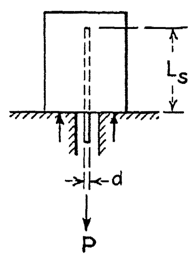


FIG. 3-1.

The function of bond in a reinforced-concrete member is somewhat analogous to that of rivets in structural steelwork. It is the force that holds the two materials together so as to develop their simultaneous and mutually helpful action. If the rods have no change of stress—and therefore no change of length—as a result of the application of a load on the member, then there will be no bond stress set up by it, but as soon as flexural action causes the steel to stretch or to compress, the bond stresses must come into action in order to cause these changes.

When the rod in Fig. 3-1 is stretched, the elongation in the length L_s is greatest at the point where the steel enters the bottom of the concrete block. It then decreases to zero at or somewhere below the top end of the rod. A little reflection will show that the intensities of the bond stresses along the rod must vary somewhat in proportion to the stretching of the rod inside the concrete block unless the bond is broken. Probably the bond stresses are very high near the point at which the rod enters the concrete. It is also likely that they are very high (probably to the point of local failure) at the cracks pictured in Fig. 3-6. The distribution

of the stresses is very uncertain, yet, for analysis and design, it is generally considered to be uniform, but one must realize that the bond will develop the rod as quickly as possible so that a part of the rod which is a long way (over about $30d$) from the point of entry of the rod in Fig. 3-1 may have no stress at all; in other words, anchorage far from the point where the rod is needed may be ineffective.

An expression for the magnitude of the bond stress can be found readily. Referring to Fig. 3-1, let o be the perimeter of the cross section of this rod; let d = its diameter, if round, or the length of its side, if square; let L_s = the length of embedment; and let u = the average bond unit stress. It is clear that the total strength of the bond per inch of rod equals ou. It is also apparent that the embedded length of the rod should be great enough to develop the required tensile (or compressive) strength of the bar in order to avoid having it pull out of the concrete. Then, for a square rod that is subjected to a tensile force P,

$$P = d^2 f_s = 4duL_s \qquad \text{or} \qquad L_s = \frac{df_s}{4u}. \tag{3-1}$$

Also, for a round rod,

$$P = \frac{\pi d^2 f_s}{4} = \pi duL_s \qquad \text{or} \qquad L_s = \frac{df_s}{4u}. \tag{3-2}$$

This shows that, for round or square bars, the length of embedment that is needed to develop the strength of the bar in tension (or in compression) does not depend upon its shape but increases with the strength of the steel and decreases with an increase in the magnitude of the permissible bond stress.

Theoretically, many small rods are better than a few very large ones as far as bond is concerned. This is evident if one studies Eqs. (3-1) and (3-2), bearing in mind that the cross-sectional area increases as the square of d, whereas the surface of the bar per inch of length varies as the first power of d. However, there must be adequate space between adjacent rods—also between the rods and the forms—so that the concrete will completely fill the forms and encase the steel. See Table 8, Appendix, for recommended spacing of rods.

If it is practicable, the rods should be "anchored" in regions where the concrete is under compression because the pressure

itself may help to hold the bars to some extent. When they must be anchored in the portion of the beam that is subjected to tensile stresses, the rods should extend across the cracks as in Fig. 3-6; they should not be even approximately parallel to the lines in which the tensile cracks in the concrete will occur; in other words, the rods must not depend upon the tensile strength of the concrete to anchor them. In the case of a beam such as that which is shown in Fig. 3-6, the tensile reinforcement runs across these cracks, and the bond stresses along the rods develop the tensile forces T by transferring the longitudinal shear in the concrete to the rods.

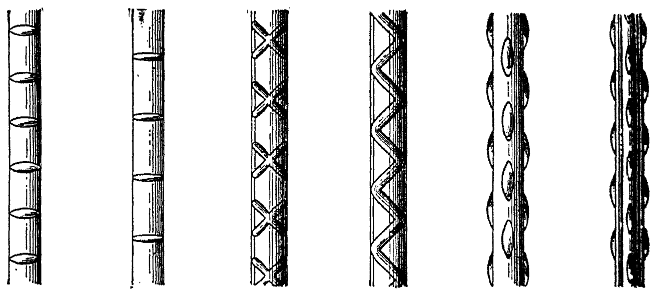


FIG. 3-2.—Typical deformed rods.

All rods must be clean and free from dirt, grease, scale, and loose rust. Anything that destroys the ability of the concrete to grip the steel may prove to be serious because it will prevent the stress in the latter from being fully developed, and therefore it will keep the steel from performing its function properly.

The permissible bond stress along the reinforcement can be increased by making the surface of a rod rough or irregular. Such rods are called "deformed" rods, and they are used generally. Some of the common types of such bars are pictured in Fig. 3-2. The lugs, or corrugations, produce a mechanical bond which helps to lock the concrete and the steel together. Although the lugs do not seem to cause much increase in the bond stress at which a rod will have its initial slip[1] when tested, the wedging effect of the projections has a considerable effect in raising the

[1] Gilkey, Chamberlin, and Beal, The Bond between Concrete and Steel, A.C.I. *Jour.*, September, 1938.

ultimate strength of the bond provided the cover of concrete over the steel is sufficient to avoid stripping or spalling.

Tests of the bond stresses that are developed by plain and deformed rods have been made at Lehigh University.[1] These experiments indicate that deformed bars develop 45 to 50 per cent more ultimate bond stress than do plain rods in beam tests with 3,000-lb. concrete. Furthermore, the form of the lugs seems to be important in "pull-out" tests, but this variation is not so apparent in the tests of beams. Any type of deformation that is equivalent to a raised rib that extends directly or diagonally around the rod appears to be better than narrow ridges running lengthwise of the bar.

The magnitude of the ultimate bond stress does not vary directly with the ultimate compressive strength of the concrete f'_c. However, Table 1-5 allows a bond stress of $0.05f'_c$ for deformed rods and $0.04f'_c$ for plain bars, with a maximum value of 200 or 160 lb. per sq. in., respectively.

3-2. Hooks. In many cases, it is not feasible or possible to extend straight rods far enough to develop their strength sufficiently by bond alone. A common way to remedy this trouble is to bend or hook the rods so as to obtain additional length for their development through bond. When a hooked rod is pulled, it tries to slip or slide around the curve. The hook provides a certain amount of mechanical locking of the steel into the concrete, but this is too indefinite to be relied upon.

Certain principles should be followed in making hooks. Figure 3-3 shows some types of anchorage that are often specified in designs. Sketch (*a*) is a sharp, right-angular bend. When the rod is pulled downward, the bent portion produces a compressive stress in the concrete. However, this arm usually has insufficient strength as a cantilever to spread the load over the entire length *AB*. Therefore, it tends to crush the concrete locally at *A*. It is clear also that a downward pull on the rod cannot produce a horizontal motion of the portion *AB*, and therefore it cannot develop its bond resistance, until the rod begins to pull out below *A* and until it crushes a fillet in the concrete. Consequently, this type of anchorage should be made with a reasonably large radius at *A*—usually at least $3d$ for the inside of the curved rod

[1] George Robert Wernisch, Bond Studies of Different Types of Reinforcing Bars, A.C.I. *Jour.*, November–December, 1937.

for minor details but larger for heavy, longitudinal rods unless the bend is merely for the purpose of catching the end of the bar enough to clinch it moderately.

The anchorage shown in Fig. 3-3(*b*) is a modification of that pictured in (*a*). It is not an improvement, because the acute angle tends to make the compression at point *C* even worse than that at *A* in Fig. 3-3(*a*). Also, the concrete may not fill the triangular space at *C* completely, so that the portion *CD* may be of little use until the bond below *C* is broken. In fact, the larger the rod the worse the condition becomes.

Figure 3-3(*c*) shows another kind of bend in which the rod is hooked back upon itself with a small radius at the bend. This is not much better than the two previous types. It is preferable

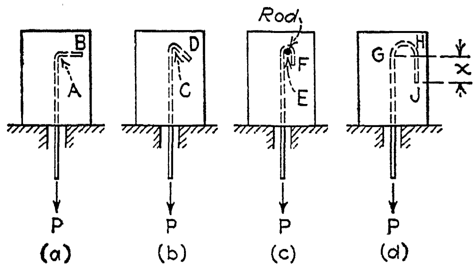


Fig. 3-3.

to go a little farther and to bend the rod as pictured in Fig. 3-3(*d*). This gives sufficient concrete area inside the bent portion of the rod to withstand the compression which is caused by the tension in the steel. It is also desirable to provide a straight portion beyond the bend as an additional anchorage. For ordinary sizes, the diameter of the inside of the bend should be about six times that of the rod. The portion *HJ* should be about four times the diameter of the rod. If the details are arranged in this way, the bond strength of the full length of the rod may be considered to be effective. In fact, it is important to make all bends in reinforcing rods so that they will have a reasonably large diameter. Table 10 in the Appendix shows some typical hooks for rods.

When the hooks pass around a longitudinal rod, as pictured in Fig. 3-3(*c*), the condition is improved somewhat because the pull of the former is transferred at least partially into the latter as a

small beam. This is a sort of mechanical connection, but such hooks on small rods are not strong. Bond must be the chief means of developing the hooked rod because the rods cannot act in one way while the concrete around them acts in another way—unless the bond is broken.

A hook should not be depended upon to anchor a rod subjected to compression, because the bend merely accentuates the tendency of the bar to buckle.

Problem 3-1. Analyze the anchorage of a rod if the type is that shown in Fig. 3-3(d) with a radius of $3d$ inside the curve. Assume that the rod has a diameter of $\frac{1}{2}$ in., a tensile stress of 20,000 lb. per sq. in., and an allowable bond stress of 125 lb. per sq. in.

From Eq. (3-2), the length of rod required for anchorage is

$$L_s = \frac{df_s}{4u} = \frac{0.5 \times 20{,}000}{4 \times 125} = 20 \text{ in.}$$

The length of HJ is $4d = 2$ in.

$$GH \text{ is } \frac{\pi(6d + d)}{2} = \pi \times 1.75 = 5.5 \text{ in.}$$

Therefore, the length required below G is

$$L_x = 20 - (2 + 5.5) = 12.5 \text{ in.}$$

Using Eq. (3-2), the maximum tensile stresses at H and G that can be developed by bond are

$$f_s \text{ at } H = \frac{4uL_s}{d} = \frac{4 \times 125 \times 2}{0.5} = 2{,}000 \text{ lb. per sq. in.}$$

$$f_s \text{ at } G = 2{,}000 + \frac{4 \times 125 \times 5.5}{0.5} = 7{,}500 \text{ lb. per sq. in.}$$

The greatest pressure on the concrete at the inside of the hook of this rod is likely to occur near G. The actual stress condition is uncertain, but an arbitrary and theoretical maximum value for this pressure may be found by assuming that the action is similar to that of a hoop or pipe which is subjected to normal pressure from the inside for which

$$T = pr.$$

T = the tension in the rod, p = the normal pressure in pounds per linear inch of the rod, and r = the radius of the curve in inches—in this case r is assumed to be the radius of the inside of the rod. From the foregoing calculations, at G,

$$T = A_sf_s = pr. \tag{3-3}$$

$$0.2 \times 7{,}500 = p \times 1.5, \qquad \text{or} \qquad p = 1{,}000 \text{ lb. per lin. in.}$$

Then the unit compressive stress in the concrete is

$$f_c = \frac{p}{d} = \frac{1{,}000}{0.5} = 2{,}000 \text{ lb. per sq. in.}$$

This is a large compressive stress, but it is localized.

3-3. Bond of Multiple Layers of Rods. Let Fig. 3-4 represent a beam with two layers of square rods having side dimensions equal to d. Let the cover of concrete on the sides be s, and let the clear distance between rods be m. When the beam is loaded it bends, and the portion below the neutral axis O-O elongates, thus stretching the steel. The increment of tension must be transferred to the rods by the concrete. If the bond strength of the bottom and the side surfaces of the lower rods is to be developed, it must be done by shear in the concrete across the section AB. The bond of the top surfaces of these rods is developed directly. Therefore, let v_L = the allowable unit stress in the concrete (longitudinal shear); $(T_1' - T_1)$, the increment of tension in the bottom rods; and $(T_2' - T_2)$, the increment in the upper rods per inch of length; then

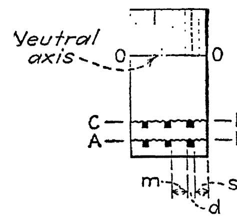


FIG. 3-4.

$$0.75(T_1' - T_1) = 3(3du) = (2s + 2m)v_L. \qquad (3\text{-}4)$$

The section CD must develop the entire lower set of rods and the three sides of the upper set, or

$$(T_1' - T_1) + 0.75(T_2' - T_2) = 3(4d + 3d)u = (2s + 2m)v_L. \quad (3\text{-}5)$$

It must be noted that u in the foregoing cases may not be the full allowable value for the bond stress, but it will be as much as is required to develop the increment of tension. However, the concrete must be strong enough in shear to act in the manner shown. This is a possible weakness in short beams with heavy reinforcement.

For round rods the procedure is similar to that already explained, except that the sections AB and CD may be considered to pass through the centers of the rods.

A further explanation of the bond stresses resulting from beam action will be given in Art. 3–6 and in the next chapter.

3-4. Splices. The ordinary method of splicing reinforcement is by the lapping of the rods past each other so that the bond stresses will transfer the load out of one rod into the concrete and thence into the other rod. These rods might be hooked, but it is not always practicable or desirable to bend them. The length of lap should be at least that given for L_s in Art. 3-1. However, such splices should not be made at points of maximum bending unless they are in the portions of the beam in which compressive stresses exist and in which the steel is not the principal stress-carrying part of the section. In general, it is best to locate splices near points of contraflexure. Many times, they can be staggered so that all splices do not come at the same point.

When rods are closely spaced, it is sometimes difficult to make splices by lapping them past each other in the same horizontal plane without closing up the clear distance between the rods so that the concrete cannot pass through the constricted opening. If they are lapped on top of each other, one set of rods will be higher than the other, thus causing a different effective depth in the two portions. When two square rods are in contact, it is also probable that the concrete will not be able to penetrate between the flat surfaces. These details require careful thought and good judgment.

Sometimes it is desirable (but expensive) to splice large rods by welding. Butt splices or lapped splices may be used, the latter being made by having welds along both sides of the rods at their junction. Such splices may be conservatively designed on the assumption that 1 lin. in. of fillet weld has a strength of 800 to 1,000 lb. for each $\frac{1}{8}$ in. of thickness of the weld itself. Ordinary fillet or bead welds are $\frac{1}{4}$ to $\frac{1}{2}$ in. thick.

Still another method of splicing heavy, plain rods is by threading them and connecting the parts by means of couplings—similar to piping. However, this arrangement is expensive, and it is used only in special cases—if at all.

3-5. Development of Longitudinal Reinforcement. The longitudinal reinforcement in a beam should be fully developed if the design is to be economical and if the steel is to be effective. Of course, the required number of rods is determined by the maximum bending moment. All of them need not be extended for the full length of the beam. They can be discontinued as fast as

the decrease in the bending moment will permit—somewhat in the same manner as the cover plates on steel girders.

For instance, if a beam has a concentrated load at its center, the bending-moment diagram is as shown in Fig. 3-5. Assume that four $\frac{3}{4}$-in. round rods are required. Then it is customary to assume that each rod carries an equal share of the maximum bending moment DC. It is therefore possible to stop one rod at each of points E, F, and G without overloading the remaining ones. However, it is also customary (and generally advisable) to extend the rods as shown by EH, FJ, and GK so as to develop them by bond (at least partially) as given in Eqs. (3-1) and (3-2), before they reach the point at which they are really needed to help in carrying the load. These rods may be either straight or

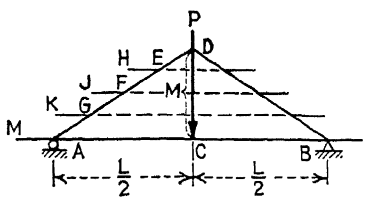
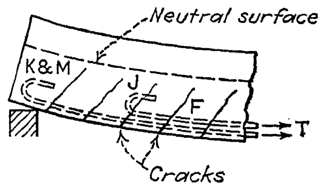


FIG. 3-5.



IG. 3-6.

bent, as shown in Fig. 4-10(a) of the next chapter. It is customary to stop or to bend them in pairs (for symmetry) unless the number is odd or unless the conditions are unusual.

To illustrate the anchorage of these rods more clearly, let Fig. 3-6 represent the anchorage of the longitudinal rod at point F of Fig. 3-5. Assuming that it is bent, then the hook FJ causes a pressure on the concrete of the lower portion of the beam which is in tension and which may be cracked. The longitudinal shearing strength of the concrete must resist this force. Thus it is easy to see that the hook is of little or no advantage because of the conce. `ration of the pull at an inadvisable point in the beam. It is therefore better to have the rod straight and extending an adequate distance beyond F or to bend it up and hook it into the compression side of the beam.

The rods of Fig. 3-5 which extend to K and M should be hooked as shown in Fig. 3-6 in order to keep them from being pulled out. These hooks should have adequate cover on the sides because, otherwise, the concrete may spall off owing to its

inadequate strength in tension due to the wedging effect at the hook.

A report of tests[1] made to ascertain the bond strength along horizontal bars gives, in part, the following tentative but interesting conclusions:

1. Straight, horizontal rods fixed in position near the tops of beams may be weak in bond because the wet shrinkage or settlement of the concrete tends to cause voids under the bars. Vibration of the forms may help to remedy the situation. This feature may be a source of weakness in continuous beams.
2. Straight, horizontal rods released so as to "float" in the concrete near the tops of beams develop good bond strength.
3. Horizontal rods with hooked ends but rigidly held vertically may develop harmfully large slip before the anchorage will be effective.
4. Straight, horizontal rods in the bottoms of beams (and vertical ones) develop good bond.
5. End anchorages beyond the points of inflection of continuous beams are likely to permit large deflections of the beams before these anchorages become effective.
6. Completely watertight forms tended to improve the bond along the top rods.

The preceding data are especially worthy of thought in connection with such beams as those pictured in Figs. 3-7, 4-17, and 4-26. They seem to show the desirability of bending the rods so that they can, before failure, serve somewhat the same function as the cables in a suspension bridge.

3-6. Bond Stresses on Longitudinal Rods. A formula for the bond stresses in reinforced-concrete beams is developed in Art. 4-4 of the next chapter. Temporarily it will be accepted. It is the following:

$$u = \frac{V}{(\Sigma o)\,jd} \tag{3-6}$$

where V = total transverse shear in the beam at the cross section being considered and Σo = total surface area of the rods per inch of length at that location—numerically equal to the total perimeters of the bars. From Eq. (3-6) it is clear that the bond stresses are greatest where the shear is the largest or where the total surface area of the rods is least.

[1] R. C. Robin, P. E. Olsen, and R. F. Kinnane, Bond Strength of Reinforcing Bars Embedded Horizontally in Concrete, *The Journal* of the Institution of Engineers, Australia, Vol. 14, No. 9, September, 1942; and Highway Research Abstracts, February, 1943, by Highway Research Board.

In order to illustrate the computation of bond unit stresses in a beam, assume the case shown in Fig. 3-7, which pictures half of a heavy, continuous beam 24 ft. long. Using Eq. (3-6) and

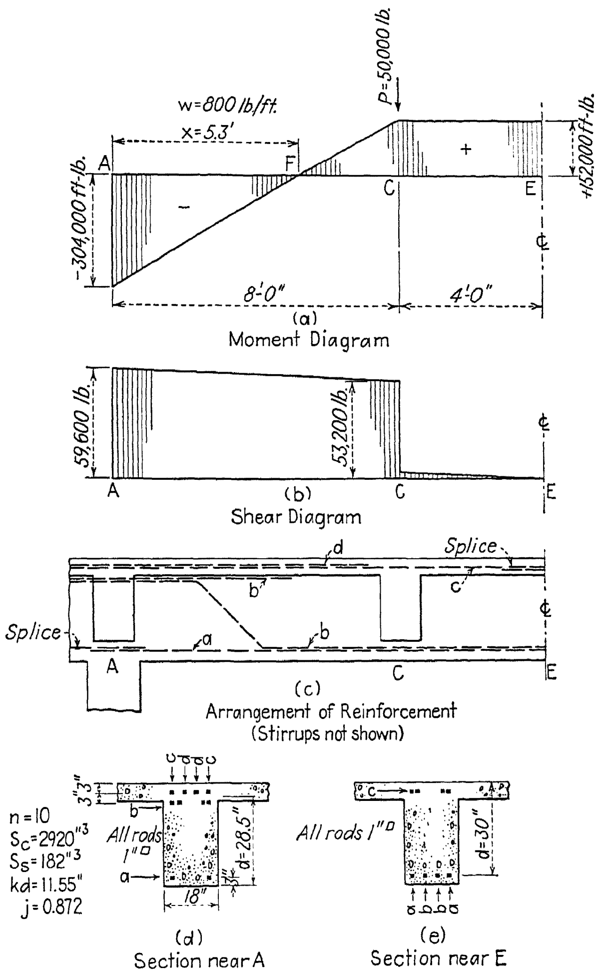


FIG. 3-7.

the values in Fig. 3-7, but with d measured to the center of gravity of the group of rods, the bond stress on the eight 1-in. square rods at A is

$$u_A = \frac{V}{(\Sigma o)jd} = \frac{59{,}600}{32 \times 0.872 \times 28.5} = 75 \text{ lb. per sq. in.}$$

Also, approximately,

$$u_C \quad 16 \times 0.88 \times \quad = 126 \text{ lb. per sq. in. just left of}$$

There is another way of looking at this question of bond unit stresses. Figure 3-7(a) shows that a point of contraflexure is at F, the distance x from A to this point scaling 5.3 ft. Obviously, the stress in the rods at F should be zero. Using the moment at A and the properties of the section given in Fig. 3-7(d), the unit stress in these rods at A is

$$f_s = \frac{304{,}000 \times 12}{182} = 20{,}000 \text{ lb. per sq. in.}$$

Therefore, this stress must be imparted to the steel through bond in the distance x. Since the shear is nearly constant from A to F, assume that this "pick-up" of tension is at a constant rate. Therefore, on this basis, the bond stress on one top rod is

$$u = \frac{f_s A_s}{x(\Sigma o)} = \frac{20{,}000 \times 1}{5.3 \times 12 \times 4} = 79 \text{ lb. per sq. in.}$$

This is only slightly different from the result given by Eq. (3-6). Exact agreement of these figures is not to be expected—nor is it generally important.

The foregoing method is particularly useful in computing bond stresses in members subjected to longitudinal thrusts or tensions combined with bending. The basic idea is to compute the stress in the steel at a given point, then calculate it a foot (or two) from the first point. Obviously, the bond stress must be

$$u = \frac{\text{difference in unit stress} \times \text{area of a rod}}{\text{distance between points in inches} \times \text{perimeter of rod}}. \qquad (3\text{-}7)$$

Unless the direct forces are large compared with the bending, their effects upon the bond stresses are not important. Equation (3-6) generally gives results that are satisfactory for practical purposes in the design of ordinary beams.

One must not forget that bond stresses also act upon rods in compression. However, the lower unit stresses generally existent in such bars seldom cause trouble with the bond. For

instance, Fig. 3-7(*d*) shows two 1-in. rods near the bottom where they are subjected to a compressive unit stress of 9,250 lb. per sq. in. From Eq. (3-7), and considering the distance x in Fig. 3-7(*a*),

$$u = \frac{9{,}250 \times 1}{5.3 \times 12 \times 4} = 36 \text{ lb. per sq. in.}$$

The designer is not so much interested in learning what the magnitudes of the bond stresses are as he is in making sure that they are not excessive. Bond generally becomes critical only in the case of short, heavily loaded members such as footings, beams

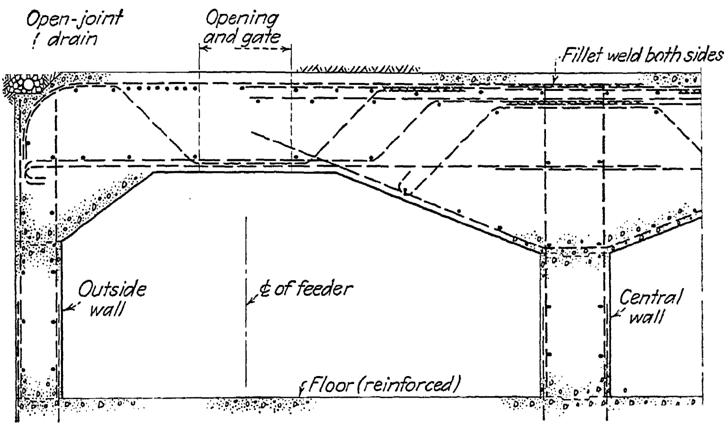


FIG. 3-8.—Roof of feeder compartment under 8,000-ton ore bin. (*Courtesy of Cananea Consolidated Copper Company, Cananea, Sonora, Mexico.*)

carrying offset columns, and other cases where the shears are relatively large compared with the bending moments.

When bond stresses are too high, or when the designer does not want to depend upon bond alone, it is often feasible mechanically to fasten rods to something that anchors them thoroughly. An example of this in industrial construction is shown in Fig. 3-8. The roof of this feeder compartment must support a possible depth of ore of about 70 ft.; hence it must be very strong. In order to ensure against bond failure, some of the critical splices are welded so as to form a partial truss action and to make the rods continuous from one side to the other.

3-7. Bond under Impact and Reversal of Stress. Apparently the resistance of the bond of concrete to reinforcement under

static or gradually applied loads is one thing; under shocks from suddenly applied loads it seems to be something else—and far less reliable. More data are needed on this subject, but experiences with reinforced-concrete structures under bombing appear to show that the concrete is likely to disintegrate and fly to pieces, leaving the rods behind as a grille of steel. This action may be due to the brittleness of the concrete and to the action of severe and rapidly traveling deformations of the steel, causing progressive local bond failures and cracking which "run" along the rods from point to point as the "wave" of sudden tensile elongation progresses through the steel. At any rate, structures designed to resist any such loads should be knitted together with a sort of steel cage having the junctions of main rods mechanically fastened in all directions—preferably by welding. In other words, the concrete should be a sort of filler between the rods, but the bond strength should be depended upon only slightly.

Rapid increases in the stresses in reinforcement, and quick reversal of these stresses, may also cause breaking of the bond. This may occur in continuous girders carrying heavy trucks, railroad trains, and cranes. Pending more reliable information than is now available, the designer should be unusually careful in checking into the bond situation. He should be conservative, basing his decisions upon the nature and importance of each special case.

3-8. Bond of Concrete to Structural Plates and Shapes. When a large I-beam is covered with concrete, a strange combination results. The former is a strong, ductile member, whereas the latter is relatively rigid and brittle. Obviously, the two materials do not willingly act in unison. The bond of the concrete to the steel must be so great that it will compel the latter to act as a piece of reinforcement in the former, or else the two will break apart.

There is much uncertainty regarding the magnitude of the bond stress that can be developed when such heavy steel members are encased in this way. The cross section of the steel may be large compared to its surface area; the surface of the steel is generally flat and smooth; there may be dirt, rust, or grease on the steel when it is embedded; the details of the concrete section may be such that it tends to break into isolated parts or chunks so that it becomes merely a filler. Therefore, the steel member

must be properly anchored to the concrete, or it must be thoroughly encased. The greatest permissible value for the bond stress in such a case is the same as for plain rods, $0.04f'_c$.

The methods to use in designing such members are described in Chap. 5.

Practice Problems

Problem 3-2. Find the necessary length through which two 1-in. round rods must be overlapped in order to splice them fully if f_s in the rods = 18,000 lb. per sq. in. and the allowable bond stress = 150 lb. per sq. in.

Problem 3-3. Two $1\frac{1}{8}$-in. square rods are lapped 30 in. If the allowable bond stress equals 125 lb. per sq. in., what stress can be developed safely in the rods by such a lapped splice?

Problem 3-4. Two $\frac{5}{8}$-in. round rods are lapped 24 in. The rods are stressed to 20,000 lb. per sq. in. What is the intensity of the bond stress that must be developed at the splice?

Problem 3-5. A $\frac{3}{4}$-in. round rod has a hooked end similar to that of Fig. 3-3(d). If HJ = 3 in., the allowable bond stress = 150 lb. per sq. in., and the radius of the inside of the bend = $4d$, find the point at which the bond alone can be said to have developed 18,000 lb. per sq. in. in the rod.

Ans. L_s = 8.9 in. from tangent point.

Problem 3-6. Find the maximum crushing stress on the concrete at the beginning of the hook of the rod in Problem 3-5, point G of Fig. 3-3(d), using principles illustrated in Problem 3-1.

Problem 3-7. Assume the beam shown in Fig. 2-19. If the maximum shear = 46,000 lb. and f'_c = 3,000 lb. per sq. in., is the bond stress satisfactory?

Discussion. From Table 1-5, the allowable $u = 0.05f'_c$. Assume d to the center of gravity of the group of rods, 23 in. Call j = 0.88.

Ans. u = 160 lb. per sq. in. and is therefore excessive.

Problem 3-8. Assume a continuous beam having the dimensions shown in Fig. 2-22. The top row of steel is four 1-in. round rods; two other 1-in. round rods are located 3 in. below these. Is the bond stress on the top tensile steel satisfactory if f'_c = 3,000 lb. per sq. in., max. $u = 0.05f'_c$, j = 0.88, d = 26 in., and V = 42,000 lb.?

Ans. Yes, since u = 98 lb. per sq. in.

Problem 3-9. Using the beam of Prob. 3-8, assume that the bending moment at the support is 2,000,000 in.-lb. and that it is 1,500,000 in.-lb. 12 in. away from that support. If n = 10, kd = 8.86 in., $d - kd$ = 18.14 in., S_c = 2,130 in.^{3}, S_s = 104 in.^{3}, and the top rods are as given in Prob. 3-8, compute the bond stresses on one of the top rods and on one of the rods in compression near the bottom, using the idea of bond developing the "pick-up" of stress.

Ans. u = 100 lb. per sq. in. at top; u = 35 lb. per sq. in. at bottom.

CHAPTER 4

SHEAR AND WEB REINFORCEMENT IN BEAMS

4-1. Introduction. The determination of the magnitudes and the distributions of the shearing stresses in reinforced-concrete beams is one of the most troublesome problems that face the student of elementary reinforced-concrete design. Many tests have been made, and these have determined the ultimate shearing strengths of certain beams. Since their results can be accepted as being correct, it is clear that the total safe shear for any given member must be that which the tests show it to be, but the question to settle is that of the most probable action of the beam and the most correct method of finding the shearing stresses in it. In fact, much more experimental information is needed by engineers in order to ascertain the actual conditions of shearing stress in such members.

There are two different sets of shearing forces which must be considered in a study of the action of a beam when it is loaded. The first set consists of the transverse shearing forces which tend to break off the member at right angles to its longitudinal axis. The second set is composed of the longitudinal shearing forces which are due to the internal bending action of the beam and which act in a direction parallel to its longitudinal axis.

Reinforced-concrete beams are not homogeneous. Therefore, when they are subjected to these shearing forces, they behave in a manner that is peculiar to themselves. The subsequent portions of this chapter are parts of a general attempt to ascertain the nature of this behavior; to present a logical and understandable theory to be used in the analysis of the shearing stresses in such beams; and to determine how to proportion the reinforcement for them. The methods used herein are designed for the purpose of enabling one to visualize clearly the action of each element of the member—or at least to understand what that action may become before the beam will fail.

4-2. Determination of Shearing Stresses. Try to visualize a reinforced-concrete beam that is loaded in any manner that

causes shearing and bending stresses in it. Under this action, the beam will curve as shown in Fig. 4-1(*a*), producing compression in the top and tension in the bottom part of the member. The portion of the beam above the neutral axis will not crack open prior to real failure. However, as pictured for the lower region, cracks will form when the concrete is unable to elongate sufficiently to equal the deformation of the rods. Therefore, these two portions of the beam must act in different ways.

Furthermore, it can be seen that there are two cases to consider, viz., the shearing stresses in the uncracked beam and those in the member after the cracks have formed.

First, investigate the uncracked condition. Let Fig. 4-1(*b*) represent a small prism of concrete having a length equal to b,

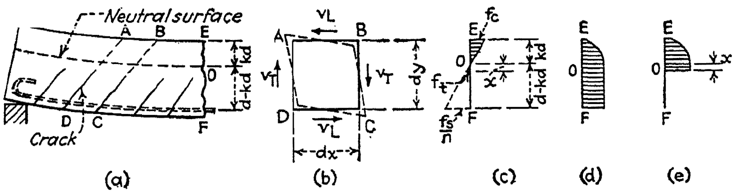


FIG. 4-1.

the width of the beam of Fig. 4-1(*a*). Assume a vertical or transverse shearing unit stress v_T acting upon the face BC and an equal but opposite stress acting on AD. These two forces constitute a couple having a moment equal to $v_T b(dy)(dx)$. Since the conditions for equilibrium require that there must be no rotation of this prism, there must be another couple which counteracts the first one. The latter must be composed of another set of shearing forces which act longitudinally on faces AB and DC. Assuming the intensity of the latter to be v_L, then

$$v_T b(dy)(dx) = v_L b(dx)(dy), \qquad \text{or} \qquad v_T =$$

Equation (4-1) indicates that at any point in the beam there must be vertical and longitudinal shearing stresses of equal intensity. These tend to distort the material as shown in Fig. 4-1(*b*), causing a compression on the diagonal section AC and a tension on the other diagonal section DB. The latter is called "diagonal tension," and its direction is assumed to be inclined at an angle of 45° from the beam's axis unless affected by other conditions. The intensities of these diagonal compressions and

tensions on AC and DB are each equal to v_L. Before any cracking of the concrete occurs, these shearing stresses may be distributed over the entire depth of the member somewhat the same as in homogeneous beams.

Now, consider the cracked condition of the beam. Referring to Fig. 4-1(a), it is clear that, when the beam bends sufficiently, the concrete will fail in tension somewhere below the neutral axis. To illustrate this, let $n = 10$, the allowable $f_s = 20{,}000$ lb. per sq. in., and the ultimate tensile strength of the concrete $f_t = 400$ lb. per sq. in. Then, from similar triangles in Fig. 4-1(c),

$$f_t : \frac{f_s}{n} :: x : (d - kd).$$

$$400 : \frac{20{,}000}{10} :: x : (d - kd) \qquad \text{or} \qquad x = \frac{d - kd}{5}.$$

It is obvious that the concrete will crack below this limit x.

There are two kinds of tensile cracks to consider. The first ones are those which occur in regions of large bending moments but where the shear is zero. They are due to the elongation of the rods and to the concrete's inability to stretch equally. These cracks are usually normal to the axis of the beam. Figure 4-19[1] shows some of these cracks between the two loads on the beams. The second kind of cracks consists of those which are caused primarily by the diagonal tension, as explained previously.

Prior to the forming of these cracks, the diagonal tension, which is due to the shearing forces, can exist within the concrete. However, when a crack has formed, the shearing and diagonal tensile stresses cannot be transmitted across the opening. Therefore, the solid part of the beam above the crack must be the thing which prevents the failure of the member, and the resistance to the transverse shearing forces must be confined to this region.

The distribution of these stresses is questionable, but, because of the foregoing, it is not reasonable to assume that the transverse shearing stresses can be distributed over the effective depth as shown in Fig. 4-1(d). Furthermore, it is improbable that they will be distributed as shown in Fig. 4-1(e) which implies that their intensities increase as the ordinates of a parabola from the

[1] Frank E. Richart, An Investigation of Web Stresses in Reinforced Concrete Beams, Engineering Experiment Station, University of Illinois, *Bull.* 166.

top fibers to the neutral axis and that, below the distance x from this axis, they suddenly cease. However, it seems to be satisfactory to assume that, after the cracks have opened up, the transverse shearing stresses are uniformly distributed over the portion of the beam that is above the neutral axis. Then, in any cross section, they are limited to the area bkd. Therefore, upon this basis,

$$v_T = \frac{V}{bkd}$$

where v_T is the average transverse shearing stress, and V is the total transverse shear in the beam. This is not the conventional formula for shearing stresses, but its significance will be apparent as one studies the subject further.

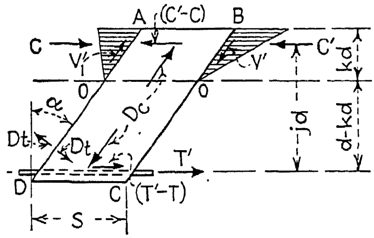


FIG. 4-2.

Next, investigate the longitudinal forces that are set up in the member when it is loaded. As the beam of Fig. 4-1(a) bends, it must develop the tension in the rods. To understand this, assume that the portion of this beam marked $ABCD$ is cut out as pictured in Fig. 4-2. Assume that the length s is short. Call the inclined shearing forces in the uncracked concrete V' and V'_1. The differences between the forces T' and T—also between C' and C—represent what might be called the "pickup" of stress due to the increase of the bending moment from D to C. Since the body is in equilibrium under the action of all of these forces, taking moments about the intersection of V'_1 and $C' - C$ gives

$$(T' - T)jd = V's \qquad \alpha). \tag{4-3}$$

However, the increase of tension in the rods must be caused by the bond stresses, the sum of which must be equal to the longitudinal shearing forces in the concrete. Furthermore, this sum must be equal to the increment of compression $C' - C$. There-

fore, it seems that there must be in the concrete a resultant diagonal compression D_c, the longitudinal component of which is $T' - T = C' - C$. Otherwise, the block $ABCD$ of Fig. 4-2 will not be in equilibrium. In other words, the pickup of stress, due to the increase of the bending moment, causes a diagonal compression in the concrete which makes the latter act as an inclined strut, the longitudinal component of this compression being equal to the increase in the tension in the rods. If the intensity of the longitudinal shear in the concrete is v_L, it is clear that

$$(T' - T) = v_L bs.$$

Substituting this value in Eq. (4-3), and assuming that $V' = \dfrac{V}{\cos \alpha}$ gives

$$v_L = \frac{V}{bjd} \tag{4-4}$$

which is the conventional and general formula for the magnitude of the shearing stresses in a reinforced-concrete beam. However, in this text it will be associated primarily with the longitudinal shearing stresses in order to avoid confusing it with Eq. (4-2). It also indicates that v_L may be considered to be distributed over the cross section of the beam somewhat as pictured in Fig. 4-1(*d*), this being in contrast to the assumed distribution of v_T. The effective depth d used in Eqs. (4-2) and (4-4) should be the dimension used in computing k and j or, when their magnitudes are assumed, d may be measured to the extreme row of rods or to the center of gravity of the group. The latter gives somewhat larger results.

What constitutes the vertical or transverse component of D_c? The fact that a reinforced-concrete beam, with longitudinal rods only, does not always fail as soon as one of these cracks appears may be due largely to the strength of the portions like *OCDO* of Fig. 4-2 acting as short cantilever blocks below the neutral surface *O-O* (locus of neutral axes). These blocks may transmit the transverse shears up into the uncracked concrete, the resultant acting as the transverse component of D_c. On the other hand, a bending moment at *O-O* may counteract the tendency of the tensile forces to break off the portion of the beam between the cracks. For the purpose of facilitating a visualization of

the general tendency, the latter action seems to be the more satisfactory.

Referring to Fig. 4-2 again, it should be noted that the diagonal compression is drawn parallel to the cracks that form in the concrete and that it acts "against," or in opposition to, the increase in the bending stresses. This must be so because the cracks naturally will not cross the lines of compression; otherwise they would be closed by the pressure. On the other hand, the elongation of the steel and the diagonal tension make them open up somewhere.

The relationship between the magnitudes of the transverse and the longitudinal shearing stresses for purposes of design, as given in Eqs. (4-2) and (4-4), is

$$v_T : v_L : : \frac{V}{bkd} : \frac{V}{bjd}. \tag{4-5}$$

$$v_T = v_L \frac{j}{k}. \tag{4-6}$$

If $j = 0.87$, and $k = 0.38$, Eq. (4-6) will give

$$v_T = \frac{0.87}{0.38} v_L = 2.29 v_L.$$

However, v_T should be looked upon as a pure or punching shear, whereas v_L is a measure of diagonal tension. The allowable unit stress for the former may be relatively high. According to Table 1-5, the Code allows a maximum shearing stress of $0.12f'_c$ upon the area bjd if the beam has properly designed web reinforcement and special anchorage of longitudinal steel. Therefore, in the foregoing example, the ultimate magnitude of v_T might equal $0.12f'_c \times 2.29 = 0.275f'_c$. However, it will be assumed to have a limiting value of $0.2f'_c$ in all cases, this being more conservative. On the other hand, the Code (Table 1-5) allows only $0.02f'_c$ for v_L in beams without special anchorage or web reinforcement and $0.03f'_c$ for v_L when the tensile steel is specially anchored.

Generally, v_T is not critical in the design of ordinary beams except when k is small or when the shears are unusually large. However, the longitudinal shear v_L, which is the measure of diagonal tension, must be investigated carefully. Of course the cracked condition of the beam is the critical one for both v_T and v_L.

The shearing stresses and diagonal tension in T-beams can be calculated in the same manner as for rectangular beams, but b', the width of the stem, must be substituted for b in the formulas. The flange portion of the T-beam should not be relied upon for resistance to shear.

Briefly reviewing the matter, notice that the real weakness of concrete in beams seems to be its lack of ductility. Its strength in punching shear is very great. However, when the steel elongates and the cracks open, the latter seem to extend farther and farther up toward the compression side of the beam until it fails. This cracking really accompanies a shifting of the neutral axis, and it causes a decrease of the real area of concrete which can resist the transverse shear and the compression. Strictly speaking, such a failure is not directly due to transverse shear alone but is the result of the longitudinal shearing forces and the elongation of the steel.

When designing a reinforced-concrete beam with longitudinal reinforcement only, and when a maximum shearing stress V/bjd is specified, a trial value can be assumed for j equal to 0.87. When this is substituted in Eq. (4-4), along with the known or trial values of v_L, V, and b, the magnitude of the trial value of the effective depth d is readily found. After the final beam is decided upon, v_L can be computed. However, great refinements in computing the magnitude of j are not justified because of the uncertainties that surround this whole question of shearing stresses. Unless the member is unusual, it is sufficiently accurate for one to assume that $j = 1 - \frac{k}{3}$, or even $j = 0.87$ or 0.88, when calculating v_L.

4-3. Direction of Cracks. Reinforced-concrete beams of various types crack in the same general manner. The cracking of some test specimens is shown in Fig. 4-19. These pictures do not present the conditions of the beams at working loads, but they show what happens shortly before and at the time of failure.

In order to understand the conditions more fully, examine the action of a beam with uniformly distributed loading as pictured in Fig. 4-3(a). The bending-moment and shear diagrams are shown in Figs. 4-3(b) and (c). The tangent of the angle θ, which a line tangent to the bending-moment diagram at A will make with the base line, gives the rate of increase of the bending

moment at A and, therefore, of the longitudinal shear and the total increase of tension in the rods at this point. At the center, this rate is zero so that the cracks must be vertical, since there is no longitudinal shear in the concrete and all of the tensile stresses act horizontally, the cracks being merely the result of the elongation of the rods.

The cracks near the ends of the beam are in the regions where the shear and the rate of increase of the bending moment are large but the horizontal tensile stresses are small. Diagonal tension will cause these cracks to be inclined at about 45° from the axis of the beam. At other points, where the shear diagram shows that the longitudinal shears and the diagonal tensions are small, the cracks may have varying slopes due to irregularities in the concrete, but they generally will be normal to the direction of the resultant of the horizontal and diagonal tensions in the beam.

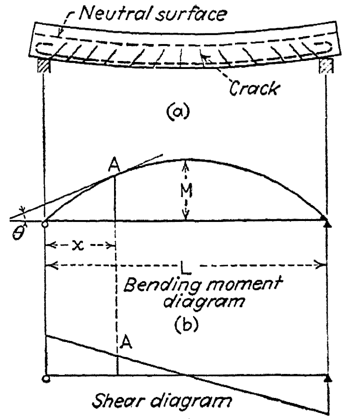


FIG. 4-3.

The spacing of the cracks pictured in Fig. 4-3(a) is dependent upon the elongation of the steel and the ability of the bond stresses to prevent the slipping of the rods, thus forcing the concrete to open up. Obviously there must be some failure of the bond at and near the cracks.

4-4. Bond Stresses on Longitudinal Rods. Since Eq. (4-4) gives the longitudinal shearing stress in the concrete of a beam, it can also be used to investigate the bond stress along the longitudinal rods. For each inch of length of the beam, the total longitudinal shearing force $F_L = v_L b = Vb/bjd$, or $F_L = V/jd$. The total area of the surface of the rods which must pick up this force for each inch of length of the beam is Σo. Then the intensity of the bond stress at any section is

$$u = \frac{F_L}{\Sigma o} = \frac{V}{(\Sigma o)jd} \text{ (not exact if rods are of mixed sizes).} \qquad (4\text{-}7)$$

This formula has already been used in Art. 3-6. When two or

more rows of steel are used, as in Fig. 4-12(g), it is satisfactory to measure d to the center of gravity of the group or to the extreme row.

Problem 4-1. Assume a simply supported slab 9 in. thick which has a span of 9 ft. and which has $\frac{5}{8}$-in. round rods 5 in. c.c. located 2 in. above the bottom. It carries a uniformly distributed live load of 578 lb. per sq. ft. If $k = 0.37$, $j = 0.88$, $n = 12$, and $f'_c = 2{,}500$ lb. per sq. in., find the maximum vertical and longitudinal shearing stresses in the slab; also find the critical bond stress.

The weight of the slab equals

$$\cdot \times 150 = 112 \text{ lb. per sq. ft.}$$

$$\text{Max. } V = \frac{wL}{2} = (112 + 578)\frac{9}{2} = 3{,}110 \text{ lb.}$$

From Eq. (4-4),

$$v_L = \frac{V}{bjd} = \frac{3{,}110}{12 \times 0.88 \times 7} = 42 \text{ lb. per sq. in.}$$

From Eq. (4-2),

$$v_T = \frac{V}{bkd} = \frac{3{,}110}{12 \times 0.37 \times 7} = 100 \text{ lb. per sq. in.}$$

The total circumferential area of the rods in 1 in. of length of the slab is

$$\Sigma o = \tfrac{12}{5} \times 1.96 = 4.7 \text{ sq. in.}$$

From Eq. (4-7), the maximum bond stress (at the end of the slab) is

$$u = \frac{V}{(\Sigma o)jd} = \frac{3{,}110}{4.7 \times 0.88 \times 7} = 107 \text{ lb. per sq in.}$$

4-5. Vertical Stirrups. If the loads acting upon a beam are great enough to exceed the safe shearing stress of the concrete when it is used with longitudinal reinforcement only, then it becomes necessary to strengthen the beam. This can be done by adding "web reinforcement" which will prevent the cracks from spreading and from causing failure of the structure.

One type of such web reinforcement is shown in Figs. 4-4(a) and (b) which represent a reinforced-concrete beam with vertical rods, called "stirrups," placed in it as illustrated by EF. These stirrups are carried under the longitudinal reinforcement at E and are anchored into the concrete near F. The presence of these vertical rods makes considerable difference in the strength of the beam.

Examine what happens as this beam bends under increased loading. Before the concrete cracks, the stirrups are subjected

to a stress of only n times that which is in the concrete beside and parallel to them. They have very little effect because of their small area compared to that of the concrete. In this case, the latter resists a large part of the longitudinal tension which is caused by bending. It also resists most of the diagonal tension which is caused by the shearing forces. However, as previously explained, the deformations accompanying the usual tensile stresses in the steel are so great that they compel the concrete

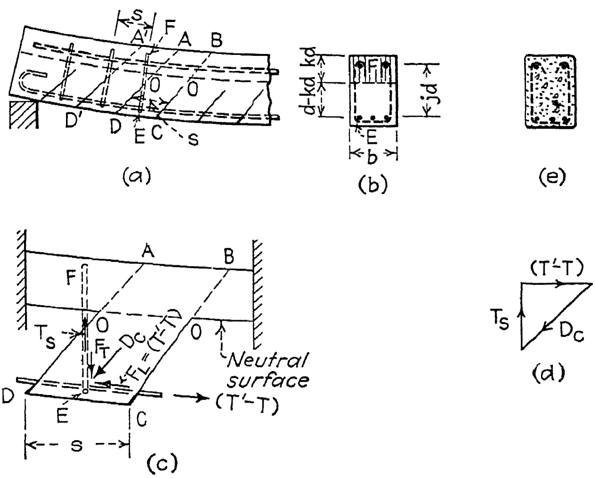


FIG. 4-4.

to open up in "hair" cracks. Then the stirrups come into action prominently, and they resist the spreading of the cracks.

Next, analyze the action of these stirrups. To do so, assume that Fig. 4-4(c) represents the piece $ABCD$ of the beam of Fig. 4-4(a). Imagine that it is cut out along with the top portion of the adjacent parts of the beam and that it is placed between fixed supports. Then assume that the increment of tension $(T' - T)$ is applied as a horizontal pull on the longitudinal rods. This pull tends to break off the portion $OCDO$ about the neutral surface O-O, or about the faces AO and BO, an action that the concrete cannot resist well. Then, if the stirrup EF is in place, it will serve as an anchor or tie to stop this rotation. Thus it becomes clear that the force $(T' - T)$ sets up a diagonal com-

pression D_c in the concrete and a tension T_s in the stirrup. The horizontal component of the diagonal compression will be called F_L, whereas its vertical component is F_T. Therefore, the stirrup EF, if acting without help from the concrete, must resist a pull T_s equal to F_T. In other words, $(T' - T)$, D_c, and T_s are three forces that are in equilibrium and that meet at a point. Therefore, their magnitudes must be proportional to the sides of a triangle, as shown in Fig. 4-4(d).

From all of this, referring to Fig. 4-4(a), it is apparent that each stirrup acts as a tie to prevent the portion or portions that it occupies from breaking off. Each stirrup must withstand a force equal to the sum of the stresses acting upon the area that affects it, or $A_v f_v = F_T = F_L$ when D_c is inclined at 45°. Therefore,

$$A_v f_v = v_L(bs) = \frac{Vs}{jd} \tag{4-8}$$

where A_v is the area of the rods composing one complete stirrup, and f_v is the intensity of the tension in the stirrup. This assumes that the concrete does not help the stirrup. However, this matter will be discussed further in the next article.

Equation (4-8) shows that the loads on the stirrups must vary with their spacing and increase with the vertical shear. Furthermore, from Fig. 4-4(c), one can see that the stirrups must not be too far apart. A practical maximum value for their spacing, according to the Code, is about $s = d/2$.

The details of one type of stirrup can be seen by examining Fig. 4-4(b). The bottom is looped under the longitudinal reinforcement so as to get a mechanical grip around it and thus obtain a better chance to develop the stirrup. The top ends are hooked but are not overlapped, thus permitting the placing of the main rods after the stirrups are set in the forms. The reason for these hooks is obvious. The ability of the concrete to develop the necessary bond along the stirrup below the neutral axis is decreased because of the cracks. The rods should be developed by bond in and near the region of compression. It seems to be safe to assume that the maximum depth of this portion that can be relied upon for bond is $0.6d$—the Code limits it to $0.5d$. Since this distance is relatively short, the hooks are needed in order to provide the necessary length of rod, which may be found from

the formula

$$T_s = A_v f_v = (\Sigma o) u L_s \tag{4-9}$$

where L_s is the length of embedment which is needed to develop the allowable tensile unit stress in the stirrup. Generally, the requirements for bond strength compel the use of rather small rods for stirrups so as to have a large ratio of surface area to cross-sectional area. It is also advantageous to use longitudinal tie rods as pictured in Fig. 4-4(*b*) so as to help develop the stirrups and to hold them in position during the placing of the concrete. If such rods are used, the stirrups may be bent as shown in Fig. 4-4(*e*), a method that may help to hold them in better line at the top.

Vertical stirrups are so simple, they can be arranged so readily in cases of varying shear, and they can be set in the forms with the other rods so easily that they constitute one of the most practical systems of web reinforcement.

The truss action of stirrups is likely to modify the transverse shearing stresses in the uncracked portion of the beam, but this change is difficult to compute with any certainty. Therefore, v_T will be assumed to be confined entirely to the uncracked concrete.

Sometimes, specifications refer to the percentage of web reinforcement in beams. This simply means the cross-sectional area of the whole stirrup divided by the area (in plan) of the portion of the beam that it reinforces. In the case of vertical stirrups,

$$p_s = \frac{A_v}{bs}.$$

4-6. Spacing of Vertical Stirrups. Stirrups should be used when the longitudinal shearing stress ($v_L = V/bjd$) exceeds $0.02f'_c$ unless the longitudinal rods are hooked or otherwise properly anchored, in which case the Code permits the safe shearing stress without stirrups to be increased to $0.03f'_c$.

Furthermore, tests reported by Prof. Richart[1] seem to indicate that the stirrups do not become stressed so highly as Eq. (4-8)

[1] Frank E. Richart, An Investigation of Web Stresses in Reinforced Concrete Beams, Engineering Experiment Station, University of Illinois, *Bull.* 166.

would indicate. This may be caused by the fact that the concrete continues to resist the shearing forces and the diagonal tension as much as it is able to do, acting as it would if no stirrups were present. However, this combined action is very uncertain.

These tests also seem to indicate that the stirrups near a load, as pictured in Fig. 4-5, have smaller stresses than those farther away. This is evidently due to local compression and to the fact that the diagonal tension does not fully affect the nearest stirrups. Furthermore, as the ends of the beam are approached, there seems to be another decrease in stirrup stresses. This is obviously due to the fact that some of the 45° pressure lines reach the support

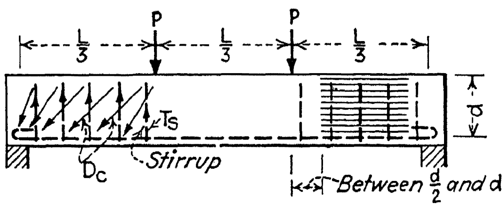


FIG. 4-5.

without fully affecting the adjacent stirrups. The left-hand portion of Fig. 4-5 is an attempt to picture this action by indicating the lines of diagonal compression in the concrete and the tension in the stirrups which appear to form the web system of a sort of Howe truss. The shaded area of the right-hand part of this beam shows the region where the stirrup stresses may be the greatest.

In designing stirrups in reinforced-concrete beams, it is customary to assume that the concrete and the stirrups will act together and that the former will resist the shearing forces up to its allowable working stress of $0.02f'_c$ (or $0.03f'_c$ for anchored reinforcement). To illustrate the procedure, let Fig. 4-6 be a diagram that shows the intensity of the longitudinal shear in one-half of a beam which carries a uniform load. (It must be remembered that the diagonal tension is the force that really makes the stirrups necessary.) The value of the maximum ordinate AC is

$$v_L = \frac{V}{bjd}.$$

If v'_L is the intensity of the longitudinal shearing stress that the

specifications permit in the concrete without stirrups, then this value can be represented by AB. Therefore, the stirrups must resist the remaining shear which is pictured as the triangle BCD. It is clear that, from the center E to the point D of Fig. 4-6, no stirrups are needed theoretically. From D to B the longitudinal shear steadily increases. Therefore, it is theoretically necessary to use stirrups of different strengths at uniform spacing or else to use stirrups of one given size with varying spacing. The

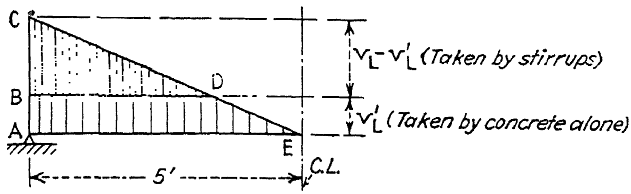


FIG. 4-6.

former method is not generally advisable. To apply the latter plan, assume a size of rod, and find $A_v f_v$. Then from Eq. (4-8),

$$- v'_L)bs \qquad (4\text{-}10)$$

where $(v_L - v'_L)$ is the *excess* longitudinal shearing stress for which the stirrups are to be proportioned.

The advisability of an arbitrary allocation of a certain part of the shear to the concrete itself seems to be questionable, because the number and the size of the cracks under a small loading probably change considerably under the heavy loads that make the stirrups necessary. However, it is wasteful to disregard the concrete entirely.

Any beam that carries relatively large moving live loads, or one that is subjected to a reversal of stress, should have stirrups throughout its length, their spacing being not over $\frac{1}{2}d$. In extreme cases, the stirrups may be designed for the total longitudinal shear (or diagonal tension), but the permissible unit stress in them may be increased to about 25,000 to 30,000 lb. per sq. in. if the shears are not too high, thus allowing something arbitrarily for the strength of the concrete.

The Code states that, if v_L exceeds $0.06f'_c$, the web reinforcement should be designed to withstand the entire shear. It is conservative to use a limiting stress of 16,000 lb. per sq. in. in this rein-

forcement for such heavily loaded members, but the higher permissible unit stresses recommended herein seem to be justified for ordinary construction. The Code also states that, when v_L exceeds $0.06f'_c$, vertical stirrups should not be used alone but should be combined with bent-up rods or inclined stirrups. The former combination is preferable. In such cases, the maximum spacing of the stirrups should not exceed about $\frac{1}{4}d$; the bent-up rods should be spaced at about $\frac{3}{8}d$.

Some engineers assume that the minimum shearing stress in any beam is from 25 to 50 per cent of the maximum stress. In such a case, the diagram in Fig. 4-6 would be a trapezoid, and the length in which stirrups are required would be extended.

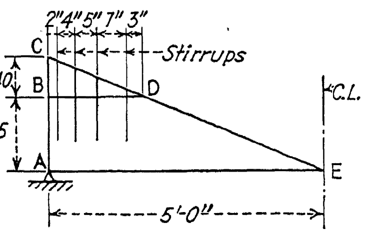


FIG. 4-7.

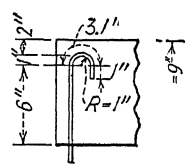


FIG. 4-8.

Problem 4-2. Design the vertical stirrups for a simply supported beam 10 ft. long with a uniformly distributed total load equal to 3,000 lb. per lin. ft. Assume $b = 10$ in., $d = 15$ in., $k = 0.38$, $j = 0.87$, and $f'_c = 2{,}500$ lb. per sq. in. Also assume that the allowable working stresses are $f_v = 18{,}000$ b. per sq. in., $u = 125$ lb. per sq. in., and $v'_L = 75$ lb. per sq. in.

$$V = \frac{wL}{2} = 3{,}000 \times 5 = 15{,}000 \text{ lb.}$$

$$\frac{V}{bjd} = \frac{15{,}000}{10 \times 0.87 \times 15} = 115 \text{ lb. per sq. in.}$$

Then, referring to Fig. 4-7, $BC = 115 - 75 = 40$ lb. per sq. in.

To find the point D where the stirrups must start, utilize the similar triangles CDB and CEA. Then

$$CB:CA::BD:AE.$$

$$BD \quad \frac{40 \times 60}{115} \quad 20.9 \text{ in., say 21 in.}$$

Assume U-shaped stirrups composed of $\frac{1}{4}$-in. round rods. From Eq. (4-10), $A_v f_v = (v_L - v'_L)bs$, or

$$2 \times 0.05 \times 18{,}000 = 40 \times 10 \times s.$$

Therefore $s = 4.5$ in. at the support. Halfway along BD, this spacing will be twice as great, or 9 in., because $v_L - v'_L = 20$ lb. per sq. in. Remember-

ing that the stirrups should not be farther apart than $\frac{1}{2}d$, they can be arbitrarily spaced as shown in Fig. 4-7, but, to secure the simplicity that is desired in ordinary practice, it would be better to use five stirrups uniformly spaced at 4 or $4\frac{1}{2}$ in. c.c. Notice that one can either assume a desired size of rod to use for stirrups and then find the required spacing or choose a spacing and solve for the necessary size of material to be used.

The length of embedment of the stirrups must be, from Eq. (4-9),

$$0.05 \times 18{,}000 = 0.78 \times 125 \times L_s \qquad \text{or} \qquad L_s = 9.2 \text{ in.}$$

If the stirrup is made as shown in Fig. 4-8, the available length for bond L_s is 10.1 in.

Compare the foregoing results with Fig. 12 and Table 7 in the Appendix.

The transverse shearing stress is, from Eq. (4-2),

$$v_T = \frac{V}{bkd} = \frac{15{,}000}{10 \times 0.38 \times 15} = 263 \text{ lb. per sq. in.}$$

This is less than $0.2f_c' = 0.2 \times 2{,}500 = 500$ lb. per sq. in., the arbitrary limit.

In case stirrups shaped like the letter W are used, in order to have four rods that will be effective in the stirrup, then the two center parts will pull against each other. However, the radius of the bend at the top of the central loop must be adequate. The hooks of the outer rods must be just the same as for a U-shaped stirrup. A better arrangement of stirrups for heavy beams is the use of two U-shaped stirrups placed in the same general cross-sectional plane with the two inner sides overlapping by several inches, thus providing four effective rods for A_v. This is helpful because it is not wise to use stirrups of more than $\frac{5}{8}$- or $\frac{3}{4}$-in. diameter because they are difficult to bend, the hooks become large, and the anchorage may be difficult to secure.

4-7. Inclined Stirrups. Another type of web reinforcement is that pictured in Fig. 4-9(a), which shows a reinforced-concrete beam with the stirrups placed in an inclined position. Let Fig. 4-9(b) represent the portion $ABCD$ when it is cut out and fixed in position as shown. Obviously, the stirrups serve the same function as the vertical ones previously explained. The increment of tension in the longitudinal reinforcement is again equal to $(T' - T)$. This tensile force, the diagonal compression, and the tension in the stirrup constitute a system of three forces which are in equilibrium and which meet at a point. Their magnitudes must be proportional to the sides of a triangle, as pictured in Fig. 4-9(c).

The triangle of forces in this case, assuming no help from the concrete, can be constructed by laying off the longitudinal force $(T' - T) = v_L bs$, drawing a line at 45° from one end parallel to the assumed diagonal compression, and then drawing the closing line from the other end, making it parallel to the stirrups.

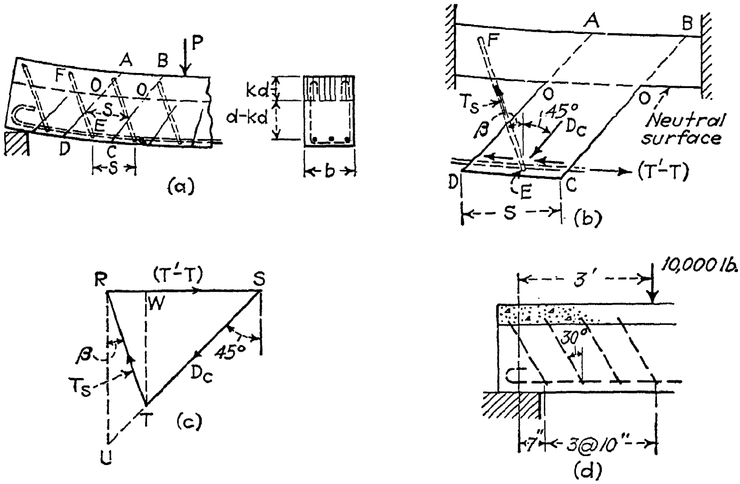


FIG. 4-9.

Examining Figs. 4-9(*b*) and (*c*) discloses some interesting facts. The stirrup *EF* withstands part of the horizontal force $(T' - T)$ as shown by *RW*. If *RU* represents the stress in a vertical stirrup, then the stress in the inclined stirrup is *RT*, which is less than *RU*. The diagonal compression can be assumed to be at an inclination of 45° from the vertical. If the stirrups are also placed so that the angle β equals 45°, then the diagonal compression *TS* is 0.5*US*, or one half of what it would be with vertical stirrups. The tension in the stirrup *RT* is 0.7*RU*, which means that it is seven-tenths of what it would be if the stirrups were vertical. The value of *RT* for any value of the angle β can be found graphically or by using the following formula:

$$\frac{RT}{RS} = \frac{\sin 45°}{\sin (45° + \beta)}$$

$$T_s = A_i f_i = \frac{0.7(T' - T)}{\sin (45° + \beta)} = \frac{0.7 v_L bs}{\sin (45° + \beta)} \tag{4-11}$$

$$A_i f_i = \frac{0.7Vs}{jd \sin (45° + \beta)} \tag{4-12}$$

where A_i = the area of the inclined stirrup, and f_i = the intensity of stress in it.

Allowing the concrete to take its share of the load, and remembering that v'_L represents the assumed safe longitudinal shearing stress in the concrete and V_c is the safe vertical shear which it can withstand, then the preceding formulas become

$$A_i f_i = \frac{(v_L - v'_L)}{\sin (45° + \beta)}, \tag{4-13}$$

$$A_i f_i = \frac{\quad}{jd \sin (45° +} \tag{4-14}$$

Equations (4-11) and (4-13) will be used in this text in preference to Eqs. (4-12) and (4-14) because the former are expressed directly in terms of the longitudinal shearing stresses.

Although the inclined stirrups are more efficient than the vertical ones, there are practical considerations which offset this advantage. The stirrups should be mechanically fastened by welding them or otherwise connecting them to the longitudinal reinforcement in order to be sure that bond failure will not cause them to slip along the main rods. This work is expensive and troublesome. The stirrups must be held in place more firmly to prevent displacement of them during the depositing of the concrete. Furthermore, near the ends of beams, special short stirrups may have to be provided in order to get them in place at all.

4-8. Spacing of Inclined Stirrups. The spacing of inclined stirrups may be determined in the same general way as for vertical ones, using Eq. (4-13) instead of Eq. (4-10). Advantage may be taken of the increased length of embedment due to the slope of the stirrups when considering the length that is required to develop them through bond.

The maximum horizontal spacing of the inclined stirrups may be increased over that for vertical ones. A satisfactory limit may be set at $\frac{1}{2}d \sec \beta$, with a maximum of $\frac{3}{4}d$.

Problem 4-3. Design U-shaped stirrups inclined at 30° from the vertical for the simply supported T-beam shown in Fig. 4-12(*g*). It has a span of 15 ft., a uniform load of 800 lb. per lin. ft. including its own weight, also

concentrated loads of 10,000 lb. located 3 ft. and 5 ft. from each end as pictured in Fig. 4-12(a). Assume $n = 12$ and the allowable f_s, f_c, u, and $v'_L = 15{,}000$, 900, 125, and 75 lb. per sq. in., respectively.

By the methods of Chap. 2, k is found to be 0.28. Then j equals 0.91 (approximately).

The end shear equals $V = 800 \times 7.5 + 10{,}000 + 10{,}000 = 26{,}000$ lb. Therefore,

$$v_L = \frac{V}{b'jd} = \frac{26{,}000}{12 \times 0.91 \times 22} = 108 \text{ lb. per sq. in.}$$

Under the first concentrated load, 3 ft. from the support,

$$v \quad \frac{26{,}000 - 2{,}400 - 10{,}000}{12 \times 0.91 \times 22} = 57 \text{ lb. per sq. in.}$$

Thus, stirrups are needed only in the part of the beam from the end to this first concentration.

Assume $\frac{3}{8}$-in. round rods for the stirrups. Then, from Eq. (4-13),

$$2 \times 0.11 \times 15{,}000 = \frac{0.7(108 -}{\sin (45^\circ}$$
$$s = 11.5 \text{ in.}$$

To place these stirrups, start with the bottom of the first one about 7 in. from the support as shown in Fig. 4-9(d). Then add three more at 10-in. spacing horizontally. This will bring the last one a little beyond the first concentration, which is ample. The difficulty of getting the first stirrups in place is obvious—unless the support is rather wide. Admittedly, this arrangement is chosen arbitrarily, but such decisions are very common in the design of reinforced concrete.

The necessary length to develop the required bond at the top of the stirrups is

$$L_s = \frac{A_i f_i}{ou} = \frac{0.11 \times 15{,}000}{1.18 \times 125} = 11.2 \text{ in.}$$

4-9. Bent-up Rods. Inasmuch as the longitudinal reinforcement in a beam is determined by the maximum bending moment, it is obvious that all the rods are not needed where the bending moment is smaller. Furthermore, elimination of part of the rods will compel the remaining ones to withstand the tension at any given section if they can do so safely and if the bond is sufficient to transfer the stress to them. This fact can be utilized to provide web reinforcement in the beam by bending the rods upward and anchoring them into the concrete. Thus they will perform the function of stirrups.

To illustrate this stirrup action, let Figs. 4-10(a) and (b) represent a beam in which the longitudinal rods are bent up—

two at a time. By cutting out the portion $ABCD$ as shown in Fig. 4-10(c), it is obvious that the bent-up rods serve the same function as the inclined stirrups which were considered in Art. 4-7, as far as their resistance to the breaking of the piece $OCDO$ is concerned.

However, these rods differ from the stirrups in a very important respect. When the rods EF are bent up at E, they already have a certain tensile load T_L represented by SX in Fig. 4-10(d),

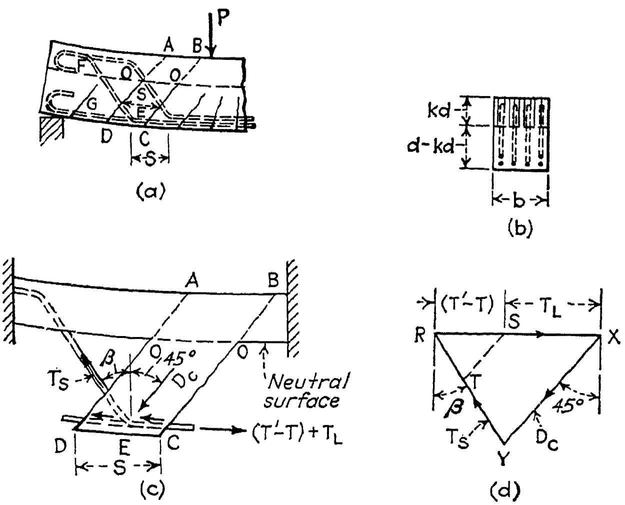


FIG. 4-10.

because they are continuous and the shearing strength of the concrete, together with the bond stresses, will compel all of the rods to have practically the same deformation before they are bent and, therefore, practically the same unit stress.

T_L may be computed approximately by dividing the bending moment M' at E by jd. It is sufficient to assume that $j = 0.87$ in all cases regardless of the number of rods remaining in the bottom of the beam if its actual magnitude is not known, because of the many unknown and secondary stresses at this point. Furthermore, T_L for the bent-up rods must be the same proportion of the total tension in all of the rods that the number of bent-up bars at E is to the total number of rods at E. For instance, if there are six rods, two of which are bent up, then

T_L for these rods must equal one-third of the total tension in the entire set of rods.

This initial longitudinal tension should be combined with the stirrup action of the bent-up bars. Referring to Fig. 4-10(d), the increment of tension $(T' - T)$ in the space CD is represented by RS; T_L, by SX. These must be added together. Again, there are three forces which are in equilibrium, which meet at a point, and which must be proportional to the sides of a triangle. Therefore, drawing XY at a slope of 45°, parallel to the assumed diagonal compression, and drawing RY parallel to the bent-up portion of the rods, $XY = D_c$, and $RY = T_s$. Therefore,

$$T_s = A_i f_i = \frac{0.7[(T' - T) + T_L]}{\sin (45° + \beta)}. \tag{4-15}$$

If N_b is the number[1] of rods that are bent up at E, and N_t is the total number of rods at E, then

$$T_L = \frac{M'N_b}{jdN_t}. \tag{4-16}$$

Combining these values, Eq. (4-15) becomes

$$T_s = A_i f_i = \frac{0.7\left[v_L bs + \dfrac{M'N_b}{jdN_t}\right]}{\sin (45° + \beta)} \tag{4-17}$$

or, allowing for the longitudinal shear which can be taken by the concrete alone,

$$A_i f_i = \frac{0.7\left[(v_L - v'_L)bs + \dfrac{M'N_b}{jdN_t}\right]}{\sin (45° + \beta)}. \tag{4-18}$$

It must be remembered that T_s is the total stress in all of the rods that are bent up at the given point. The stress thus computed is that which exists at or just above the bend at E. Naturally, through bond, this stress is gradually reduced and is transferred to the concrete.

The second term in the numerator of Eq. (4-18) often represents a very real force, and it should not be overlooked. The value of M' to be used in it can be scaled from the bending-moment diagram, or its magnitude can be quickly approximated by calculation.

[1] Use relative areas if rods are not all the same size.

Bent-up rods are usually relatively large in size. Therefore the bond stresses are likely to be excessive. In no case can such a rod pick up more longitudinal shear than the bond strength between faces *OC* and *OD* will permit. It is easy to see how large the local shearing and bond stresses really may be near such bends. The tensile stress caused by T_s is seldom excessive, but this does not mean that it should not be investigated.

There are some practical considerations to bear in mind. The diagonal compression is not serious in the main portion of the concrete, but there is a high local pressure next to the bend in the rods. The radius used at this point should be large enough so that the bearing of the curved portion of the rod will not crush the concrete. Care must also be used to see that the bending of the rods at any point does not overstress the remaining longitudinal reinforcement which must withstand the bending moment in the beam. Another point of possible weakness is in the region marked *G* in Fig. 4-10(*a*) where there may be no bent-up rods but stirrups may have to be added. Furthermore, there are frequently insufficient rods to permit an arrangement that will reinforce all of the required portion of the beam against failure by diagonal tension because they should be bent up in a symmetrical arrangement—generally in pairs. Furthermore, if heavy, square rods are bent up, they often tend to twist in a way that makes the bending more difficult to do.

Two advantages of using bent-up rods for web reinforcement are the anchoring of the longitudinal rods themselves and the shifting of the rods from the bottom of the beam to the top where they may be needed to resist negative bending moments, as for continuous beams and frames. Rods *b* and *c* in Fig. 4-11(*c*) may almost be looked upon as cradled between anchorages at both ends of the beam.

From the standpoint of the tension in the bent-up steel due to the bending moment, it is best to avoid bending the rods at too great an angle because this causes very large local concentrations of stress at the bend. Ordinarily, a slope of 45 to 60° from the vertical is satisfactory.

4-10. Spacing of Bent-up Rods. The bending-moment diagram is the first thing to investigate in order to determine points at which longitudinal rods may be bent up to act as web reinforcement. For instance, let Fig. 4-11(*a*) represent the bending-

moment diagram for a simply supported beam when it has a concentrated load applied at D; let Sketch (d) be a similar diagram for uniform loading. Assume that the maximum moments are equal in both cases and that there are six rods in the bottom of the beam, as pictured at the left of Sketch (c).

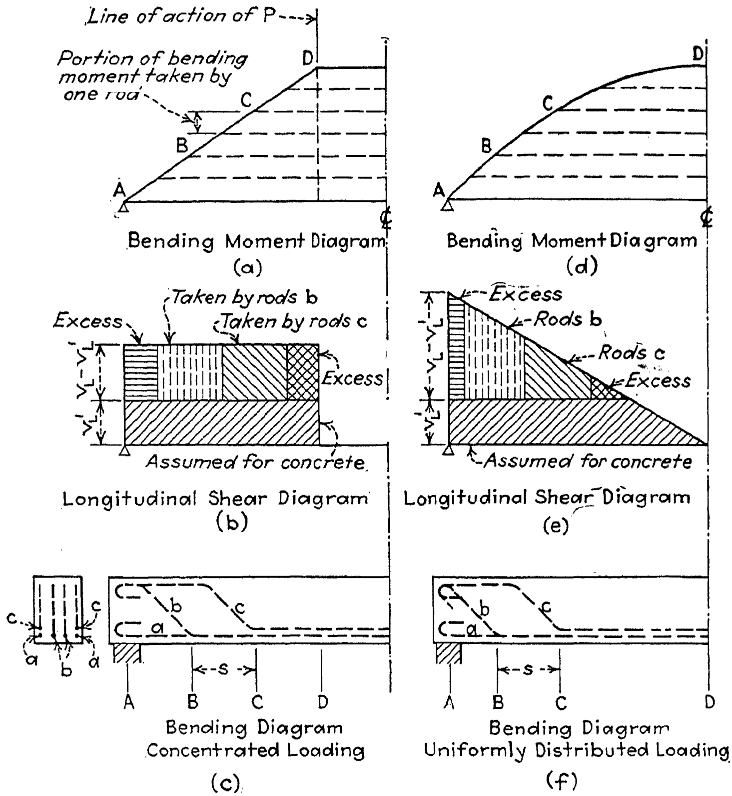


FIG. 4-11.

The shear diagram is the next thing to investigate. Therefore, compute the intensity of the longitudinal shear (the measure of the diagonal tension) from Eq. (4-4), and plot the diagrams for both loading conditions as in Sketches (b) and (e). If the concrete alone is relied upon to withstand a stress of v'_L, the remainder of the shearing stresses must be resisted by the bent-up rods.

The rest of the procedure is as follows:

1. Assume that the rods can be bent up at 45° and that these rods are to be bent up in pairs, for symmetry.
2. Assume that each rod withstands the same proportion of the bending moment, as shown by the horizontal dotted lines in (*a*) and (*d*).
3. The first two rods *c* should be bent up at or to the left of the point C where they are no longer needed to resist the bending moment.
4. Project from point C downward to the other figure to locate the permissible points for the first bends.
5. Project down from B in a similar manner to find the points for the bending of the second pair of rods.

The distances between bends should not exceed about $\frac{1}{2}d$ times the secant of the angle of inclination of the bent rod measured from the vertical—with an arbitrary maximum limit of $\frac{3}{4}d$—unless other web reinforcement is used or unless the rods are not needed as web reinforcement at all but are bent up in order to have them available for resisting negative bending moments.

Some of the difficulties that accompany bent-up rods are shown in Fig. 4-11. The parts of the longitudinal shear diagrams that are labeled "excess" have no rods in this case to reinforce them; the spacing s exceeds $\frac{3}{4}d$ in the pictures, indicating that the bends must be closer together; at least a few stirrups must be added to strengthen the member.

Problem 4-4. Bend up at 45°, for web reinforcement, the rods of the beam that is shown in Fig. 4-12(*g*) if the condition of loading is that which is pictured in Sketch (*a*). Assume the following allowable stresses: f_s = 20,000 lb. per sq. in., u = 125 lb. per sq. in., f'_c = 2,500 lb. per sq. in., and $v'_L = 0.03f'_c$ = 75 lb. per sq. in. Let $n = 12$, $k = 0.28$, $j = 0.91$, and d = 22 in.

Most of the parts of Fig. 4-12 are self-explanatory. However, in (*c*), the bending moment is assumed to be resisted equally by all six rods, although the three upper ones are less effective than the lower row. The left portion of (*c*) is reproduced in (*e*) in order to represent it to a larger scale.

In planning this work, examine Figs. 4-12(*f*) and (*g*). Assume that rod *a* will be bent up first, then *b*, and finally both rods *c*—the last being bent up together for symmetry. Rods *d* should extend the full length of the beam as shown. The maximum spacing of the bends is $0.5d$ (sec 45°) = 15.5 in., but, in order to distribute them well, they will be spaced as shown in Fig.

By scaling from Fig. 4-12(*e*), M' at A = 820,000 in.-lb. Similarly, from Fig. 4-12(*d*), $(v_L - v'_L)$ = 24 lb. per sq. in. Substituting these values in Eq. (4-18) gives

$$0.79 f_i \qquad \frac{0.7\left[\,24 \times 12 \times 11 \qquad \dfrac{820{,}000 \times 1}{0.91 \times 22 \times 6}\right]}{\sin (45^\circ + 45^\circ)}$$

$$f_i = \frac{0.7(3{,}170 + 6{,}820)}{0.79} = 8{,}850 \text{ lb. per sq. in.}$$

This is the unit stress in rod a. If j had not been computed previously, a value of 0.87 would have been satisfactory.

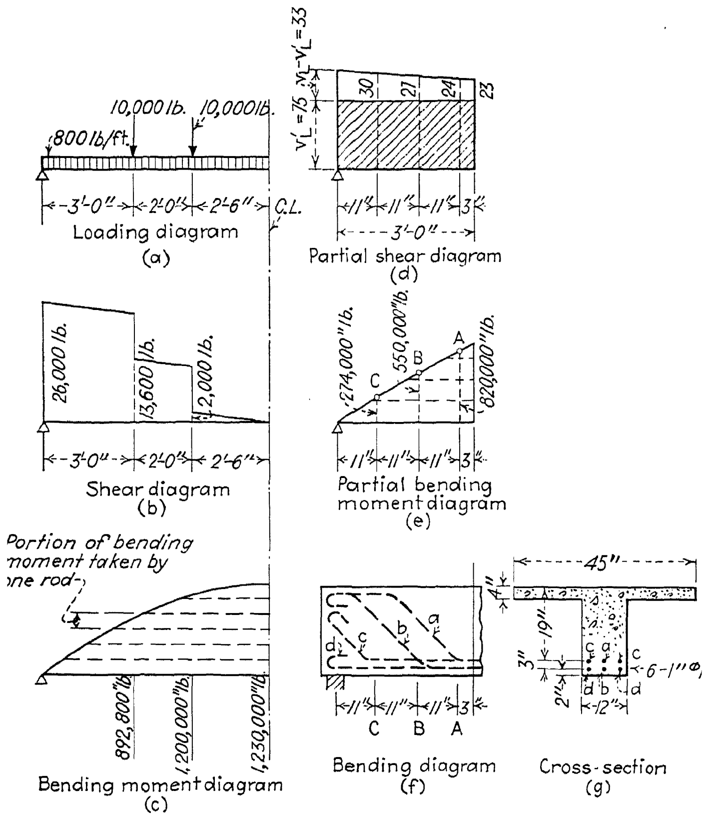


FIG. 4-12.

The first term of the foregoing equation shows that rod a must develop $0.7(v_L - v_L')b \div \sin 90^\circ = 0.7 \times 24 \times 12 = 202$ lb. per lin. in. if it is to pick up the force required by the longitudinal shear. The resulting bond stress is

$$u = \frac{202}{3.14} = 64 \text{ lb. per sq. in.}$$

The length of embedment in the upper part of this beam which is required to develop this rod is

$$L_s = \frac{A_i f_i}{\Sigma o)u} = \frac{0.79 \times 8{,}850}{3.14 \times 125} = 17.8 \text{ in}$$

This length must be provided above a plane about $0.5d$ from the top of the beam.

Similarly, for rod b,

$$f_i = \frac{0.7\left[27 \times 12 \times 11 + \dfrac{550{,}000 \times 1}{0.91 \times 22 \times 5}\right]}{0.79} = \frac{0.7(3{,}560 + 5{,}500)}{0.79} =$$

8,030 lb. per sq. in.

Then, for rods c,

$$f_i = \frac{0.7\left[30 \times 12 \times 11 + \dfrac{274{,}000 \times 2}{0.91 \times 22 \times 4}\right]}{2 \times 0.79} = \frac{0.7(3{,}960 + 6{,}840)}{2 \times 0.79}$$

4,780 lb. per sq. in.

It is not ordinarily necessary to correct the value of j for each of these calculations, but it can be done when it seems to be justified. However, the margin of safety that it is advisable to provide in concrete structures will usually take care of such inaccuracies.

Investigation of the transverse shearing stress in the concrete shows that

$$v_T = \frac{V}{bkd} = \frac{26{,}000}{12 \times 0.28 \times 22} = 350 \text{ lb. per sq. in.}$$

$$(\text{max. allowable} = 0.2f'_c = 500 \text{ lb. per sq. in.})$$

To find the approximate compression in the concrete under the bend of the rod, assuming the diameter inside the bend to be $8d$, use Eq. (3-3). Call the tension in the rod equal to that previously found, viz., 8,850 lb. per sq. in. Then

$$T = A_s f_s = pr.$$

$$0.79 \times 8{,}850 = p \times 4.5 \times 1, \qquad \text{or} \qquad p = 1{,}550 \text{ lb. per lin. in.}$$

Therefore, $f_c = p/d = 1{,}550/1 = 1{,}550$ lb. per sq. in. This is high but it can be accepted because it is localized.

4-11. Web Reinforcement at Supports of Continuous Beams.

A large portion of reinforced-concrete construction utilizes the advantage of continuity or restraint of members in order to decrease the maximum bending moments. The shears—both transverse and longitudinal—are affected by these conditions to a much less extent than are the bending moments.

As an illustration of the effect of continuity and restraint, let Fig. 4-13(a) picture a simply supported beam with a concentrated load at its center. The bending-moment and shear diagrams are self-explanatory. Then let the ends of the beam be fixed as in Fig. 4-13(b). The bending-moment and shear diagrams for this new condition are again easily understood. The maximum bending moment in the second case is only one-half that of the first one; also, the distribution of the bending moment is different.

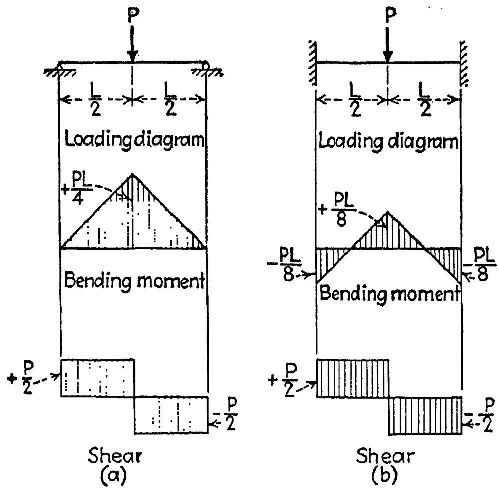


FIG. 4-13.

The rate of increase of bending moment in both cases is $\frac{PL}{4} \div \frac{L}{2}$. This is a constant. This fact is shown in the shear diagrams, which are also identical. Therefore, the required strength of the beam against shearing forces remains unchanged, although the ends are fixed in the second case. This means that the relative importance of the shear is greater than it is in a simply supported beam.

Now, examine the conditions at a point of support where the bending moment is negative. Figure 4-14(a) is an exaggerated picture of such a case, with inclined stirrups. The elongation of the longitudinal reinforcement causes cracking of the concrete in the same manner as for simply supported beams except that, in this case, the top fibers are elongated. If the piece $ABCD$

is cut out and gripped in rigid supports, as shown in Fig. 4-14(*b*), a brief examination of it shows that it is the same as Fig. 4-9(*b*) if the latter is rotated 180°. The tension in the top longitudinal reinforcement increases from D to the support S. Therefore, the increment of stress which is picked up by the piece $OCDO$ must act toward the left. After computing $(T' - T) = v_L bs$, the diagonal compression and the stirrup tension can be found as before by constructing the force triangle of Fig. 4-14(*c*).

Therefore, vertical stirrups, inclined stirrups, and bent-up rods serve the same purpose whether or not they are in simply

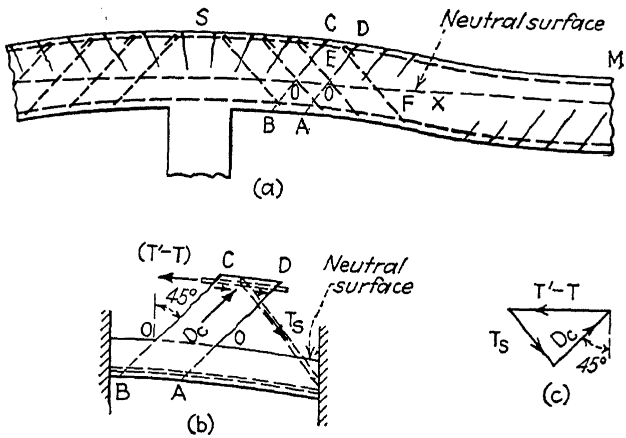


FIG. 4-14.

supported, continuous, or fixed-end beams. All of them can be designed by the same general method because the fundamental force to be considered is the longitudinal shear (or the diagonal tension) in the concrete. The diagrams showing the intensity of the longitudinal shear and the bending moment should be constructed; the excess of the longitudinal shear above that which is permissible in concrete beams without web reinforcement should be found; the tension in the longitudinal rods should be calculated when bent-up rods are to be used; then the size, spacing, and details of the stirrups and the bent-up rods should be determined just as they were in the previous cases.

It is important to study the right-hand portion of Fig. 4-14(*a*). At the support S, the neutral axis is usually below the middle of the depth of the beam because the tension is at the top. At

the center of the span M, the neutral axis is above the middle of the depth. Somewhere in the region of the point of contraflexure X, the neutral axis shifts (probably rather gradually) from the lower to the higher position. This indicates that the neutral surface is not a plane. Furthermore, the transverse shear is resisted largely as punching shear by the lower portion of the beam from the support S to about X, by the whole beam near X where the tensile forces are too small to crack the concrete, and by the upper portion of the beam from the vicinity of X to the center M.

Problem 4-5. The beam shown in Fig. 4-15(a) is continuous; it has a span of 20 ft.; it carries a uniformly distributed load of 2,400 lb. per lin. ft.;

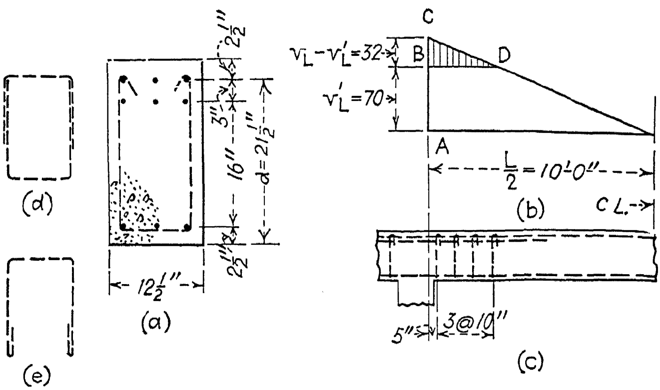


FIG. 4-15.

its reinforcement at the support is pictured in (a); $n = 12$; $f'_c = 2{,}500$ lb. per sq. in.; and the allowable f_v, u, and $v'_L = 18{,}000$, 120 and 70 lb. per sq. in., respectively. Design the required vertical stirrups for this beam, using U-shaped rods.

The value of $k = 0.37$ and $j = 0.88$.

$$v_L = \frac{V}{bjd} = \frac{2{,}400 \times 10}{12.5 \times 0.88 \times 21.5} = 102 \text{ lb. per sq. in.}$$

$$v_L - v'_L = 102 - 70 = 32 \text{ lb. per sq. in.}$$

BD in Sketch (b) $\dfrac{10 \times 12 \times 32}{102} = 37.6$ in. Call it 38 in. Assume ⅜-in. round rods for the stirrups. Then, from Eq. (4-10),

$$0.11 \times 2 \times 18{,}000 = 32 \times 12.5 \times s, \qquad \text{or} \qquad s = 9.9 \text{ in. at } A.$$

Use a spacing of 10 in. with the arrangement that is shown in Fig. 4-15(c).

A check calculation for the transverse shearing stress gives, from Eq. (4-2),

$$v_T = \frac{2{,}400 \times 10}{12.5 \times 0.37 \times 21.5} = 242 \text{ lb. per sq. in}$$

The length of embedment required for the stirrups is

$$L = \frac{A_v f_v}{\Sigma o u} = \frac{0.11 \times 18{,}000}{1.18 \times 120} \quad 14 \text{ in.}$$

Difficulties are encountered in arranging the details of stirrups at points of negative bending moments in rectangular beams. A stirrup like that in Fig. 4-15(*a*) is not anchored thoroughly across the tension side of the beam; when it is inverted, it is difficult to get the hooks under the longitudinal rods if the stirrups are erected last or to place the main reinforcement if the stirrups are in position first. Sometimes two U-shaped rods without hooks are arranged as shown in Fig. 4-15(*d*), but the laps must be long enough to develop the stirrups, and it is wise to have as much of the lap as possible in the compression zone of the beam. Still another type is that which is pictured in Fig. 4-15(*e*). The hooks are parallel to the main reinforcement, permitting the stirrups to be slid down after the longitudinal rods are in place. However, the anchorage of these stirrups is poor because the entire hook is near the surface of the beam, a fact that may cause spalling of the concrete; the compressive reinforcement is not tied in, a precaution that should be taken in heavy members if the compressive stresses in the concrete at the support are large.

The stirrups in the stems of T-beams may be detailed as shown in Fig. 4-15(*a*) because the concrete of the slab, and its "negative" reinforcement which crosses the top of the stem, will support the latter adequately.

4-12. Combination of Stirrups and Bent-up Rods. As stated in Art. 4-10, and as shown in Fig. 4-11, it is often difficult to arrange the details of bent-up rods so that they will reinforce properly all parts of the web. The deficiency can be made up by adding stirrups in the otherwise unreinforced portions. It is not necessary to explain this in detail because both types of web reinforcement act in accordance with the same general principles. However, good judgment must be exercised in combining them.

Theoretically it seems that inclined stirrups are to be preferred when the number of bent-up rods is large in order to have the tensile forces in the web reinforcement acting in the same direction. However, when used in combination with bent-up rods, they are very troublesome in the field if they are properly fastened to the longitudinal reinforcement. Therefore, vertical stirrups are more practical, but they will generally overlap some of the inclined portions of the main rods. Also, when only a few bars are bent up, it seems to be advisable to reinforce the web with vertical stirrups alone, neglecting the bent-up rods and letting their strength add to the factor of safety of the beam. Another advantage of this last arrangement is that, in continuous beams, the rods can be bent to meet the requirements of the bending moments rather than those of the longitudinal shears.

Problem 4-6. Assume a continuous T-beam with two equal spans and loaded as shown in Fig. 4-16(*a*). In working out the problem, assume that the allowable unit stresses f_s, u, and v_L' are 18,000, 120, and 70 lb. per sq. in., respectively. Let $f_c' = 2{,}400$ lb. per sq. in. The vertical shear and bending-moment diagrams are given in Figs. 4-16(*b*) and (*c*). The effective sections and reinforcement at C and B are shown in Figs. 4-16(*d*) and (*e*), respectively. For the former, $k = 0.21$, and $j = 0.93$. For the latter, $k = 0.43$, and $j = 0.86$ (approx.). However, as average or approximate values for all points in the beam, use $j = 0.87$ throughout the problem. Bend the rods up at an angle of 60° from the vertical.

At A,

$$v_L = \frac{V}{bjd} = \frac{12{,}400}{10 \times 0.87 \times 18} = 79 \text{ lb. per sq. in. } (d \text{ measured from top}).$$

At B,

$$v_L = \frac{20{,}600}{10 \times 0.87 \times 18} = 132 \text{ lb. per sq. in. } (d \text{ measured from bottom}).$$

Figure 4-16(*f*) shows, by the hatched portions, the part of the longitudinal shear or diagonal tension that must be taken by the web reinforcement.

The critical points for bending of the rods can be scaled from Fig. 4-16(*g*). The bending moment at any point can also be scaled from this diagram.

The maximum allowed spacing of the bent-up rods is $0.75d = 0.75 \times 18 = 13.5$ in. Assume $s = 12$ in. Bend rod d first, starting arbitrarily a little beyond the point of permissible bending (on the side of safety). Then bend rods e and c as shown in Fig. 4-16(*h*). Since rod d is obviously the one that is most heavily stressed near B, it is the only one that needs to be analyzed. Then, scaling the values $v_L - v_L'$ and M' from sketches (*f*) and (*g*), Eq. (4-18) gives

$$0.6 \times f_i = \frac{0.7\left[49 \times 10 \times 12 + \dfrac{522{,}000 \times 1}{0.87 \times 18 \times 6}\right]}{\sin\,(75°)}$$

$$f_i = \frac{0.7(5{,}880 + 5{,}560)}{0.6 \times 0.97} \quad 13{,}800 \text{ lb. per sq. in.}$$

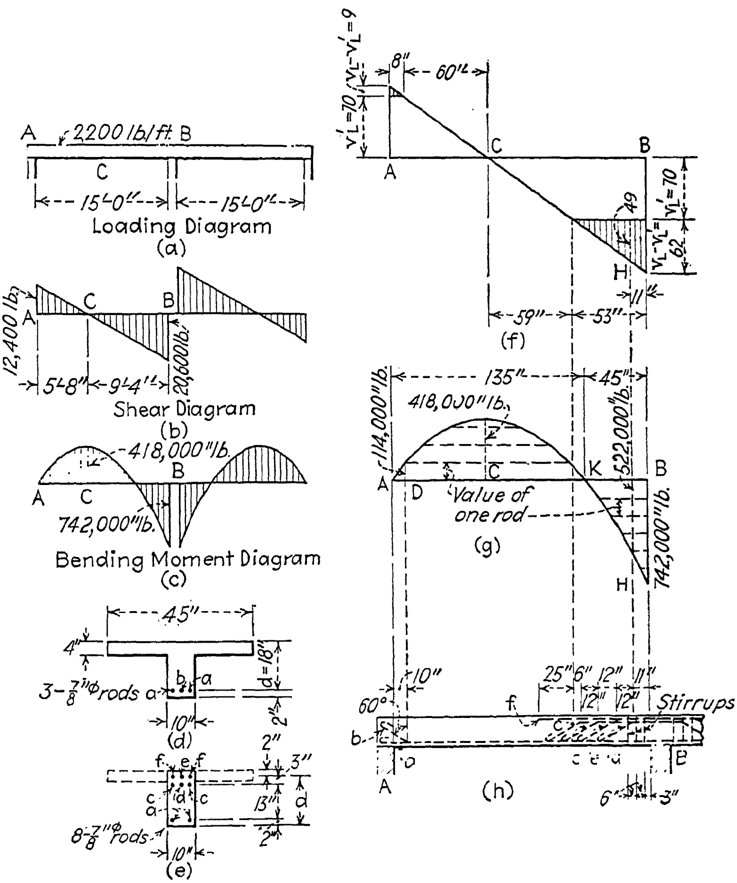


FIG. 4-16.

The first term of the foregoing equation shows that the pickup of stress that must be provided per inch of beam at the point H equals $\frac{(v_L - v'_L)b \times 0.7}{\sin 75°} = 49 \times 10 \times 0.7 \div 0.97 = 354$ lb. The required value of the bond per inch of the length of the rod is $354 \div \Sigma o = 354 \div 2.75$

= 129 lb. per sq. in. This is near enough to the allowable value to permit its acceptance. Bend the other rods as shown, including *b*.

To reinforce the remainder of the web not cared for by the rods, add $\frac{3}{8}$-in. round vertical U-shaped stirrups. Then, from Eq. (4-10),

$$s = \frac{2 \times 0.11 \times 18{,}000}{62 \times 10} = 6.4 \text{ in.}$$

For safety, use three of these stirrups as shown in Fig. 4-16(*h*).

The greatest transverse shear is at *B*. Its magnitude is

$$v_T = \frac{V}{bkd} = \frac{20{,}600}{10 \times 0.43 \times 18} = 266 \text{ lb. per sq. in., which is less than } 0.2f'_c.$$

Some practical points should be noticed in the arrangement of the reinforcement. Rods *f* are extended beyond point *K* of Fig. 4-16(*g*) a sufficient distance to develop about 60 per cent of their strength through bond.

$$L_s(\Sigma o)u = L_s \times 2.75 \times 120 = 0.6 \times 18{,}000, \qquad \text{or} \qquad L_s = 33 \text{ in.}$$

Rods *a* are rather long for easy handling in the shop and in the field. Therefore, they should be spliced by lapping them in the region of compression at the support *B*. If the conditions permitted rod *b* to come in one length, then it could be continuous, and it could replace rod *d* if it were not for the fact that it is difficult to make a series of bends in a long, heavy rod and to make these bends correctly. Furthermore, the hooks at the bottoms of rods *c* extend rather far into the region of tensile stresses at the bottom of the beam. It would be better to straighten the hooks out and to extend the rods far enough for proper development. This requires considerable weight of steel that is not particularly useful. Furthermore, the arrangement of rods is based upon the diagrams of Figs. 4-16(*f*) and (*g*). It might be wise to add a few stirrups near *A* to allow for possible variations of loading in the future, thereby also eliminating the bend in *b*.

Problem 4-7. Figure 4-17(*a*) shows an end span of some of the continuous-girder viaduct construction of the New Jersey approach of the Lincoln Tunnel at Weehawken, N.J. The cross section of the girder is pictured in Sketch (*b*); the shear diagram, in (*c*). Design vertical stirrups for this girder, assuming f'_c = 2,500 lb. per sq. in., v'_L = 75 lb. per sq. in., u = 125 lb. per sq. in., and the allowable stress in stirrups f_v = 16,000 lb. per sq. in. Since the girder is a very deep one, assume $\frac{5}{8}$-in. round stirrups in order to have them strong enough as columns to support the top longitudinal rods and to secure their reasonable spacing. Because of the negative moments at the supports, use inverted-U stirrups in the portion *DE* as shown in Fig. 4-17(*b*) and for the reasons that have been explained in Art. 4-11.

1. *Section AB.* The critical section is at *B* where $d = 72 + 3\frac{1}{4} - 3\frac{1}{2} = 72$ in. (approx.).

It is not necessary to compute *j* for all of the varying depths in such a member as this one. The fact that it is a T beam generally indicates that *k* will be less than 0.38. Therefore, if *j* is assumed to be equal to 0.87 instead

of a larger value, the fact that it occurs in the denominator of Eq. (4-8) indicates that its use will yield conservative results.

$$v_L = \frac{}{b'jd} = \frac{111{,}000}{27 \times 0.87 \times 72} = 66 \text{ lb. per sq. in.}$$

which is less than v'_L; hence no stirrups are required theoretically.

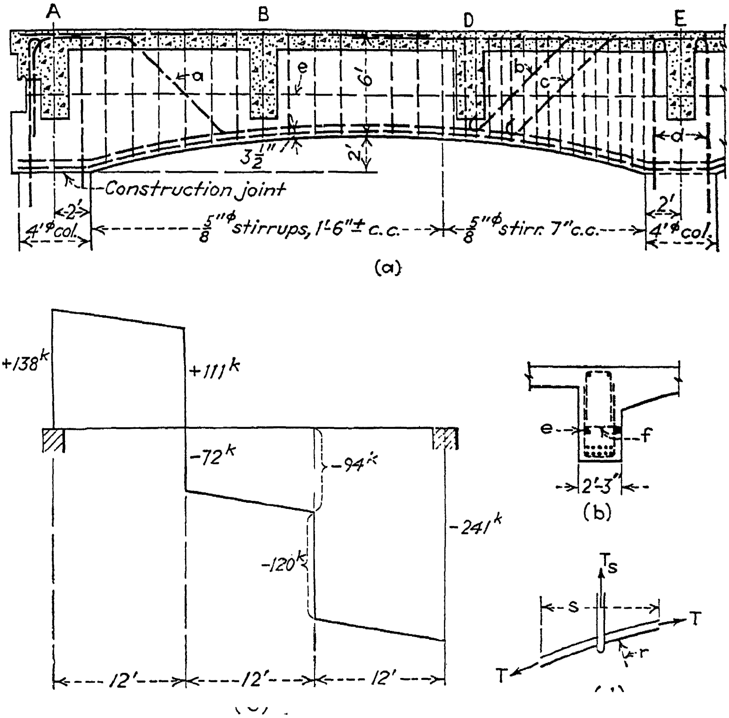


FIG. 4-17.—Girder used in New Jersey approach to Lincoln Tunnel, New York City.

2. *Section BD.* The shear is less than it is in Section *AB*, so that no stirrups are necessary for resisting the shearing forces.

3. *Section DE.* The critical section is at *D*.

$$v_L = \frac{V}{b'jd} = \frac{214{,}000}{27 \times 0.87 \times 72} = 127 \text{ lb. per sq. in.}$$

and, from Eq. (4-10),

$$s = \frac{A_v f_v}{(v_L - v'_L)b'} = \frac{2 \times 0.31 \times 16{,}000}{(127 - 75)27} = 7.1 \text{ in.}$$

Therefore, use stirrups at 7 in. c.c.

The top and bottom portions of the stirrups should be overlapped a distance

$$L_s = \frac{A_v f_v}{(\Sigma o)u} = \frac{0.31 \times 16{,}000}{1.96 \times 125} = 20.2 \text{ in., or, say, } 24 \text{ in.}$$

4. *Practical Details.* The following practical details should be noticed in Figs. 4-17(*a*) and (*b*):

a. The bottom longitudinal rods, except those near E, will be in tension. As shown in Sketch (*d*) they will tend to straighten out and to spall off the concrete below them. Therefore, stirrups must be added to hold them back. (A similar situation occurs if compressive rods are bent around a corner when the stresses tend to buckle them outward.)

The stresses in these stirrups may be approximated by using Eq. (3-3), $T = pr$. In this case, $r = c^2/8m = 32^2/8 \times 2 = 64$ ft. (approx.).[1]

Assuming the maximum tension in all of the bottom rods to be 330 kips, and using the typical $\frac{5}{8}$-in. round stirrups at 18 in. c.c.,

$$p = \frac{T}{r} = \frac{330{,}000}{64} = 5{,}150 \text{ lb. per lin. ft.}$$

$$T_s \text{ in Fig. 4-17}(d) = ps = 5{,}150 \times 1.5 = 7{,}700 \text{ lb.}$$

$$f_v = \frac{T_s}{A_v} = \frac{7{,}700}{2 \times 0.31} = 12{,}400 \text{ lb. per sq. in.}$$

These stirrups must be strong enough, as a sort of combined tie and beam, to carry the radial pull of the rods near the center of the stem [see Sketch (*b*)]. This is mostly a matter of judgment; but if the width of the member is such that any of the main rods are more than 10 or 12 in. from the vertical part of a stirrup, additional supports should be provided.

Therefore, these stirrups will be used throughout the girder from A to D. As a practical matter, a few stirrups are often desirable in similar straight girders, too, in order to tie the member together thoroughly and to hold the main reinforcement in position during the pouring of the concrete.

b. Rods *a* are bent up, and then they are curved down at the end so as to reinforce the top corner.

c. Rods *b* and *c* are bent down to anchor them and to use them as extra web reinforcement—an arbitrary but conservative procedure.

d. The splicing of the bottom longitudinal rods is made at the top of the column at E; a few of the top rods are spliced near D and extended to A.

e. Rods *d* represent the column reinforcement which is extended up into the girder in order to reinforce the joint for continuous frame action.

f. The longitudinal rods *e* and the hooked ones *f* of Sketch (*b*) are used to tie all of the stirrups together in both directions.

4-13. Web Reinforcement in Beams Carrying Moving Loads. The floors of bridges and warehouses, longitudinal girders under railroad tracks, crane girders, and similar structures usually carry

[1] Approximately mid-ordinate $m = c^2/8r$.

moving live loads of large magnitude. Therefore, these structures must be designed to withstand the maximum possible combination of these loads. Such conditions often cause severe stresses in the web reinforcement. The bending moment and the rate of increase of the bending moment—and therefore the longitudinal shearing stresses—differ with various positions of the loads. This means that, for the design of each section, the critical positions of the loads must be ascertained and the resultant stresses must be determined to see if the structure is safe. The reader should review Art. 3-7 because some of the principles stated there are applicable for shearing strength as well as for bond.

The use of influence lines will greatly facilitate the design of members that carry a series of moving live loads. A discussion of the theory of these diagrams and their construction will not be included here.

When more than one moving, concentrated load is used, the maximum shear can be found at enough points to enable one to plot a curve showing the maximum longitudinal shear at all points in one-half of the beam. The excess of these values over the assumed allowable stress v'_L for the unreinforced web of the beam can then be used in the design of the web reinforcement.

Because of the rapid changes in the magnitudes of the web stresses as the live loads pass over such beams, it is conservative to have web reinforcement throughout the length of the member, using nominal sizes in the central portions. Vertical stirrups are generally most suitable for this purpose. It also seems advisable to proportion the stirrups to carry all the shearing stresses but to use a theoretical unit stress of about 25,000 to 30,000 lb. per sq. in. in them. It is difficult to predict the shearing strengths of the portions of such beams that have tension in their tops under one position of the loads whereas there is tension in their bottoms for other positions of these same loads almost immediately thereafter, and vice versa. At such points, bent-up rods (if well anchored) are especially beneficial in preventing disintegration of the beams and in providing steel "hangers" to knit them together.

Problem 4-8. Assume that Fig. 4-18 is part of a small, simply supported highway bridge. The loading diagram in (*a*) is to represent one set of

wheels of a very short, heavy truck. The figures include the allowance for impact. Design vertical stirrups for this beam, using the allowable stresses f_v, u, and v'_L = 20,000, 150, and 75 lb. per sq. in., respectively. Accept the values of f'_c = 3,750 lb. per sq. in., n = 8, k = 0.227, and j = 0.92.

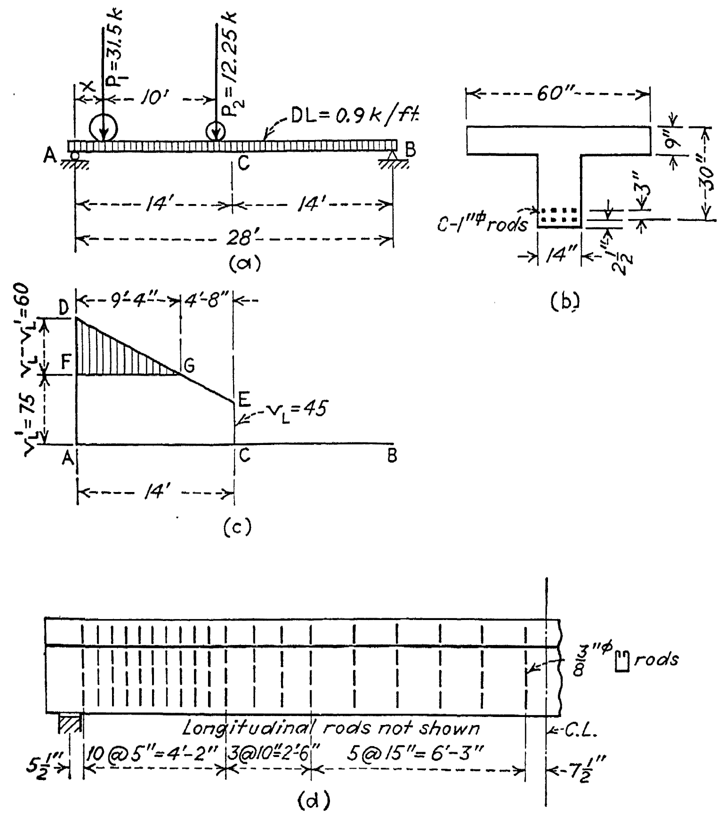


FIG. 4-13.

Then, if x = 0 in Fig. 4-18(a), the shear at A equals 39.38 + 12.6, or 51.98 kips. This figure includes the dead load of 900 lb. per lin. ft. Also, if x = 14 ft., the shear at the left of C is 17.5 kips.

Therefore, using j = 0.92,

$$\text{at } A = \frac{V}{bjd} = \frac{51{,}980}{14 \times 0.92 \times 30} = 135 \text{ lb. per sq}$$

and

$$v_L \text{ at } C = \frac{17{,}500}{14 \times 0.92 \times 30} = 45 \text{ lb. per sq. in.}$$

These values are used to construct Fig. 4-18(*c*). It is obvious that *DE* is a straight line. Also, it can be seen readily that the maximum longitudinal shear at any point between *A* and *C* is the ordinate from the line *AC* to *DE*. Therefore, allowing 75 lb. per sq. in. for v'_L, the longitudinal shear to be taken by the stirrups is represented by the triangle *DFG*.

Assume U-shaped stirrups made of $\frac{3}{8}$-in. round rods. Then, from $A_v f_v = (v_L - v'_L)bs$, the maximum spacing of these stirrups at *A* is

$$s = \frac{2 \times 0.11 \times 20{,}000}{60 \times 14} = 5.24 \text{ in.}$$

The spacing halfway between *D* and *G* may be 10.48 in. The maximum value of *s* is assumed to be $0.5d = 15$ in.

Furthermore, although stirrups are not theoretically needed from *G* to *E*, they will be used at the maximum spacing of 15 in. because the concrete may be cracked in this region owing to deformation of the longitudinal steel.

The complete arrangement of the stirrups is shown in Fig. 4-18(*d*), but the longitudinal rods and the reinforcement in the slab are not shown.

The greatest transverse shear in the beam will occur at *A*. Its magnitude is

$$v_T = \frac{V}{bkd} = \frac{51{,}980}{14 \times 0.227 \times 30} = 545 \text{ lb. per sq. in.,}$$

or less than $0.2 \times 3{,}750 = 750$ lb. per sq. in.

The use of a tensile stress of 20,000 lb. per sq. in. in these stirrups when the concrete is relied upon for 75 lb. per sq. in. of the longitudinal shear may lead to excessive cracking of the concrete. A stress of 16,000 lb. per sq. in. is more advisable for important structures.

For the sake of comparison, assume that this beam is to be designed so that the stirrups carry all the shear at a unit stress of 25,000 lb. per sq. in. Then, using $\frac{3}{8}$-in. round U-shaped stirrups,

$$s = \frac{2 \times 0.11 \times 25{,}000}{135 \times 14} = 2.9 \text{ in.}$$

Since a larger spacing is preferable, use $\frac{1}{2}$-in. round stirrups. The minimum spacing will be 5.3 in.—use 5 in.

4-14. General Considerations. The allowance that may be made for the strength of the concrete alone (v'_L) in Eqs. (4-10), (4-13), and (4-18) is somewhat indefinite, and certainly it is arbitrarily established. Sometimes it is specified; sometimes it must be based upon judgment. At best, it is merely an empirical method of allowing for the strength of the beam without web reinforcement. The designer must decide whether or not the importance of the structure justifies assuming that the concrete alone may be relied upon to develop a safe longitudinal shearing

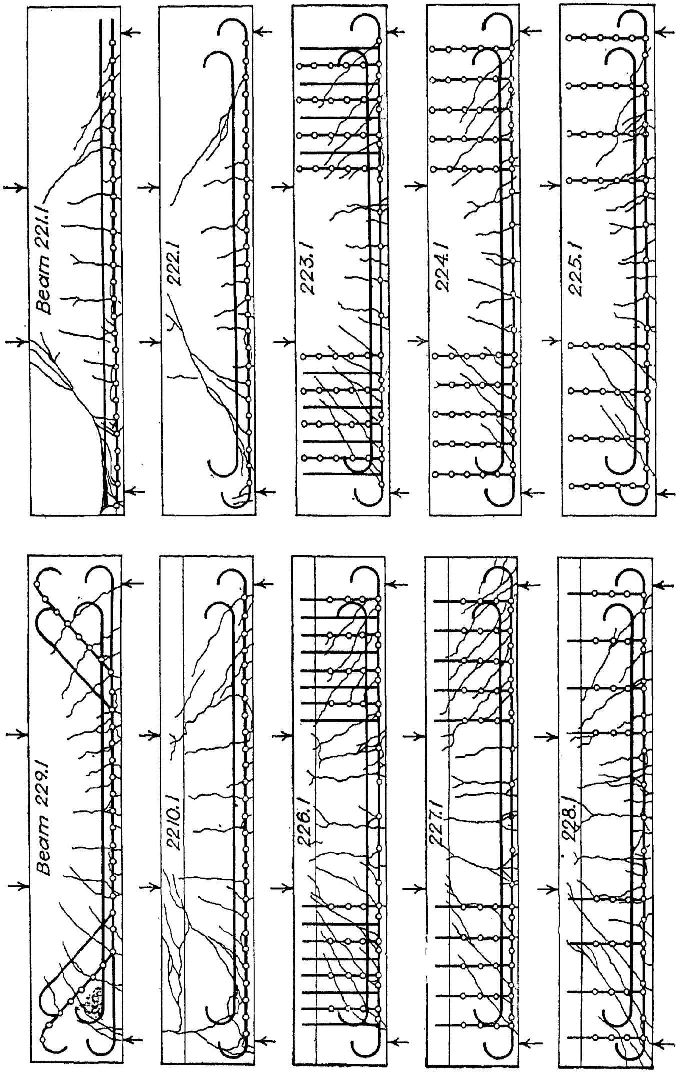


Fig. 4-19.—Sketches of beams that were tested at the Engineering Experiment Station, University of Illinois.

stress of not over $0.03f'_c$ when the longitudinal reinforcement is anchored adequately.

It is very instructive for a student to design reinforced-concrete beams and then to test them to failure so that he can observe their behavior. However, this is not always possible.

Figure 4-19 shows some beams with various types of web reinforcement which were tested to failure. The arrangement of the cracks should be studied carefully in order to give the student a clear idea of how these beams acted under load. For instance, bearing in mind the previous discussions, notice the following:

1. The sketches show the tendency of the hair cracks to be vertical between the loads where the shear is zero, but to be inclined at about 45° outside this central region.
2. The T-beams, No. 2210.1, etc., show a large number of cracks which extend to or above the bottom of the flange. This cracking is more severe than for rectangular beams, and it clearly indicates the rise of the neutral axis toward the top because of the greater area of concrete which can resist the compression.
3. Beams 221.1, 222.1, and 2210.1 indicate the progressive failure of the concrete due to diagonal tension.
4. The beams that have heavy, closely spaced web reinforcement seem to have finer cracks which are more uniformly spaced than those of the other members.
5. The stripping of the unanchored longitudinal rods away from the concrete is shown at the left end of beam 221.1.

Such considerations as these will help one to visualize more clearly the probable actions of the structures that he designs. This ability to look upon the members of a structure as parts of an almost living whole, each part acting and deforming in accordance with known laws, is one of the attributes of the expert designer.

4-15. Design of a Beam-and-girder Floor System. In order to apply the principles illustrated in this chapter and the two preceding ones to practical work, a portion of a warehouse floor will be designed. The work is arranged as it might appear on a designer's calculation sheets, these being shown in Figs. 4-20 to 4-23. The sketches showing the sizes and arrangements to be used by the draftsmen in detailing the various parts are perhaps more fully illustrated than might customarily be the case, but

they are given to show the reader how to plan and picture such things.

The framing plan in Fig. 4-20 indicates that the structure contains three 25-ft. bays crosswise whereas it consists of a considerable number of 20-ft.

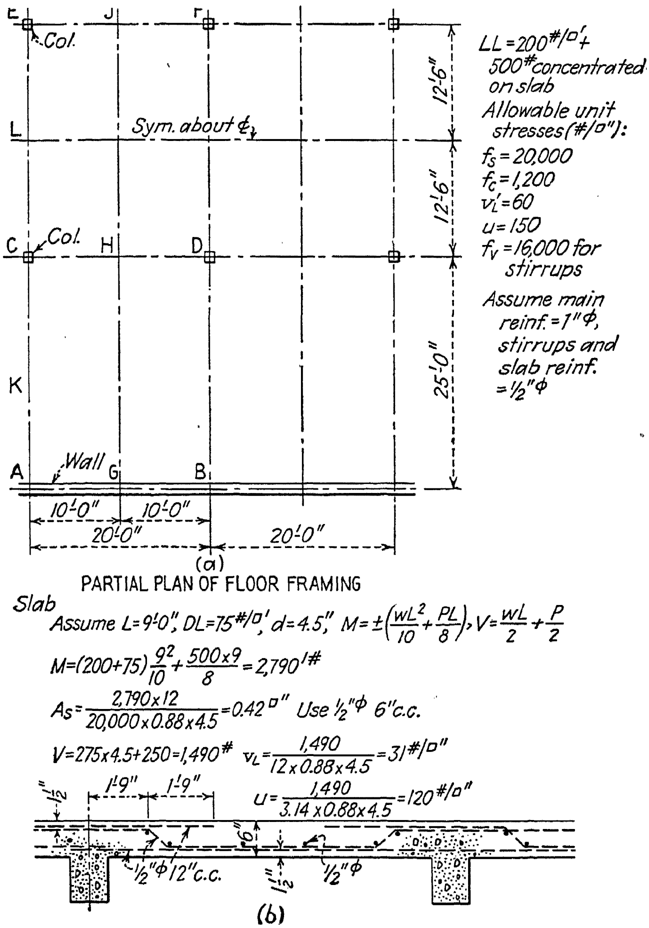


FIG. 4-20.—Design of a beam-and-girder floor system.

bays lengthwise. Assume that the construction under consideration is the first floor of a multistory building, the outer portions being supported on concrete basement walls. The span lengths and general arrangement are dictated by the desired clearances for occupancy. The beams and slabs are assumed to be continuous except at the outer walls.

Beam AC

Assume $M_A=0$, $M_C=-\frac{wL^2}{10}$, $M_K=+\frac{wL^2}{12}$, $V_A=\frac{3}{8}wL$. $V_C=\frac{5}{8}wL$, $b'=15''$

$d=24''$, DL = 400#/' for stem, $L=24'$

$M_C=(275\times \overset{3150}{10}+400)\frac{24^2}{10}=181{,}000$ '#

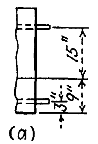
$A_s=\frac{181{,}000\times 12}{20{,}000\times 0.88\times 24}=5.1$ □" Use 7-1"φ min.

Call $kd=\frac{3}{8}\times 24=9''$ $\qquad \frac{x}{20{,}000}=\frac{6}{15}$ $\qquad x=8{,}000$ #/□"

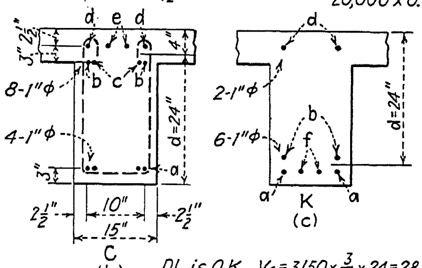
$C=\frac{181{,}000\times 12}{0.88\times 24}=103{,}000$#

$C_c=\frac{15\times 9\times 1{,}200}{2}=\frac{-81{,}000}{22{,}000}$# = Comp. in steel

$A_s'=\frac{22{,}000}{8{,}000}=2.75$□" Use 4-1"φ



$M_K=3150\times\frac{24^2}{12}=151{,}000$ '# $\qquad A_s=\frac{151{,}000\times 12}{20{,}000\times 0.88\times 24}=4.3$□" Use 6-1"φ



$V_C=3150\times\frac{5}{8}\times 24=47{,}200$#

$v_L=\frac{47{,}200}{15\times 0.88\times 24}=149$ #/□" (Say 150)

$s=\frac{0.4\times 16{,}000}{(150-60)\,15}=4.7''$ Use $4\frac{1}{2}$

$u_C=\frac{47{,}200}{25.1\times 0.88\times 24}=89$ #/□"

DL is O.K. $V_A=3150\times\frac{3}{8}\times 24=28{,}400$#, $v_L=\frac{28{,}400}{15\times 0.88\times 24}=90$#/□"

$s=\frac{0.4\times 16{,}000}{(90-60)\,15}=14.2''$ Use 12" $\quad u=\frac{28{,}400}{12.6\times 0.88\times 24}=107$#/□" for 4-1"φ

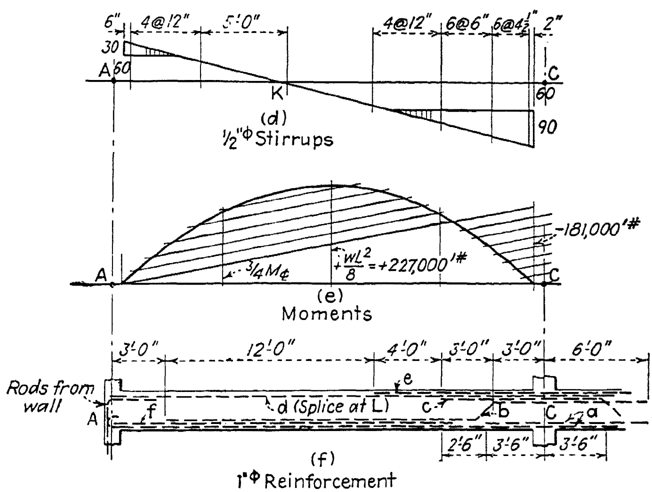


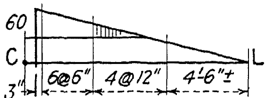
FIG. 4-21.—Design of a beam-and-girder floor system (continued).

Beam CE

$$M_L = \frac{wL^2}{12} = \frac{3150 \times 24^2}{12} = 151{,}000'^{\#} \quad A_S = \text{same as } K \text{ in } AC = 6\text{-}1''\phi$$

$$V = 3150 \times 12 = 37{,}800^{\#} \quad v_L = \frac{37{,}800}{15 \times 0.88 \times 24} = 119^{\#}/\square''(\text{Say } 120)$$

$$\text{For } \tfrac{1}{2}''\phi, \quad s = \frac{0.4 \times 16{,}000}{(120-60)\,15} = 7.1'' \quad \text{Use } 6''\text{c.c.} \quad u < u_c$$



$\frac{1}{2}''\phi$ Stirrups

(a)

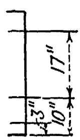
Beam CD

$$\text{Assume } M_C = -M_H = -\frac{wL^2}{12} - \frac{PL}{8}, \; V = \frac{wL}{2} + \frac{P}{2}, \; b = 16'', \; d = 27'', \; DL = 400^{\#}/', \; L = 19'$$

$$M_C = (400 + 275 \times 1.33)\frac{19^2}{12} + 3150 \times 23.67\,(0.5 + 0.625)\frac{19}{8} = 222{,}000'^{\#}$$

$$A_S = \frac{222{,}000 \times 12}{20{,}000 \times 0.88 \times 27} = 5.6\,\square'' \quad \text{Use } 8\text{-}1''\phi$$

$$\text{Call } kd = \tfrac{3}{8} \times 27 = 10'' \qquad \frac{x}{20{,}000} = \frac{7}{17} \qquad x = 8{,}200^{\#}/\square''$$



(b)

$$C = \frac{222{,}000 \times 12}{0.88 \times 27} = 112{,}000^{\#} \quad A'_S = \frac{16{,}000}{8{,}200} = 1.95\,\square'' \quad \text{Use } 4\text{-}1''\phi$$

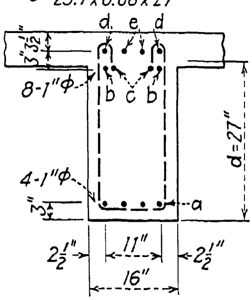
$$C_c = \frac{16 \times 10 \times 1200}{2} = \frac{-96{,}000}{16{,}000^{\#}}$$

$$V = \overset{7{,}300}{(400 + 275 \times 1.33)\,9.5} + \overset{41{,}900}{\underline{3{,}150 \times 23.67 \times 1.125}} = 49{,}200^{\#}$$

$$\qquad \text{stem} \quad \text{slab+LL} \qquad\qquad\qquad 2$$

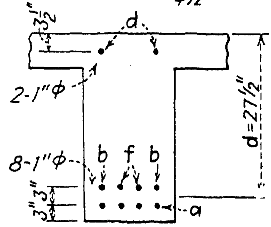
$$v_L = \frac{49{,}200}{16 \times 0.88 \times 27} = 129^{\#}/\square'' \quad s = \frac{0.4 \times 16{,}000}{(129-60)\,16} = 5.8'' \quad \text{Use } \tfrac{1}{2}''\phi \; 6''\text{c.c.}$$

$$u_c = \frac{49{,}200}{25.1 \times 0.88 \times 27} = 82^{\#}/\square'' \text{ for } 8\text{-}1''\phi \qquad v_L \text{ at } H = 129 \times \frac{419}{492} = 110^{\#}/\square''$$



C

(c)



H

(d)

FIG. 4-22.—Design of a beam-and-girder floor system (continued).

The live loads, permissible unit stresses, and sizes of reinforcement for the job are specified. The 500-lb. concentration is to be used for the slab only, being applied at the center of a 1-ft. strip in order to allow for local excessive piling of goods beyond the intended limits or for moving live loads. The assumptions regarding span lengths for design are shown in each part of the computations. Purposely they are made less than the center to center distances in order to illustrate how to handle the calculations in such cases.

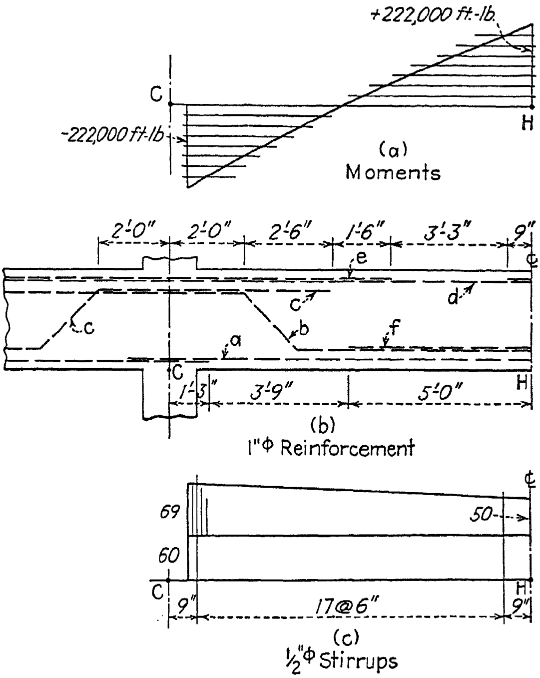


FIG. 4-23.—Design of a beam-and-girder floor system (continued).

The maximum positive and negative bending moments are computed by means of assumed coefficients. More refined methods for the calculation of such moments will be covered in Chaps. 13 and 14. The main reinforcement is limited to 1-in. rounds in order to show how to adjust the members for the use of one standard size and to illustrate the arrangement of the rods when double rows are needed.

The computations are made by slide rule and by the approximate methods of design previously described. As a matter of interest and for illustration of the results, the stresses at C in beams AC and CD were checked by using the transformed-section method, giving the following answers:

Member	*AC*	*CD*
S_c	1,940 in.^{3}	2,440 in.^{3}
S_s	120 in.^{3}	138 in.^{3}
f_s	18,100 lb. per sq. in.	19,300 lb. per sq. in.
f_c	1,120 lb. per sq. in.	1,090 lb. per sq. in.
j	$0.87\pm$	$0.88\pm$

The reader should notice the necessity for careful planning of the arrangement of the reinforcement at such a point as *C* in Fig. 4-20. The rods have

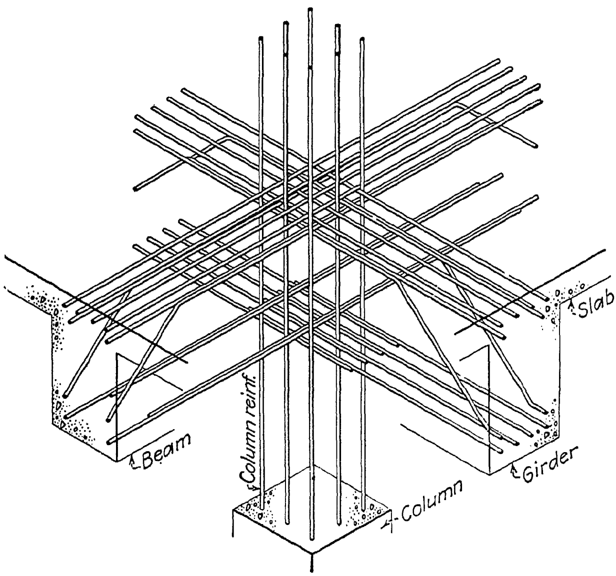


FIG. 4-24.—Isometric sketch showing reinforcement at junction of column and continuous beam and girder, omitting stirrups, column ties, and rods in floor slab.

real thickness. It is easy to make the lines on a drawing so that the rods supposedly go in certain positions in one member regardless of what may be the detail of an intersecting one. However, this cannot be done in the field. If the rods are cut and bent to improper dimensions, the men in the field may have great difficulty in assembling the steel. Figure 4-24 is made to illustrate the packing and arrangement of the bars at *C* of Fig. 4-20. The "girder" is member *CD*. The reader can thus see why the cover over the top rods in Figs. 4-21 and 4-22 differ by 1 in. The tendency to form a screen where the rods cross is self-evident. One can imagine the erector's troubles if the work is unwisely or carelessly planned.

The reader will also notice in the calculations the use of the bending moment diagrams for the determination of the bend points or termination of the main reinforcement. In actual design, an engineer will not always be willing to plot these diagrams. In order to show a safe, approximate, and

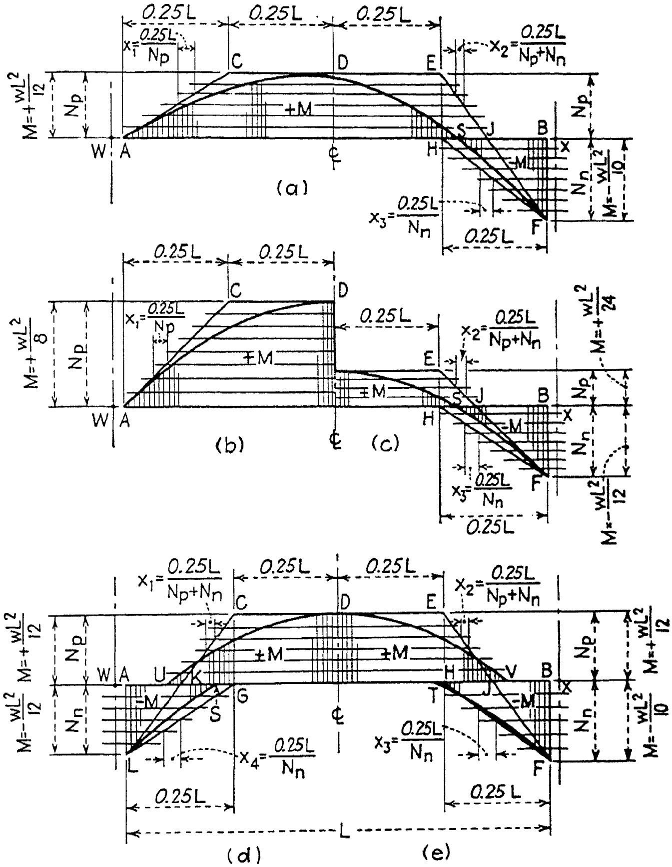


FIG. 4-25.—Approximate locations of permissible cutoff points for reinforcement of continuous beams with uniformly distributed loads.
(Add at least 12 diameters for anchorage.)

easily applied substitute method to use in finding where the rods are no longer needed, Fig. 4-25 has been prepared. Sketch (*a*) is like the diagram shown in Fig. 4-21; (*b*) is a simple span; (*c*) is a fixed-end beam; and (*d*) and (*e*) are assumptions frequently made. The horizontal lines represent the portions of the moments carried by the rods assumed to be used. N_p indicates the number of rods used for positive reinforcement; N_n denotes

those used for the negative reinforcement. The suggested procedure is as follows, referring to Fig. 4-25:

1. For a beam with unrestrained ends:

Stop the rods in the outer quarter of the span beyond the line AC of Sketches (a) and (b). For instance, assume the third and fourth rods to be discontinued at $0.25L + 4 \times 0.25L \div N_p$, then add some anchorage.

The Code specifies that at least one-third of the positive reinforcement must be extended into the supports.

2. For the positive reinforcement in a beam with end restraint:

Stop the rods in the outer quarter of the span beyond the lines EJF and LKC of Sketches (a), (c), (d), and (e). The theoretical horizontal distance between cutoffs may be $0.25L \div (N_p + N_n)$. This generally requires these rods to be stopped well beyond the usual theoretical bending moment diagram, a procedure which is desirable because of the fact that any yielding at the supports will cause the negative bending moments to decline whereas the positive ones increase toward the condition shown in Sketch (b). The Code's further requirement that one-fourth of the positive reinforcement be extended into the supports is well justified.

3. For the negative reinforcement in a beam with end restraint:

Stop the rods in the outer quarter of the span but inside of the lines HF and LG of sketches (a), (c), (d), and (e). The theoretical horizontal distance between cutoffs may be $0.25L \div N_n$. The Code requires that one-third of the negative reinforcement must extend beyond the point of inflection sufficiently to develop one-half the allowable stress in the bars. It may be wise to extend a few of these rods clear through as shown in Figs. 4-21 and 4-23.

4. In any case, the Code properly requires that the anchorage for such rods beyond the theoretical cutoff be 12 diameters or not less than $L/20$.

The theoretical spans used in Figs. 4-21 and 4-22 are assumed to be 1 ft. less than the center to center distances. Practice in this respect varies greatly. The Code permits the computation of bending moments on the basis of the center to center spans with the negative moment reduced by an amount equal to the shear times one-third of the width of the support.

Practice Problems

Problem 4-9. Assume a rectangular beam for which $b = 16$ in., $d = 27$ in., $k = 0.36$, $j = 0.88$, and the maximum shear $V = 25{,}000$ lb. Find v_L and v_T. If $v'_L = 50$ lb. per sq. in., is the beam safe without web reinforcement?

Discussion. Remember that, as used herein, v_L = intensity of longitudinal shear, v_T = intensity of transverse shear when confined to the depth kd, and v'_L = allowable intensity of longitudinal shear when no web reinforcement is used.

Problem 4-10. Find the spacing of $\frac{1}{4}$-in. round vertical U-stirrups at the end of the beam of Problem 4-9 if the stress in the stirrups f_v is 16,000 lb. per sq. in. and if they are to withstand the excess shear over v'_L.

Problem 4-11. Find v_L and v_T for a rectangular beam for which $b = 18$ in., $d = 32$ in., $k = 0.33$; $j = 0.89$, and $V = 40{,}000$ lb. If $v'_L = 50$ lb. per

sq. in., find the tensile stress in $\frac{3}{8}$-in. round vertical U-stirrups spaced 8 in. c.c. to withstand the excess shear over v'_L.

Problem 4-12. Assume that a simply supported beam is 20 ft. long and that it carries a total uniformly distributed load of 1,000 lb. per lin. ft. Assume that $b = 12$ in., $f'_c = 3{,}000$ lb. per sq. in., $j = 0.88$, and $v'_L = 0.02f'_c$. Find the depth of the beam that will be safe with no web reinforcement or special anchorage of the steel.

Discussion. Find the end reaction; then substitute v'_L for v_L in Eq. (4-4), and solve for d. Add 2 in. for cover at the bottom.

Problem 4-13. Assume that the beam of Problem 4-9 is simply supported, the span = 25 ft., the load = 2,500 lb. per lin. ft. uniformly distributed, $v'_L = 50$ lb. per sq. in. Draw the shear diagram, and design $\frac{3}{8}$-in. round vertical U-stirrups to act as web reinforcement, using $f_v = 16{,}000$ lb. per sq. in.

Problem 4-14. Assume a simply supported T-beam with a span of 24 ft. The moving live loads and the dead load cause a shear at the end equal to 30,000 lb.; that at the center of the span equals 10,000 lb. Assume that this shear varies uniformly between these points. If $b' = 15$ in., $d = 27$ in., $j = 0.92$, and $v'_L = 60$ lb. per sq. in., find the required spacing of $\frac{3}{8}$-in. round vertical U-shaped stirrups if the permissible tension in the steel = 15,000 lb. per sq. in. *Ans.* $s = 10.7$ in. (use 10 in.).

Problem 4-15. Assume that the beam of Problem 4-13 is to have the same stirrups designed to take all of the shear, using a unit stress of 25,000 lb. per sq. in. in them. Make a detailed layout of the stirrups for this case. What is the maximum spacing near the center regardless of the stress?

Problem 4-16. Assume a T-beam which is continuous (call it "fixed") over a series of supports. Let one interior span be 30 ft. If the total load is 3,500 lb. per lin. ft. (uniformly distributed), detail and space $\frac{1}{2}$-in. round vertical U-shaped stirrups to provide for the shear in excess of $v'_L = 60$ lb. per sq. in., using $f_v = 16{,}000$ lb. per sq. in., $b' = 15$ in., $d = 27$ in., and $j = 0.88$.

Discussion. Draw the shear and moment diagrams. In the regions where tensile stresses exist at the top of the beam, use lapped U-stirrups as shown in Fig. 4-15(d); elsewhere, use standard U-shaped ones. Throughout the central portion, use stirrups at $\frac{1}{2}d$ c.c.

Problem 4-17. Design $\frac{1}{2}$-in. square U-shaped stirrups which are inclined 30° from the vertical for Problem 4-16.

Problem 4-18. Design the web reinforcement for a simply supported T-beam, using bent-up rods where they are feasible, and using $\frac{3}{8}$-in. round U-shaped vertical stirrups elsewhere. Assume the following data: $b' = 12$ in.; $d = 30$ in.; $j = 0.94$; the tensile reinforcement = six 1-in. round rods in two rows 3 in. apart; the total load = 2,500 lb. per lin. ft.; span = 24 ft.; the two outer rods of the bottom row are to be straight throughout; $v'_L = 50$ lb. per sq. in.; $f_v = 15{,}000$ lb. per sq. in.; the seat under the beam is assumed to be 1 ft. long.

Discussion. Draw the shear and bending-moment diagrams as in Fig. 4-12; determine the possible bend points of the bars; find where the bent rods do not reinforce the web (if any such parts exist), and add $\frac{3}{8}$-in. vertical stirrups; make a detailed sketch of the reinforcement.

Problem 4-19. Design and detail the web reinforcement for the T-beam of Fig. 2-22, using bent-up rods and additional $\frac{3}{8}$-in. round vertical U-shaped stirrups if they are needed. Assume $j = 0.9$; $f_v = 18{,}000$ lb. per sq. in.; the top steel = six 1-in. round rods with four in the top row and two in the second row 3 in. farther down; the span = 30 ft.; the total load is 2,100

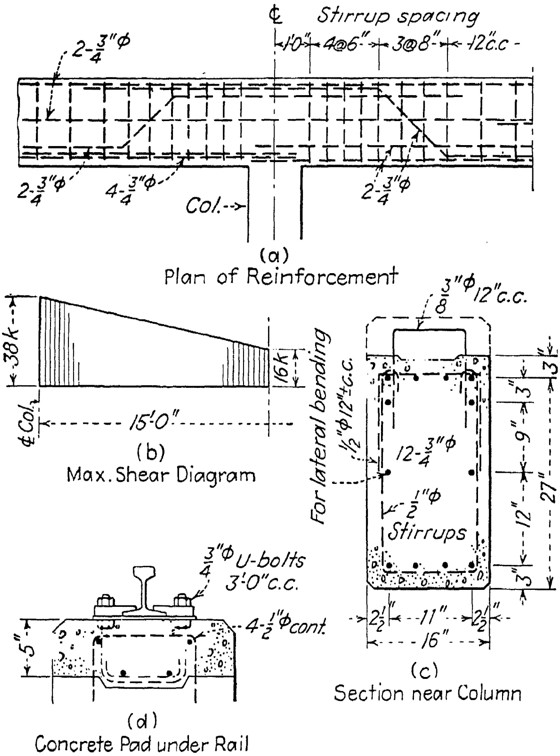


FIG. 4-26.—Reinforced-concrete crane girder used in the concentrator of the low-grade ore project. (*Courtesy of Cananea Consolidated Copper Company, Cananea, Sonora, Mexico.*)

lb. per lin. ft., uniformly distributed; the ends may be called fixed; four 1-in. round rods are used in the bottom at the center, two being straight for their full length; $v_L' = 60$ lb. per sq. in.; and the support under the beam is 18 in. long.

Problem 4-20. Redesign the web reinforcement of the beam of Problem 4-19 if the total loading is changed to 1,000 lb. per lin. ft. uniformly distributed and two concentrated loads of 15,000 lb. each at points 10 ft. from each support. (See Table 13-2.) Use the same rods as before and bend them. Add vertical U-shaped stirrups where required.

Problem 4-21. Redesign the web reinforcement of the beam of Problem 4-19 with the loading of Problem 4-20, but use $\frac{3}{8}$-in. round vertical U-shaped stirrups (or double ones), stopping the longitudinal rods or bending them as desired but not relying upon the bent rods as web reinforcement. Let $f_v = 16{,}000$ lb. per sq. in. and $v'_L = 50$ lb. per sq. in. Detail the reinforcement.

Problem 4-22. Figure 2-26 shows a tapered, cantilevered, rectangular beam. The proposed vertical stirrups are $\frac{1}{2}$-in. rounds 6 in. center to center throughout the beam. Consider the weight of the concrete, and assume the column to be 18 in. wide. Test the deep and the shallow portions, using $d = 26$ in. at the column and $d = 16$ in. at the end near the concentrated load. Assume $j = 0.88$ in both cases. (*a*) Is the beam safe if the concrete is to resist 60 lb. per sq. in. and the allowable $f_v = 16{,}000$ lb. per sq. in.? (*b*) Is it safe if the stirrups are designed to carry all the shear with $f_v = 30{,}000$ lb. per sq. in.? (*c*) What should be done to remedy the case if the beam were otherwise satisfactory?

Ans. (*a*) f_v near col. = 12,200 lb. per sq. in., O.K.; f_v near end = 26,600 lb. per sq. in., no good; (*b*) f_v near col. = 26,600 lb. per sq. in., O.K.; f_v near end = 41,000 lb. per sq. in., no good; (*c*) bend down two lower bars of tensile reinforcement on general principles, then use closer stirrup spacing toward end.

Problem 4-23. Figure 4-26 shows a continuous, reinforced-concrete crane girder used in the concentrator of the Cananea Consolidated Copper Co., Cananea, Sonora, Mexico. Are the stirrups safe if they are to carry all the shear, assuming that the allowable $f_v = 25{,}000$ lb. per sq. in. and $j = 0.88$?

Discussion. The 5-in. cap shown in Figs. 4-26(*c*) and (*d*) is for the purpose of aligning, leveling, and grouting the rail after the girders have been completed and their forms removed. Otherwise it would be difficult to set the rail and its attachments accurately. The total depth of the beam is assumed to be 30 in. Incidentally, such heavy, elevated concrete construction requires strong, and perhaps expensive, formwork.

Ans. Yes; $f_v = 24{,}000$ lb. per sq. in. max.

CHAPTER 5

COMPOSITE BEAMS

5-1. Introduction. Sometimes it becomes necessary to analyze or to design a structure that is composed of steel I-beams, girders, or light trusses that are encased in concrete. The concrete may be used as a protection against fire and corrosion, or it may be designed to act as a load-carrying element of the structure. The latter case will be illustrated. The properties of the structural-steel sections can be found in suitable handbooks.

The term "composite beam" is used to denote the second of the foregoing cases; that is, the two materials are to act together in resisting the bending moment. Such a beam is differentiated from an ordinary reinforced-concrete one, primarily because the steel is a large rolled or fabricated unit which usually has great strength in itself. The concrete adds to the strength and stiffness of the steel, but it is the weaker of the two materials.

5-2. I-beam and Thin-slab Construction. A type of construction that is common in steel-framed buildings is shown in Fig. 5-1. It consists of a rather thin concrete slab which is supported on steel I-beams. Ordinarily, the slab is poured monolithically with the encasement of the beams. When a load is placed upon such a floor, it is obvious that both the concrete and the steel will be affected.

Frequently, in such a case, the steel beam is designed to act alone in carrying the entire load. If its top flange is thoroughly embedded in the concrete of the slab—1 in. or more above the bottom of the latter—the beam is usually considered to have adequate lateral support because the concrete will not let it bend sidewise. Under such conditions, some specifications permit the use of a higher unit stress in the beam than would be allowed for one that is not encased. This is done in order to allow indirectly for the benefit from the action of the concrete in assisting or augmenting the load-carrying capacity of the steel.

This again calls attention to the very important fact that, when steelwork is encased in concrete, a material of high ductility

is covered by another one with very little ability to stretch but with great compressive strength. Therefore, when a load is applied, the two materials try to act in accordance with their own particular characteristics. The top flange of the I-beam in Fig. 5-1 tends to compress inside the concrete; the bottom flange tries to elongate; both flanges endeavor to deform about a neutral axis which is at the center of the web. Simultaneously, the concrete tries to act as a T-beam with its neutral axis close to the bottom of the slab. These two different actions must be in conflict because of the bond between the steel and the concrete. Unless this bond is broken, the steel cannot deform one way while

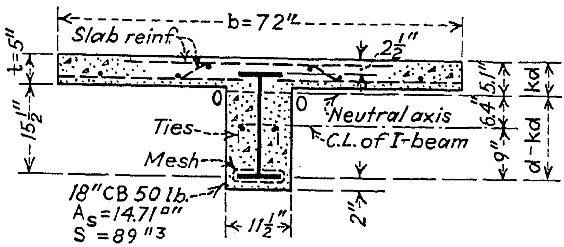


FIG. 5-1.

the concrete deforms another way. Therefore, the action is that of a composite unit.

Assuming that the bond does not fail, the beam can be analyzed by the transformed-section method. To illustrate this, analyze the beam of Fig. 5-1. Let $n = 10$ and the allowable f_s and $f_c = 20{,}000$ and 1,000 lb. per sq. in., respectively. Neglect the small tie rods in the slab.

The transformed section of the I-beam will be assumed to equal $nA_s = 10 \times 14.71 = 147.1$ sq. in. Theoretically, the portion of the I-beam above the neutral axis of the composite beam should have its area multiplied by $(n - 1)$, but although the flange area of the steel equals several square inches, it is too small compared to the concrete to warrant such refinement.

Taking the static moment about the neutral axis of the composite section, neglecting any of the stem that may be in compression,

$$bt\left(kd - \frac{t}{2}\right) = nA_s(11.5 - kd)$$

$$72 \times 5(kd - 2.5) = 147.1(11.5 - kd)$$

$$kd = 5.1 \text{ in.},\ d - kd = 15.4 \text{ in.} \qquad \text{(Near enough.)}$$

Assume that the moment of inertia of the transformed section of the I-beam about its own center of gravity is nI.

$$nI = 10 \times 800.6 = 8{,}006 \text{ in.}^4 \qquad \text{(See any steel handbook.)}$$

$$I_c = \frac{bt^3}{12} + bt\left(kd - \frac{t}{2}\right)^2 + nI + nA_s(11.5 - kd)^2$$

$$I_c = \frac{72 \times 5^3}{12} + 72 \times 5 \times 2.6^2 + 8{,}006 + 147.1 \times 6.4^2 = 17{,}200 \text{ in.}^4$$

$$S_c = \frac{17{,}200}{5.1} = 3{,}370 \text{ in.}^3$$

$$S_s = \frac{17{,}200}{10 \times 15.4} = 112 \text{ in.}^3$$

Then

$$\text{Max. } M_c = f_cS_c = 1{,}000 \times 3{,}370 = 3{,}370{,}000 \text{ in.-lb.}$$
$$\text{Max. } M_s = f_sS_s = 20{,}000 \times 112 = 2{,}240{,}000 \text{ in.-lb.}$$

Therefore, the stress in the concrete, assuming composite-beam action, is determined by the strength of the steel. When the stress in the steel is 20,000 lb. per sq. in.,

$$f_c = \frac{M_s}{S_c} = \frac{2{,}240{,}000}{3{,}370} = 665 \text{ lb. per sq. in.}$$

Considering the I-beam alone with $f_s = 20{,}000$ lb. per sq. in., its safe resisting moment, when laterally supported, is

$$M = Sf_s = 89 \times 20{,}000 = 1{,}780{,}000 \text{ in.-lb.}$$

Therefore, the permissible increase in the safe bending moment for the composite beam over that for the plain steel beam is

$$\frac{100(M_s - M)}{M} = \frac{100(2{,}240{,}000 - 1{,}780{,}000)}{1{,}780{,}000} = 25.8 \text{ per cent.}$$

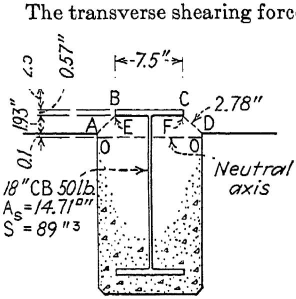


FIG. 5-2.

The transverse shearing forces are not important in this case because they are resisted by both the web of the I-beam and the concrete. The former is usually capable of carrying the entire load.

Also, the longitudinal shear cannot cause failure of the same character as that which was discussed in the preceding chapter. However, it may produce excessive bond stresses or high local shearing stresses in the concrete.

To investigate this problem, let Fig. 5-2 represent an enlargement of a portion of Fig. 5-1. The beam will be assumed to have a span of 25 ft. and a uniform load of 2,200 lb. per lin. ft., including the dead load. The longitudinal shearing stress upon any plane of any section of the beam per unit length may be assumed as

$$S_L = \frac{VQ}{I_c}$$

where S_L equals the total longitudinal shear in pounds per linear inch of the beam, V equals the transverse shear at the given section in pounds, Q is the static moment of the part beyond the plane being considered (in this case *above* $AEBCFD$ in the figure) computed about the neutral axis of the composite section, and I_c is the moment of inertia of the composite beam about its own center of gravity (or neutral) axis. Q and I_c must be expressed in terms of inch units. They must also be computed upon the basis of the transformed section in terms of concrete. Therefore, at the end of the beam,

$$S_L = \frac{(12.5 \times 2{,}200)(72 \times 5 \times 2.6 - 7.5 \times 2.5 \times 1.35 - 1.93 \times 2 \times 0.74)}{17{,}200}$$

$$S_L = 1{,}450 \text{ lb. per lin. in.}$$

$$v_L = \frac{S_L}{AEBCFD} = \frac{1{,}450}{2 \times 2.78 + 2 \times 0.57 + 7.5} = \frac{1{,}450}{14.1} = 103 \text{ lb. per sq. in.}$$

The stress on the surfaces AE and FD is shear in the concrete, whereas that on $EBCF$ is bond stress. The magnitude of the latter seems to be rather large to be relied upon for such wide, flat surfaces of steel. Furthermore, since the bond stress is really a shearing stress, it is not reasonable to assume that the shear on the sections of concrete at AE and FD can have a value that is much different from the bond stress on the steel, because all of the material must deform together until the bond fails or the concrete shears off. It is therefore advisable to use some kind of anchorage between the steel and the concrete, as illustrated in the next problem.

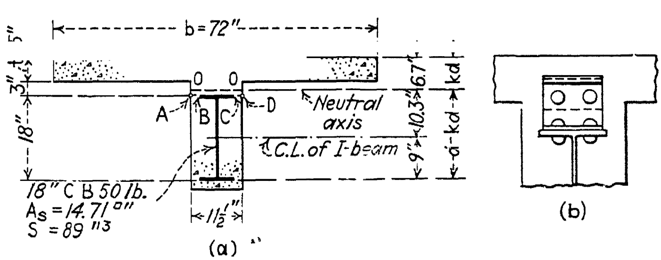


FIG. 5-3.

Problem 5-1. A modification of the problem that has previously been illustrated is shown in Fig. 5-3(a). Using the same data, find the bond stress at the top of the I-beam.

Although this arrangement appears strange, such conditions often occur in building construction because of practical considerations. They should be guarded against because a crack is likely to form along the plane *ABCD*.

Proceeding as before, an analysis of this beam gives the following values:

$$\begin{aligned}
kd &= 6.7 \text{ in.}, \; d - kd = 19.3 \text{ in.}, \; I_c = 30{,}700 \text{ in.}^4 \\
S_c &= 4{,}580 \text{ in.}^3, \qquad S_s = 159 \text{ in.}^3 \\
M_c &= 4{,}580{,}000 \text{ in.-lb.} \qquad \text{(Greater than before.)} \\
M_s &= 3{,}180{,}000 \text{ in.-lb.} \qquad \text{(Greater than before.)}
\end{aligned}$$

The computed $f_c = \dfrac{wL^2 \times 12}{8 \times S_c} = \dfrac{2{,}200 \times 25^2 \times 12}{8 \times 4{,}580} = \dfrac{2{,}060{,}000}{4{,}580} = 450$ lb. per sq. in., when the real external bending moment is carried by the composite beam.

The static moment Q of the portion *below* AD about the neutral axis O-O is

$$Q = nA_s(8 + 9 - kd) = 10 \times 14.71 \times 10.3 = 1{,}515 \text{ in.}^3$$

Therefore, using a transverse shear V equal to the end reaction of the beam,

$$S_L = \frac{VQ}{I_c} = \frac{(12.5 \times 2{,}200)(1{,}515)}{30{,}700} = 1{,}360 \text{ lb. per lin. in.}$$

or

$$v_L = \frac{S_L}{AD} = \frac{1{,}360}{11.5} = 118 \text{ lb. per sq. in.}$$

These figures show that a crack may form along the plane AD. One possible method of preventing this is by riveting double angles on the top flange as shown by Fig. 5-3(*b*), forming Z-shaped lugs which provide a mechanical bond. These angles can be spaced as required to withstand the longitudinal shear, being closer together near the ends of the beam where the shearing forces are the greatest. These lugs provide a mechanical bond which helps to resist the longitudinal shear and to enable the beam to act as a composite section. Incidentally, these angles are useful in tying the slab and the steel together so that continuity of the slab, or any other action, will not cause the slab to be pulled away from the I-beam.

The patented "Alpha" system utilizes spiral rods which are welded to the top flange of the beam to resist the longitudinal shearing forces. In some of the structures that have been built by The Port of New York Authority, the beams are set so that the reinforcing trusses in the slab can be welded directly to the top flange, thus forming a lug on the beam.

If structural-steel members—I-beams or trusses—are used to support the forms during the placing of the concrete, the stresses resulting from this dead load must be computed and deducted from the allowable stress in the steel, the remainder being the stress that is available for the steel as reinforcement in the concrete. For instance, if such dead loads caused a tension of 9,000 lb. per sq. in the I-beam of this problem, for which the allowable stress is 20,000 lb. per sq. in., then 11,000 lb. per sq. in. would be the maximum that could be used in the composite section. However, the dead load that is already carried by the I-beam should not be included again in computing the loads that cause bending moments and shears in the composite beam.

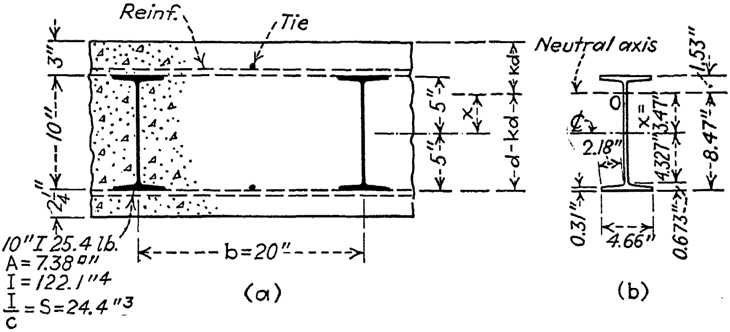


FIG. 5-4.—Floor of Lincoln Tunnel, New York City.

When composite beams frame into steel girders or columns, special care should be taken to avoid cracking along the top flanges because of deflection of the beams. Where feasible, the steel beams should be made continuous with tension plates and bottom thrust angles; otherwise use rods in the slab to carry tension across the girder or past the column. The whole framework should be looked over to find the places where deflections under live loads will be likely to cause cracks; then, if it is not practicable to make the steelwork continuous, it may be advisable to make definite joints or cuts in the concrete so that the cracking will occur at such predetermined locations.

5-3. I-beams Completely Encased in Thick Slabs. In some cases, small I-beams may be completely encased in thick concrete slabs as shown in Fig. 5-4. In such cases, it is reasonable to assume that, with proper inspection in the field, the bond and

shearing stresses will be sufficient to make the structure act as a composite unit. The sketch shows the floor of the Lincoln Tunnel which is of this type of construction (see Fig. 5-4A).

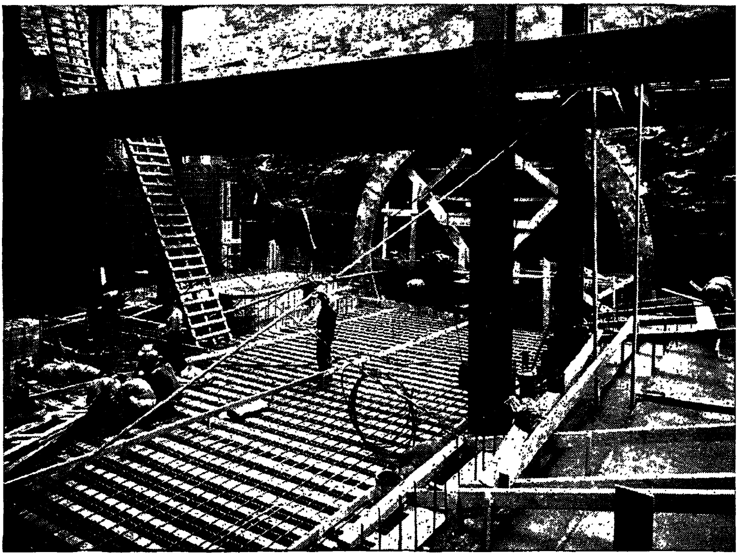


FIG. 5-4A.—Construction of the roadway slab of the Lincoln Tunnel at the New Jersey shaft.

The slab of Fig. 5-4(a) will now be analyzed, using $n = 8$ and f_s, f_c, and u = 20,000, 1,200, and 150 lb. per sq. in., respectively. The tie rods will be neglected.

$$nA_s = 8 \times 7.38 = 59.04.$$

Assuming $b = 20$ in., and taking the static moments about the neutral axis'

$$\frac{20(kd)^2}{2} = 59.04(8 - kd).$$

$$kd = 4.53 \text{ in.,} \qquad \text{and} \qquad d - kd = 8.47 \text{ in.}$$

If I equals the moment of inertia of the I-beam and x = the distance from its center to the neutral axis of the composite section, then

$$I_c = \frac{b(kd)^3}{3} + nI + nA_s(x)^2$$

$$I_c = \frac{20 \times 4.53^3}{3} + 8 \times 122.1 + 59.04(8 - 4.53)^2 = 2{,}307 \text{ in.}^4$$

$$S_c = \frac{2{,}307}{4.53} = 509 \text{ in.}^3 \qquad S_s = \frac{2{,}307}{8 \times 8.47} = 34 \text{ in.}^3$$

$$M_c = 509 \times 1{,}200 = 611{,}000 \text{ in.-lb. for a 20-in. strip}$$
$$M_s = 34 \times 20{,}000 = 680{,}000 \text{ in.-lb. for a 20-in. strip}$$

In this case, the concrete determines the safe resisting moment of the slab.

If the I-beams are assumed to carry the load by themselves without the help of the concrete, then, using the ordinary flexure formula,

$$M = \frac{sI}{c} = 20{,}000 \times 24.4 = 488{,}000 \text{ in.-lb. for a 20-in. strip.}$$

The concrete thus enables the slab to develop a resisting moment which is

$$\frac{100(M_c - M)}{M} = \frac{100(611{,}000 - 488{,}000)}{488{,}000} = 25.2 \text{ per cent}$$

greater than for the I-beams alone.

The composite action of the type of construction shown herein is much more reliable than that of the type illustrated in the previous article. The horizontal and vertical forces are resisted by both the concrete and the I-beams, although the latter will usually be found to be capable of withstanding the entire shearing force if necessary. Also, because of the thorough embedment of the steel, the bond stresses are generally unimportant. They cannot be tested accurately by the formula

$$u = \frac{V}{(\Sigma o)jd}$$

because of the continuity of the webs of the I-beams and their participation in the composite action.

When the importance of a structure requires at least an *approximate* analysis of the bond stresses, their maximum value may be found by assuming that all of the increments in the steel stresses come through the action of the bond between the concrete and the I-beams in the same manner that they do in the case of ordinary reinforcing rods.

Therefore, referring to the beam of Fig. 5-4, if dM represents the increment of the bending moment per linear inch of beam, the increment of the stress in the steel df_s will be

$$df_s = \frac{dM}{S_s} = \frac{dM}{34} \text{ for this example.}$$

Next, find the section modulus of the I-beam alone about an axis at the center of gravity of the composite beam [point O, Fig. 5-4(b)]. For this particular case, it is

$$S = \frac{I}{c} = \frac{I_{CL} + Ax^2}{c} = \frac{122.1 + 7.38 \times 3.47^2}{8.47} = 25 \text{ in.}^3$$

where I_{CL} is the moment of inertia of the I-beam about its own center of gravity. Then find the section modulus of the entire surface area of the I-beam about the same axis O, using a length of beam of 1 in. This gives

$$S \text{ for surface} = \frac{I'_{CL} + A'x^2}{c} = \frac{547 \text{ (approx.)} + 36.59 \times 3.47^2}{8.47} = 117 \text{ in.}^3$$

For $\Sigma M = 0$, the increment of moment of the steel stresses, per unit of length of the beam, about the axis O must equal the moment of the bond stresses, per unit of length of the beam, about this same axis, or

$$df_s(\text{section modulus of the steel section}) = u(\text{section modulus of surface}),$$

$$df_s(25) = u(117) \text{ for this special problem.}$$

This can be solved when the maximum increment of moment is known.

A disadvantage of the construction that is shown in Fig. 5-4 is the tendency of the I-beams to isolate the concrete into separate units. This may result in the destruction of the bond if the structure carries heavy moving loads and if the beams are relatively flexible so that the lateral distribution of the loads is poor. Furthermore, the wide, flat surfaces of the top flanges of the I-beams tend to cause cracks in the concrete above them due to weathering, if the structure is exposed.

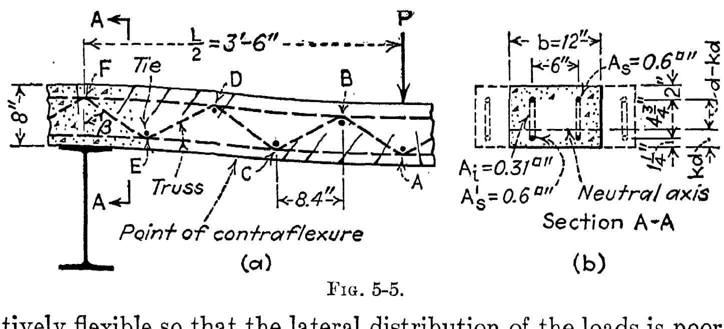


FIG. 5-5.

5-4. Fabricated Trusses in Concrete Slabs. Members that contain fabricated trusses as reinforcement for the concrete are classed herein as composite beams because these trusses are shop-fabricated units which serve the same general purpose as the I-beams of the previous article but which are designed to overcome the disadvantages of the latter. Such trusses are made in

various types. Some are merely welded reinforcing rods; some are made with chords which are composed of small structural shapes; still others are really expanded I-beams. In some cases, these trusses are made strong enough to act as supports for the forms and the wet concrete.

These trusses, as pictured in Fig. 5-5, have many practical advantages as reinforcement in concrete slabs and beams. Being fabricated units, they are easily handled in the field, especially when the members are continuous and when they require reinforcement for both positive and negative bending moments. The open-web systems eliminate the isolating effect of the webs of the I-beams, and they permit the use of effective transverse rods which thoroughly tie the structure together; the web members provide a certain amount of mechanical bond for the longitudinal reinforcement; and the diagonal members serve as an effective—partially, at least—system of web reinforcement.

This last feature may be clarified by an examination of Fig. 5-5(a). The diagonal members AB, CD, and EF function as inclined stirrups. However, BC and DE are not effective as anchors to withstand the longitudinal shearing forces which tend to break open the concrete below B and above E. They have some value as steel compression members which assist directly in carrying the vertical shear, but they should not be relied upon too much. The trouble can usually be remedied by inverting alternate trusses so that the directions of the web members of adjacent trusses will be reversed. By assuming that the trusses function in pairs, the web reinforcement generally will prove to be adequate even though the spacing or panel length exceeds that generally allowed for inclined stirrups—$\frac{3}{4}d$. Naturally, trusses with a double web system (X type) need not be alternated in this way.

Problem 5-2. Assume that a strip of the slab of Fig. 5-5 1 ft. wide carries a load P equal to 10,000 lb. at its center plus its own weight of 100 lb. per sq. ft.

For this problem, assume that the bending moment at the support is

$$M = \frac{PL}{8} \qquad \frac{wL^2}{12}$$

If n equals 8, and the details of the slab and the trusses are as shown in the illustration, find f_s, f_c, u and the stress in the diagonal EF, if alternate trusses

are shifted 8.4 in. and one diagonal rod is assumed to carry the entire stress for the 1-ft. strip with no allowance for the strength of the concrete in resisting longitudinal shear ($v'_L = 0$). Also, neglect any action of the truss as an independent member, because it is merely reinforcement in the concrete.

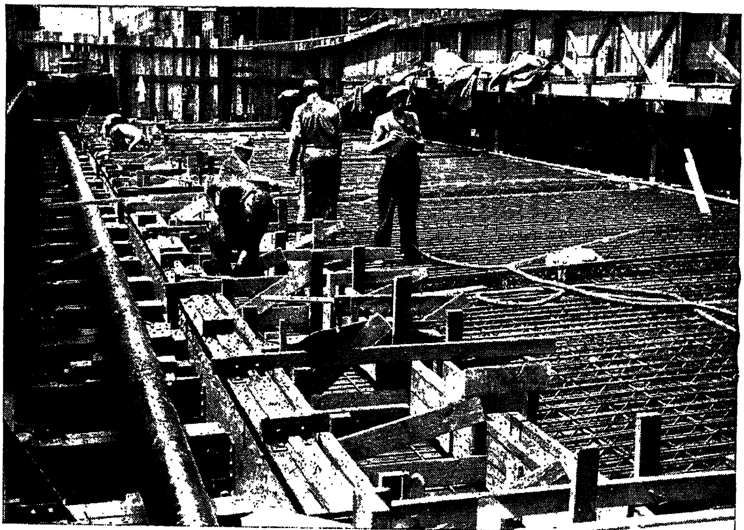


FIG. 5-6.—Trussed reinforcement in the roadway of the West Thirty-seventh St. Bridge over the New York approach to the Lincoln Tunnel.

$$M = \left(-\frac{10{,}000 \times 7}{8} - \frac{100 \times 7^2}{12}\right)12 = 110{,}000 \text{ in.-lb.}$$

$$nA_s = 8 \times 0.6 \times 2 = 9.6, \qquad (n-1)A'_s = 7 \times 1.2 = 8.4$$

$$\frac{12(kd)^2}{2} + 8.4(kd - 1.25) = 9.6(6 - kd)$$

$$kd = 2.19 \text{ in.,} \quad \text{and} \quad d - kd = 3.81 \text{ in.}$$

$$k = 0.36, \quad \text{and} \quad j = 0.88(\pm)$$

$$I_c = \frac{12 \times 2.19^3}{3} + 8.4 \times 0.94^2 + 9.6 \times 3.81^2 = 188.8 \text{ in.}^4$$

$$S_c = \frac{188.8}{2.19} = 86 \text{ in.}^3, \quad \text{and} \quad S_s = \frac{188.8}{8 \times 3.81} = 6.2 \text{ in.}^3$$

$$f_c = \frac{M}{S_c} = \frac{110{,}000}{86} = 1{,}280 \text{ lb. per sq. in.}$$

$$f_s = \frac{M}{S_s} = \frac{110{,}000}{6.2} = 17{,}700 \text{ lb. per sq. in.}$$

$$v_L = \frac{V}{bjd} = \frac{5{,}000 + 350}{12 \times 0.88 \times 6} = 84.4 \text{ lb. per sq. in.}$$

The tangent of β, the angle of inclination of EF, is

$$\tan \beta = \frac{8.4}{4.75} = 1.77 \qquad \text{or} \qquad \beta = 60°32'.$$

Therefore, the stress in this one inclined rod for the 12-in. strip is

$$A_i f_i = \frac{0.7 v_L b s}{\sin (45° + \beta)} = \frac{0.7 \times 84.4 \times 12 \times 8.4}{0.963}$$

or

$$f_i \quad \frac{0.7 \times 84.4 \times 12 \times 8.4}{0.31 \times 0.963} = 20{,}000 \text{ lb. per sq. in.}$$

The intensity of the transverse shear is

$$v_T = \frac{V}{bkd} = \frac{5{,}350}{12 \times 0.36 \times 6} = 206 \text{ lb. per sq. in.}$$

The maximum bond stress at the support, which is the point where it is the greatest for the longitudinal reinforcement, is, assuming the rods to be $\frac{7}{8}$ in. round,

$$u = \frac{v_L b}{\Sigma o} \quad \frac{84.4 \times 12}{2 \times 2.75} = 184 \text{ lb. per sq. in.}$$

This is just about the safe limit that is allowed for $n = 8$ in the Code, assuming deformed rods. Although the chords of the trusses are smooth, the mechanical bond of the web members where they are rigidly fastened to the longitudinal rods should permit the use of this maximum value.

As a matter of detail, these trusses can be spliced by locating the joint over a support, setting the trusses end to end or side by side. The web members need not be spliced in such a case, but separate rods can be wired to the chords so as to make up the necessary area and to allow the proper length for bond to develop the splice bars. In other cases, the trusses may be lapped at a floor beam.

Trusses will probably increase in use as their cost becomes less and as engineers become more accustomed to using them in their designs. The use of structural-steel supporting members in this illustration is taken from bridgework such as that of Fig. 5-6. However, trusses are just as advantageous in the case of similar structures that are made entirely of concrete.

Practice Problems

Problem 5-3. Assume that a composite beam similar to Fig. 5-1 is to be analyzed. The flange width of the T-beam $b = 66$ in.; its thickness $t = 8$

in.; the steel beam is 24 in. deep, weighs 80 lb. per ft., the flange width = 9 in., I = 2,230 in.^{4}, A = 23.54 sq. in., the section modulus = 185.8 in.^{3}; the width of the stem (encasement) = 13 in.; and the top of the beam is 3 in. below the top of the concrete slab. Find the section moduli S_c and S_s of the composite beam if $n = 10$. Find the safe resisting moment if the allowable f_c = 900 lb. per sq. in. and f_s = 18,000 lb. per sq. in.

Ans. S_c = 6,060 in.^{3}; S_s = 228 in.^{3}; M_s = 342,000 ft.-lb.

Problem 5-4. Find the critical longitudinal shearing stress along the top of the beam of Problem 5-3 if it is simply supported, has a span of 28 ft., and carries a total uniformly distributed load of 3,000 lb. per lin. ft., including the dead load.

Discussion. Find the end reaction, the static moment of the concrete outside the probable planes of failure (similar to Fig. 5-2), the longitudinal shear per inch of length ($S_L = VQ/I_c$), and the longitudinal shearing stress per square inch of surface.

Problem 5-5. Assume that the composite beam of Problem 5-3 has a span of 27 ft. and that it is simply supported. Assume that the forms are supported from the steelwork so that the dead load is carried by the steel beam alone. If the allowable f_s in the steel is 18,000 lb. per sq. in., find the uniformly distributed live load per linear foot that the composite beam will support.

Discussion. Find f_s for the dead-load bending moment for the steel beam alone; subtract it from 18,000 lb. per sq. in.; use the difference times S_s to find the available resisting moment for live loads; and then compute the uniform load that will cause this moment.

Ans. L.L. = 2,630 lb. per ft.

Problem 5-6. Recompute Problem 5-3 if b = 75 in., t = 8 in., and $n = 12$.

Problem 5-7. Compute the maximum longitudinal shearing stress at the top of the beam of Problem 5-6 if the beam is simply supported, has a span of 25 ft., and carries a total uniformly distributed load of 3,600 lb. per lin. ft.

Problem 5-8. Recompute the composite beam of Fig. 5-4 as in Art. 5-3, using exactly the same steel beams but a weaker concrete for which $n = 12$, f_c = 900 lb. per sq. in., and f_s = 18,000 lb. per sq. in. Find the safe resisting moment.

Problem 5-9. Find the safe negative bending moment for a floor slab that is reinforced as shown in Fig. 5-5, if $n = 12$. Assume that the allowable f_c = 900 lb. per sq. in. and f_s = 18,000 lb. per sq. in.

Discussion. Use the same dimensions, steel, etc., as in Fig. 5-5. Follow the procedure of Problem 5-2.

Problem 5-10. Find the safe positive bending moment for a floor slab like that of Fig. 5-5 if the total depth is 9 in. and the distance center to center of truss chords = 5.75 in. Otherwise, use all of the dimensions and the data that are given in the pictures. Let $n = 10$, the allowable f_c = 1,000 lb. per sq. in., and f_s = 20,000 lb. per sq. in.

Ans. M_c = 135,000 in.-lb.; M_s = 159,000 in.-lb.

CHAPTER 6

COLUMNS

6-1. Introduction. Members carrying direct axial loads which cause compressive stresses of such magnitude that these stresses largely control the design of the members may be included in the general classification called "columns." They may also be divided into two types, viz., "short" columns, the lengths of which are less than ten times their least lateral dimension; and "long" columns, the relative lengths of which exceed this limit. Columns may also be subjected to bending moments as well as to axial loads. Therefore, the foregoing definition is given as a general one to differentiate between a column that resists bending and a beam that carries a direct compressive load.

Because of the nature of the material, concrete columns are generally of the short-column type. Longitudinal steel rods are usually added to assist in carrying the direct loads; also, hoops and spirals serve the same general purpose; and sometimes structural-steel sections are considered to be a sort of glorified reinforcement.

Columns need not be vertical, but, to avoid confusion, it will be advisable to consider them so, using the term "strut" to describe inclined or horizontal compression members.

6-2. General Discussion of Reinforced-concrete Columns. A simple, square concrete column with eight vertical reinforcing rods is pictured in Fig. 6-1(a). Under the action of the direct compression, the concrete bulges out laterally, as shown in exaggerated manner in Fig. 6-1(b). From a consideration of Poisson's ratio, this is to be expected when a material is placed under compression. It is also obvious that the rods themselves are somewhat like very slender columns. Naturally, they tend to buckle, but they cannot bend inward against the concrete. Therefore, they will buckle in the line of least resistance, viz., away from the column's axis. This action causes tension in the outside shell of the concrete which, if the pressure becomes

sufficient, will crack open somewhat as shown, the failure usually being sudden.

The way to overcome this trouble seems to be obvious. If the column is a square one, as pictured in Fig. 6-1(c), the apparent

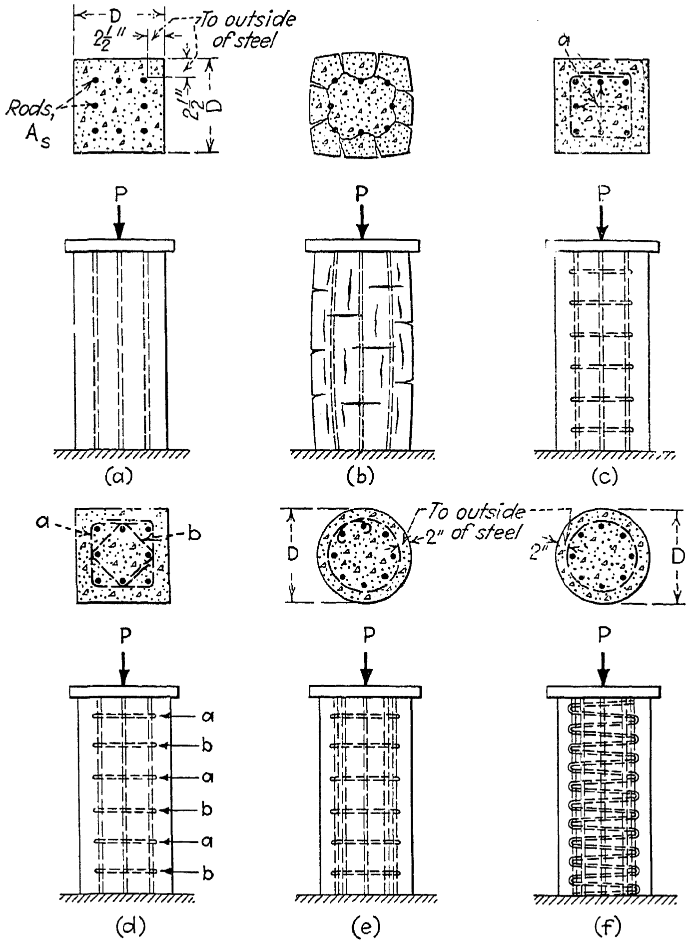


FIG. 6-1.

remedy is the placing of small rods around the longitudinal reinforcement, forming a series of bands or ties which are wired to the main rods and which are supposed to keep the latter from buckling, as well as to restrain the bulging action of the concrete.

However, if the upper view of this figure is examined carefully, it is seen that these bands are in the form of hollow squares. Therefore, when rods *a* try to bend sidewise, they exert a lateral force which is normal to the straight sides of the bands. The bulging of the concrete does likewise. However, the bands are not effective in withstanding such beam action without bending outward so much that the concrete may crack. It is therefore advisable to use two kinds of ties, placing them alternately so that one holds the corner rods while the other supports the intermediate ones, as shown in Fig. 6-1(*d*). This arrangement results in troublesome details, and it handicaps the "rodding," or compacting, of the concrete.

The next logical improvement seems to be the placing of the longitudinal rods in a circle, with hoops placed outside and wired to them as pictured in Fig. 6-1(*e*). The buckling tendency of the rods and the bulging of the inner portion of the concrete merely cause tensile stresses in these hoops, which means that they are really effective in restraining this action. However, the single hoops have to be spliced by lapping or by bending their ends around some of the main rods. This again is troublesome when a large number of hoops must be used.

The best way to support the longitudinal rods and the concrete is by means of spiral reinforcement as illustrated in Fig. 6-1(*f*). These spirals are merely long rods of small diameter which are bent around the main rods, forming a helix. In this way, small pieces are eliminated, the field work is decreased, and the waste of material in splices is avoided.

The arrangement of the main rods in a circular pattern with spiral reinforcement (or hoops) is advisable even when the cross section of the finished column must be square rather than round. Of course, the protective coating over the rods is needed to guard the steel against fire and corrosion, but the exterior surface can be shaped to suit the architectural requirements as long as a minimum cover of $1\frac{1}{2}$ or 2 in. is maintained over the outer surfaces of the rods.

It has been the general practice in the past to neglect the strength of the concrete covering that is outside the rods—the 2-in. layer shown in Figs. 6-1(*e*) and (*f*). This was done because this covering was assumed to be for protection against possible fire and because, in that event, it might spall off. Also, this

covering is outside the portion that could be restrained by the hoops or spiral reinforcement. The present tendency, however, seems to be to assume that the entire section of the concrete will participate in resisting the loads.

This assumption seems to be entirely logical because all of the material must be shortened when the member is compressed. Then all of it must also resist this deformation, especially within the range of ordinary working loads. There are differences of opinion regarding how much concrete outside of the hoops or spirals may be relied upon in design. Obviously, some practical limits must be set up. Since the Code states that, for columns which are poured monolithically with walls or piers, the effective gross section shall be assumed as that within $1\frac{1}{2}$ in. of the outsides of the spirals, this definition will be adopted herein, even though the thickness of the cover outside the spirals or hoops may exceed $1\frac{1}{2}$ in.

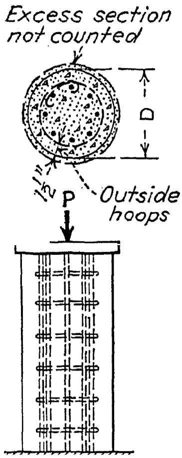


FIG. 6-2.

The arrangement of the reinforcement divides concrete columns into two general classes, viz., "tied" columns which have longitudinal rods with intermittent hoops; and "spirally reinforced" columns, which have longitudinal rods that are enclosed within steel spirals. The advisability of using rods of large diameter which will be stiff and strong as "little" columns is self-evident.

6-3. Design of Tied Columns. The design of short reinforced-concrete columns may be based upon the theory of the elasticity of the materials. If the load P is applied to the column of Fig. 6-1(a), it causes a corresponding shortening of the member. Both the concrete and the steel are squeezed down equally. If the stresses in them are below the elastic limit of the materials, the relation between the values of the unit stresses in the steel and in the concrete f_s and f_c would appear to be

$$f_s = \frac{E_s}{E_c} f_c = nf_c.$$

Therefore, the total load must equal the area of the two materials, each times its respective unit stress.

As an example, refer to Fig. 6-2. Let A_g equal the gross effective area of the column in square inches and p_g equal the ratio of the total area of the longitudinal rods to this gross effective area. Then the area of the steel is

$$A_s = p_g A_g. \tag{6-1}$$

The net area of the concrete $= A_g - A_s = A_g - p_g A_g = A_g(1 - p_g)$. Therefore,

$$P = A_s f_s + (A_g - A_s) f_c = p_g A_g f_s + A_g(1 - p_g) f_c.$$

$$\left.\begin{aligned} P &= p_g A_g n f_c + A_g(1 - p_g) f_c \\ P &= A_g f_c[1 + (n - 1)p_g] \end{aligned}\right\} \tag{6-2}$$

which gives the load in terms of the transformed section and of the stress in the concrete. This formula is based upon the elastic theory. In reality, however, it is not in accord with the recent ideas of some authorities.

Experience has shown that the action of a reinforced-concrete column under load seems to be rather peculiar. At first, the conditions are practically those given by Eq. (6-2); but if the load is large and if it is continued for a long time, plastic flow of the concrete (a tendency of the concrete to get out from under the load when the latter is continued for a long period) seems to take place, resulting in a decrease of the unit stress in the concrete but an increase of the stress in the steel because of the latter's inability to "get out from under" the load also. Thus, more of the burden of supporting the load tends to be shifted from the concrete into the rods. This action may continue until the stress in the steel reaches the yield point,[1] whereupon the rods deform appreciably without taking much further increase in stress, even though they cannot get rid of the stress already in them. Then, at this stage, the shifting of the load from the concrete to the steel must practically cease because the latter is so much more compressible than the former. If the load on the column is then increased, the steel will continue to have a stress which is at or a little above the yield point, but the concrete, by itself, must carry the increase of the load until it fails.

[1] Unless it is relieved by "bond creep" at the splices; see J. R. Shank, Bond Creep and Shrinkage Effects in Reinforced Concrete, A.C.I. *Jour.*, November, 1938.

According to these principles, a formula for the ultimate strength of a reinforced-concrete column might be

$$\left.\begin{aligned} P' &= (A_g - A_s)f'_c + A_s f'_s \\ P' &= A_g[(1 - p_g)f'_c + p_g f'_s]. \end{aligned}\right\} \tag{6-3}$$

In Eq. (6-3), P' = the ultimate load on the column and f'_s = the stress in the steel when at its yield point. Applying a suitable safety factor to these formulas gives those stated by the Code for the safe load P on tied columns:

$$\left.\begin{aligned} P &= 0.18f'_c A_g + 0.8A_s f_s \\ P &= A_g(0.18f'_c + 0.8f_s p_g) \end{aligned}\right\}$$

wherein f_s = 16,000 lb. per sq. in. for intermediate-grade steel and 20,000 lb. per sq. in. for hard grade. Theoretically, the area of the steel A_s should be deducted from A_g, but this minor effect has been considered in determining the coefficients of f'_c in Eq. (6-4).

A column that is designed in accordance with the elastic theory must have a larger cross section or more longitudinal reinforcement than it will require if it is designed to meet the Code, because the value of the second term of Eq. (6-2) is so much smaller than the corresponding one of Eq. (6-4). On the other hand, one must remember that both formulas are approximate. They are attempts to determine in advance what load a given column can support safely. A load test of the real member would give the correct result, but such a test cannot be made in most cases.

Furthermore, there are uncertainties which arise in the use of Eq. (6-4) because it merely sets a maximum safe load for a column that has a given size and one specific make-up. If a column is not fully loaded, what are the stresses in the concrete and in the steel? As the load is increased, does each stress vary directly from zero to the maximum which is set by Eq. (6-4)? It is doubtful if this is so, but from the standpoint of safety it is unimportant.

If Eq. (6-2) is used to compute the stresses in a column, one must remember that it will give concrete stresses that are likely to be higher than those which will actually occur, because of plastic flow. The computed stresses in the rods will be less than

the actual ones, but such computed stresses in the steel will generally appear to be low. For instance, assume a concrete for which $f'_c = 3{,}750$ lb. per sq. in. Then, according to the Code, $n = 30{,}000/f'_c = 8$. If $f_c = 0.18f'_c = 0.18 \times 3{,}750 = 675$ lb. per sq. in., then $f_s = nf_c = 8 \times 675 = 5{,}400$ lb. per sq. in. Even if the stress in the steel is far above this value, no harm should result, because steel is a splendid, reliable material and is well adapted for this particular use. Even when it is stressed above its yield point, it will compress sufficiently to shift the balance of the load on to the concrete. No serious consequences should come from this readjustment under the compressive forces. However, such a statement cannot be made when one refers to tensile stresses in reinforced concrete.

Of course, the ties in any column should be adequate to brace the rods. According to the Code, such ties must be at least $\frac{1}{4}$ in. in diameter. They must not be spaced over sixteen times the diameter of the longitudinal rods, forty-eight times the diameter of the ties, or the least dimension of the column.

The splices of the longitudinal rods are very important. They should be designed so as to allow for a high stress in the steel. Generally, it will be safe to lap them in the same manner and for the same unit stress as is done when the rods are subjected to the allowable tensile stresses. The rods must be far enough apart so that there will be at least $1\frac{1}{2}$ to 2 in. clear space between one set of lapped rods and the adjacent ones, because proper encasement of the steel is exceedingly important. Table 11 in the Appendix will be helpful in this connection. It also gives the specified limits for p_g.

The steps in the design of a column are as follows: (1) the determination of the quality of the concrete to be used (f'_c and n) and the allowable stress in the steel; (2) the assumption of a size and shape; (3) the assumption of p_g (or the number and size of rods); and, finally, the test of the safe load for the column by the use of Eq. (6-2) or (6-4). As illustration, the sample problems will be worked out for both the elastic theory and the Code.

Problem 6-1. Find the safe load for a short, tied column which is 12 in. in diameter and which has six $\frac{3}{4}$-in. round longitudinal rods. It also has hoops which are made of $\frac{1}{4}$-in. round rods, these hoops being 10 in. c.c. and having a cover of $1\frac{1}{2}$ in. of concrete. Assume that the ultimate strength

of the concrete = 2,500 lb. per sq. in., $n = 12$, the allowable working stress in the steel = 16,000 lb. per sq. in., and the allowable $f_c = 0.18f'_c$.

$$A_g = \frac{\pi D^2}{4} = \frac{\pi \times 12^2}{4} = 113 \text{ sq. in.}$$
$$A_s = 6 \times 0.44 = 2.64 \text{ sq. in.}$$
$$p_g = \frac{A_s}{A_g} = \frac{2.64}{113} = 0.0234.$$

The allowable $f_c = 0.18f'_c = 0.18 \times 2{,}500 = 450$ lb. per sq. in.
By the elastic theory,

$$P = A_g f_c[1 + (n - 1)p_g]$$
$$P = 113 \times 450[1 + (12 - 1)0.0234] = 64{,}000 \text{ lb.}$$
$$f_s = nf_c = 12 \times 450 = 5{,}400 \text{ lb. per sq. in.}$$

By the Code,

$$P = 0.18f'_c A_g + 0.8A_s f_s$$
$$P = 0.18 \times 2{,}500 \times 113 + 0.8 \times 2.64 \times 16{,}000 = 84{,}600 \text{ lb.}$$

The differences between the foregoing results are rather large, indicating again that the first method requires a larger column or more steel than the second one to carry the same load.

According to the limits for ties as stated previously, the diameter is the minimum—$\frac{1}{4}$ in. The spacing is 10 in., which is less than $16 \times \frac{3}{4} = 12$, or $48 \times \frac{1}{4} = 12$. They are therefore satisfactory.

6-4. Design of Spirally Reinforced Columns. When a column has reinforcement of the type that is pictured in Fig. 6-3(*a*), it is said to be "spirally reinforced." As previously stated, the concrete in such a member is supported more adequately than it is in the case of a tied column. Therefore, although the fundamental action of the materials is the same as that which was described in the preceding article, the concrete of a spirally reinforced column will have a much greater ultimate resistance to failure than that of the tied one. Its safe working load can be larger also.

The formula given by the Code for the safe load of spirally reinforced columns is

$$\left.\begin{aligned} P &= 0.225f'_c A_g + A_s f_s \\ P &= A_g(0.225f'_c + p_g f_s) \end{aligned}\right\}$$

wherein all symbols have the same meaning and value as for

Eq. (6-4) and the limiting safe loads will be 25 per cent larger than for tied columns having similar sections. Figure 13 in the

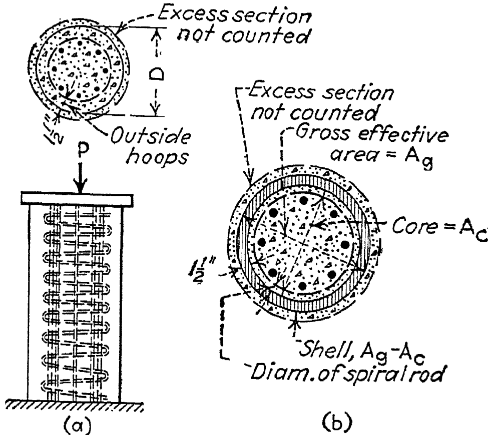


FIG. 6-3.

Appendix is useful in the practical design and analysis of spirally reinforced columns.

Of course, the increase of the safe load of the column as given in Eq. (6-5) over that in Eq. (6-4) is due to the additional strength of the concrete because of the supporting power of the spiral reinforcement. Therefore, the latter must be subjected to considerable stress. An expression for the design of such spirals, as given by the Code, is

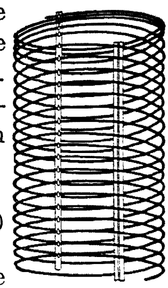


FIG. 6-4.

$$p' = \qquad (6\text{-}6)$$

where p' = volume of the spiral ÷ volume of the concrete core (out to out of spiral), f_s' = "useful limit stress" = 40,000 lb. per sq. in. for hot-rolled rods of intermediate grade or 60,000 lb. per sq. in. for cold-drawn wire, and R = gross area of the effective section ÷ area of the core = A_g/A_c. Equation (6-6) will be adopted herein. An approximate derivation of it might be made, but it is largely empirical.

Reinforced-concrete columns are made frequently with light longitudinal rods but with strong spiral reinforcement. Within reasonable limits, this is good construction, provided enough

longitudinal rods are used to hold the spiral sufficiently to prevent it from collapsing downward during the placing of the concrete, and provided that the column is not subjected to severe bending moments.

Figure 6-4 is a sketch of the arrangement of the steel in a spirally reinforced column.

Problem 6-2. By means of Eqs. (6-5) and (6-6), design a circular column to support a centrally applied load of 300,000 lb., including its own weight, using f'_c, f_s, and f'_s equal to 3,000, 16,000, and 40,000 lb. per sq. in., respectively.

The Code specifications require that p_g must be at least 0.01 but not more than 0.08. For the purposes of illustration, design this column first with $p_g = 0.06$ and then with $p_g = 0.015$.

First solution ($p_g = 0.06$):

Let D = the outside diameter of the column, and use the second form of Eq. (6-5).

$$P = A_g(0.225f'_c + p_gf_s)$$

$$300{,}000 = \frac{\pi D^2}{4}(0.225 \times 3{,}000 + 0.06 \times 16{,}000).$$

$D = 15.3$ in. (Say 16 in. in order to have column dimensions in full inches.)

$$A_g = \frac{\pi \times 16^2}{4} = 201 \text{ sq. in.}$$

$$A_s = p_gA_g = 0.06 \times 201 = 12.1 \text{ sq. in.}$$

Let d_1 = diameter of the spiral = $D - 3 = 13$ in. Then the core A_c is

$$A_c = \frac{\pi d_1^2}{4} = \frac{\pi \times 13^2}{4} = 133 \text{ sq. in.}$$

Allowing $1\frac{1}{2}$ in. for the thicknesses of spirals and main reinforcement, the circumference of the circle in which the longitudinal rods will be set inside the spiral $= \pi(d_1 - 1.5) = 3.14(13 - 1.5) = 36$ in. With 10 rods, the spacing will be $\frac{36}{10} = 3.6$ in., but the spacings and numbers of bars shown in Table 11 of the Appendix are preferable. The area of one rod must be $A_s/10 = \dfrac{12.1}{10} = 1.21$ sq. in. This requires $1\frac{1}{8}$-in. square bars, giving $A_s = 12.7$ sq. in. As the reader will realize, this reinforcing is very heavy, and the splicing of the rods will be difficult.

For the spiral,

$$p' = 0.45(R - 1)\frac{f'_c}{f'_s}$$

$$p' = 0.45\left(\frac{201}{133} - 1\right)\frac{3{,}000}{40{,}000} = 0.017.$$

The required volume of the spiral in 1 ft. of column $= 12A_cp'$ cu. in.

$$12 \times 133 \times 0.017 = 27.1 \text{ cu. in.}$$

Assuming a pitch of $1\frac{3}{4}$ in., the length of spiral in 1 ft. of column $= \dfrac{\pi \times 13 \times 12}{1.75} = 280$ in. The cross section of the spiral rod $= 27.1/280 = 0.1$ sq. in. Use $\frac{3}{8}$-in. round rods with about two extra turns at each end.

These computations for sizes of spirals seldom need to be made in practice. Using Fig. 14 of the Appendix, a satisfactory size and spacing can be obtained for the given values of p' and the core diameter. Table 9 also gives similar data for various core sizes and strengths of concrete.

Second solution ($p_g = 0.015$):

$$300{,}000 = \frac{\pi D^2}{4}(0.225 \times 3{,}000 + 0.015 \times 16{,}000)$$

$$D = 20.4 \text{ in. (use 21 in.)}$$

$$A = \frac{\pi \times 21^2}{4} = 346 \text{ sq. in.}$$

$$A_s = 346 \times 0.015 = 5.2 \text{ sq. in.}$$

$$d_1 = 21 - 3 = 18 \text{ in.} \qquad \text{and} \qquad A_c = \frac{\pi \times 18^2}{4} = 254 \text{ sq. in.}$$

$$\pi(d_1 - 1.5) = \pi(18 - 1.5) = 52 \text{ in. circumference.}$$

Try nine rods at $\frac{52}{9} = 5.8$ in. c.c. (approx.). The area of one rod $= 5.2/9 = 0.58$ sq. in. Therefore, use nine $\frac{7}{8}$-in. round rods, giving $A_s = 5.4$ sq. in.

Table 9 in the Appendix shows that $\frac{3}{8}$-in. round spiral rods with a 2-in. pitch will be satisfactory.

This second solution seems to result in a column that is more reasonably proportioned than the first. The extra concrete in the larger column is $\pi/4(21^2 - 16^2) \div 144 = 1$ cu. ft. per ft. of column. This probably offsets the cost of the additional steel in the smaller one. Architectural considerations are likely to govern the choice of sizes to be used.

Redesign by Elastic Theory. For the sake of comparison, the second case will be recomputed upon the basis of the elastic theory. From Eq. (6-2), using $f_c = 0.225f'_c$ and $n = 10$,

$$P = A_g f_c[1 + (n - 1)p_g]$$

$$p_g = 0.015, \text{ and } f_c = 0.225f'_c = 675 \text{ lb. per sq. in.}$$

$$300{,}000 = \frac{\pi D^2}{4}(675)[1 + (10 - 1)0.015]$$

$$D = 22.3 \text{ in.} \quad \text{(Use 22.5 in.)}$$

$$A_s = 398 \times 0.015 = 5.97 \text{ sq. in.}$$

$$22.5 - 3 = 19.5 \text{ in.,} \qquad \text{and} \qquad A_c = \frac{\pi \times 19.5^2}{4} = 229 \text{ sq. in.}$$

$\pi(d_1 - 1.5) = \pi(19.5 - 1.5) = 56.5$ in. circumference for the circle of longitudinal rods. Therefore, use 10 rods at $56.5/10 = 5.6$ in. c.c. (approx.).

The area of one longitudinal rod $= 5.97/10 = 0.597$ sq. in. Therefore, use ten $\frac{7}{8}$-in. round rods. The theoretical stress in the steel is

$$f_s = nf_c = 10 \times 675 = 6{,}750 \text{ lb. per sq. in.}(\pm).$$

The spiral will not be recomputed. However, the column is found to have a required diameter about $1\frac{1}{2}$ in. larger than that when calculated by the Code formula. It also has one more rod.

6-5. Composite Columns. The term "composite column" is used to denote a structural-steel column—or sometimes a cast-iron one—which is thoroughly encased in concrete that often has longitudinal reinforcement and which must have adequate spirals or hoops. Such members are often encountered in the construction of large buildings. The cross section of the concrete is generally large, and it can be relied upon to assist the steel in resisting the applied load.

This classification should not include the ordinary H-column which is encased in the minimum amount of concrete that can be used for the purpose of fire protection as pictured in Fig. 6-5. Such steel members are generally heavy; the strength of the concrete is insignificant in comparison to that of the steel; and the encasement is poorly restrained. Therefore, composite columns will be considered to be those in which the strength of the concrete is really substantial. The Code limits the area of the steel section of composite columns to 20 per cent of the total area of the column. Those like Fig. 6-5 will be called combination columns, and they will be discussed in the next article.

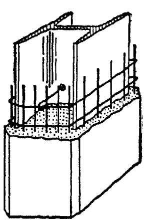


FIG. 6-5.

Figure 6-6 shows a composite column that was investigated in connection with the construction of the George Washington Bridge at New York City. The longitudinal rods were very small compared to the structural-steel member and the concrete. However, the concrete is held very well by the bands, by the outside angles of the steel member, and by the internal diaphragms. It is easy to see that the total strength of such a column is made up of three parts, viz., the strength of the structural steel, that of the longitudinal rods, and that of the concrete. Therefore,

the ultimate strength of the column might be assumed to be

$$P' = f'_c A_c + f'_s A_s + f'_s A_r \tag{6-7}$$

where A_c is the net area of the concrete in square inches, and A_s and A_r equal the cross-sectional areas in square inches of the longitudinal rods and the structural steel, respectively. The permissible thickness of shell outside the ties or spirals should be

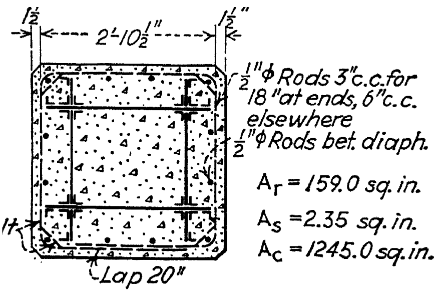


FIG. 6-6.

limited as for other columns. Preferably there should be 3 in. of concrete between the hoops or spirals and the steel core—of course circular spirals are far better than hoops.

On the basis of the elastic theory, a steel composite column may be designed by using Eq. (6-2) but p_g must be

However, the steel portion is relatively large, and it is not safe to let it be overloaded too severely. If f'_c = 3,000 lb. per sq. in., $n = 10$, and f'_s = 36,000 lb. per sq. in., the stress in the concrete when the steel is at its yield point will be $f'_s \div n = 36{,}000/10 =$ 3,600 lb. per sq. in. It will be seen that the concrete will probably fail before the steel does, unless the former is relieved by plastic flow. Even with a working stress of f_s = 16,000 lb. per sq. in. in the steel the stress in the concrete would appear to be

$$f_c = \frac{f_s}{n} = \frac{16,}{10}$$

This is too great a working stress for a concrete of the stated strength.

The Code has established a formula for the safe load on such a column. It is

$$P = 0.225A_cf'_c + f_sA_s + f_rA_r. \tag{6-8}$$

The value of f_r, the safe working stress in the structural steel, is the same as f_s in the rods, viz., 16,000 lb. per sq. in. for intermediate grade. Of course, the magnitude of f_r is less in the case of a cast-iron section (10,000 lb. per sq. in.).

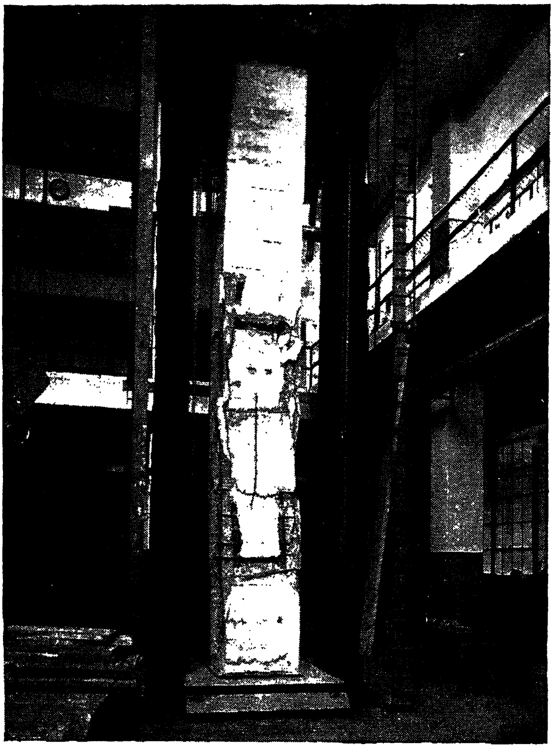


FIG. 6-7.—Concrete-encased model of tower columns of the George Washington Bridge at the Bureau of Standards, Washington, D.C.

Equation (6-8) can be used as a limiting formula. It allots the same unit stress to the concrete as Eq. (6-5) does in a spirally reinforced column. However, Eq. (6-8) is empirical, and it does not enable one to find the unit stresses in a column that is not loaded to its capacity, but the important question is whether or not the column is safe.

Figure 6-7 is a picture of one of the columns[1] that is shown in Fig. 6-6 after it had been tested to failure by the Bureau of Standards at Washington, D.C. This gives an idea of the size of the member. Figure 6-8 is a view of one end of another specimen

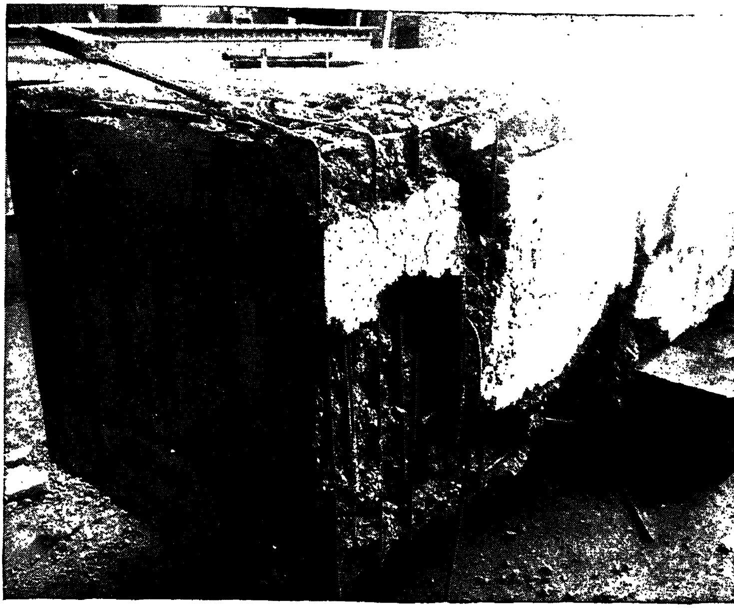


FIG. 6-8.—Composite column tested by the Bureau of Standards, Washington, D.C.

after it had been removed from the testing machine. The steel section is clearly visible.

Using Eq. (6-8) with f'_c = 3,000 lb. per sq. in., $f_s = f_r$ = 16,000 lb. per sq. in., and the other data as given in Fig. 6-6,

$$P = 0.225 \times 1{,}245 \times 3{,}000 + 16{,}000 \times 2.35 + 16{,}000 \times 159$$
$$P = 3{,}422{,}000 \text{ lb.}$$

Actually, the first vertical cracks in the concrete seem to have appeared when the measured values were as follows:

P = 3,500,000 lb.
f_c = 1,000 lb. per sq. in. (compared to 0.225 × 3,000 = 675 lb. per sq. in.)
f_s = 14,000 lb. per sq. in. (compared to 16,000 lb. per sq. in.).

[1] Stang and Whittmore, U.S. Department of Commerce, National Bureau of Standards, *Research Paper* R P 831, September, 1935.

This, in general, checks Eq. (6-8) very well as far as the total load is concerned.

A review of the results seems to indicate that the stresses in the materials actually were part way between the values that would be computed from Eqs. (6-2) and (6-8), but the lapse of a longer time for the column under load might modify these results.

The ultimate strength of the column was found to be 8,314,000 lb., with a corresponding value of $f_s' = 50{,}400$ lb. per sq. in. When this unit stress is multiplied by A_r, it gives

$$A_r f_r = 159 \times 50{,}400 = 8{,}030{,}000 \text{ lb.}$$

This, together with the appearance of the specimen in Fig. 6-8, shows that portions of the concrete failed almost completely, as should be expected.

On the other hand, the concrete encasement increased the strength of the column over that of the structural-steel member alone. Tests of similar bare steel members showed that they failed at an ultimate load of 5,853,000 lb. The increase of strength contributed by the concrete was therefore

$$8{,}314{,}000 - 5{,}853{,}000 = 2{,}461{,}000 \text{ lb., or 42 per cent.}$$

Of course, the structural-steel portions of composite columns must be spliced without relying upon the concrete. Also, base plates must be provided at the bottoms of such members.

The student should examine Figs. 6-7 and 6-8 carefully because they are extreme illustrations of what takes place in columns that are loaded to failure. The general buckling of the member, the bending and bursting of the ties, the buckling of the longitudinal rods, the spalling and crushing of the concrete—all of these are revealed in the pictures.

6-6. Combination Columns. The Code states that the allowable load on a heavy structural steel column like that of Fig. 6-5 should be computed as follows:

$$P = A_r f_r' \left(1 + \frac{A_c}{100 A_r}\right), \qquad (6\text{-}9)$$

where A_r = cross-sectional area of the steel column, f_r' = allowable stress for the unencased steel column, and $A_c = A_g - A_r$.

There may be considerable question as to when the encasement can be assumed to assist in carrying loads and when it is merely fire protection. To be in the former classification, the Code sets a minimum cover of $2\frac{1}{2}$ in. over the steel with adequate wire mesh—No. 10 gauge wires and 4×8 spacing—about 1 in. from the face of the concrete. However, a designer must judge each case to determine the advisability of relying upon the encasement.

Problem 6-3. Find the safe load on a column like that of Fig. 6-5 if the steel section is a 14*WF* 202-lb. H-section and the concrete is 22 in. square with 1-in. chamfers. Assume that the allowable unit stress in the bare steel section is 13.8 kips per sq. in. $A_r = 59.39$ sq. in.

$$A_c = 22^2 - 4 \times \tfrac{1}{2} - 59.39 = 423 \text{ sq. in.}$$

$$P = 59.39 \times 13.8\left(1 + \frac{423}{100 \times 59.39}\right) = 878 \text{ kips.}$$

6-7. Pipe Columns Filled with Concrete. Pipes may sometimes be used as columns; if filled with concrete they may be even better for that purpose. However, except as end-bearing piles, their use is rather restricted. In effect, the empirical formula given by the Code for the calculation of the allowable load on such columns is

$$P = 0.225\, f'_c A_c + \left(18{,}000 - 70\frac{h}{R}\right)\frac{f'_s A_r}{45{,}000} \qquad (6\text{-}10)$$

where A_r = area of pipe steel, h = unsupported height of column, R = radius of gyration of steel pipe, f'_s = tensile yield point of pipe material, and f'_c must equal at least 2,500 lb. per sq. in. The coefficient of A_r is an expression for the allowable unit stress in the steel pipe.

Problem 6-4. Assume a 12-in. I.D. steel pipe, $\frac{3}{8}$ in. thick, with a height of 18 ft. Let f'_c and f'_s = 3,000 and 33,000 lb. per sq. in., respectively. $A_r = 14.38$ sq. in., $R = 4.38$ in.; $A_c = 113.1$ sq. in.

$$P = 0.225 \times 3{,}000 \times 113.1 + \left(18{,}000 - \frac{70 \times 18 \times 12}{4.38}\right)\frac{33{,}000 \times 14.38}{45{,}000}$$
$$= 230{,}000 \text{ lb.}$$

6-8. Long Columns. When the unsupported length of a column exceeds ten times its least lateral dimension, it is arbitrarily classed as a "long" column. Members of such slender proportions are not used frequently in ordinary reinforced-concrete construction. Of course it is obvious that their slenderness increases the possibility that they may buckle under load in the general manner that is illustrated in Fig. 6-7. Therefore, special provision must be made to increase their strength above that which is required for a short column that carries the same axial load.

Lateral buckling causes a bending moment in the column. This moment acts simultaneously with the direct load, thus

tending to cause excessive compressive stresses in one side of the member and tensile stresses in the other. Inasmuch as the column is not composed of one homogeneous material which can resist large tensile stresses as well as compressive ones, an exact theoretical determination of the stress condition in the member due to the tendency to buckle is practically impossible. Therefore, one had better rely upon empirical data which are based upon tests and study by experts.

The recommendations of the Code will be adopted. It gives the following formula for the safe load on long columns:

$$P' = P\left(1.3 - 0.03\frac{h}{d}\right) \tag{6-11}$$

where P' is the maximum permissible axial load on the long column, P is the maximum axial load for the same column if it

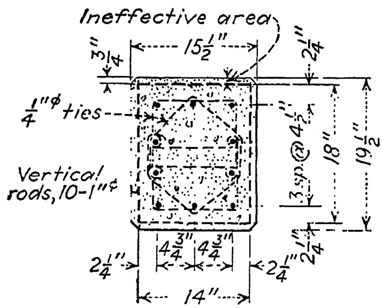


FIG. 6-9.

is "short," h is the unsupported length, and d equals the least lateral dimension of the member. Equation (6-11) applies whether the column is designed in accordance with the formulas of this chapter for direct load only or in accordance with those of the following one which include bending with the axial load. Especially in the latter case, h/d must not exceed 20.

Problem 6-5. Find the safe load for the rectangular tied column which is shown in Fig. 6-9. Use the Code [Eq. (6-4)], and assume a length of 11 ft.; then use Eq. (6-11) with an assumed length of 20 ft. Use f'_c = 3,000 lb. per sq. in. and f_s = 16,000 lb. per sq. in.

The cover over the rods is 3 in., but allow here only $1\frac{1}{2}$ in. beyond the outside of the ties in the computations for the effective area. Therefore, since the main bars are 1 in. round, $\frac{1}{2} + \frac{1}{4} + 1\frac{1}{2} = 2\frac{1}{4}$ in. beyond the centers

of the longitudinal rods is the limit for this area. Then

$$A_g = 14 \times 18 = 252 \text{ sq. in.}$$
$$p_g = \frac{A_s}{A_g} = \frac{10 \times 0.79}{252} = 0.0313.$$

For the short column, $h/d = 11 \times 12/15.5 = 8.5$. It seems to be satisfactory to use the gross width rather than the net width for d when the difference is not too great.

$$P = A_g(0.18f'_c + 0.8f_s p_g)$$
$$P = 252(0.18 \times 3{,}000 + 0.8 \times 16{,}000 \times 0.0313) = 237{,}000 \text{ lb.}$$

For the long column,

$$P' = P\left(1.3 - 0.03\frac{h}{d}\right)$$
$$P' = 237{,}000\left(1.3 - 0.03 \times \frac{20 \times 12}{15.5}\right) = 198{,}000 \text{ lb.}$$

The length of this column therefore reduces the safe load to $\frac{198}{237} \times 100 = 83$ per cent of its magnitude for a similar short column.

Practice Problems

Problem 6-6. Find the safe load for a short tied column having an outside diameter of 15 in. and five $\frac{7}{8}$-in. round longitudinal rods. The ties are $\frac{1}{4}$-in. round hoops 12 in. c.c., with a cover of $1\frac{1}{2}$ in. of concrete. Assume $f_c = 540$ lb. per sq. in. and $n = 10$. Use the elastic theory [Eq. (6-2)]. What is the stress in the steel?

Problem 6-7. Compute the safe load for the column of Problem 6-6 using the Code [Eq. (6-4)]. Let $f'_c = 3{,}000$ lb. per sq. in. and $f_s = 18{,}000$ lb. per sq. in. Compare the results of the two solutions.

Problem 6-8. Design a square, short, tied column to carry a load $P = 500{,}000$ lb. Assume $f_c = 0.18f'_c$, $f'_c = 3{,}000$ lb. per sq. in., $n = 10$, and p_g = about 3 per cent. Detail the column. Use the elastic theory.

Problem 6-9. Design a square, short, tied column to support a load $P = 700{,}000$ lb., using Eq. (6-4). Let $f'_c = 3{,}500$ lb. per sq. in., $f_s = 18{,}000$ lb. per sq. in., and p_g = about 2 per cent. Detail the column.

Problem 6-10. Find the safe load on a spirally reinforced column whose outside diameter = 24 in., using the elastic theory [Eq. (6-2)], if $p_g = 2$ per cent, $f_c = 675$ lb. per sq. in., and $n = 10$. The spirals are assumed to have a cover of $1\frac{1}{2}$ in. Design the spiral reinforcement for this column. $f'_c = 3{,}000$ lb. per sq. in.; $f'_s = 40{,}000$ lb. per sq. in.

Problem 6-11. Find the safe load for the column of Problem 6-10 if p_g is increased to 5 per cent, all other data remaining unchanged, using Eq. (6-2).

Problem 6-12. By the Code [Eq. (6-5)], design a circular, spirally reinforced column to support a centrally applied load of 380,000 lb., using

$f'_c = 2{,}500$ lb. per sq. in., $f'_s = 40{,}000$ lb. per sq. in., and $f_s = 18{,}000$ lb. per sq. in. Use p_g = about 3 per cent. Design the spiral reinforcement for this column. Compare with Fig. 13 in the Appendix.

Problem 6-13. Find the safe load for a tied column 25 ft. high and 20 by 24 in. in cross section, using the Code [Eqs. (6-4) and (6-11)]. Let $f'_c =$ 3,000 lb. per sq. in. and $f_s = 18{,}000$ lb. per sq. in. There are eighteen $\frac{7}{8}$-in. round longitudinal rods. The ties have a cover of $1\frac{1}{2}$ in. of concrete.

Problem 6-14. Find the safe load for a spirally reinforced column 20 ft. high and 18 in. in diameter if $f'_c = 3{,}500$ lb. per sq. in., $f_s = 18{,}000$ lb. per sq. in., and the reinforcement = ten $\frac{7}{8}$-in. round rods. Use Eqs. (6-5) and (6-11), and assume that the spiral has $1\frac{1}{2}$ in. of cover.

Problem 6-15. Design a square, tied column to support a load of 400,000 lb. if its height is 26 ft., f'_c and f_s = 3,000 and 16,000 lb. per sq. in., respectively. Use the Code [Eq. (6-4)], also Eq. (6-11) if necessary. Use a cover of 2 in. over the ties. Choose sizes and spacing of main rods and ties.

Discussion. From Fig. 13 in the Appendix, with $f'_c = 3{,}000$, find a trial A_g and width, first dividing P by 0.8 and choosing a tentative value for p_g—say 0.02. Test to see if Eq. (6-11) affects the case. Add more area if it seems to be needed. Analyze the trial section, then modify the design until it is satisfactory. Remember to have $\frac{1}{2}$ in. more cover than the minimum $1\frac{1}{2}$ in.

Ans. One satisfactory column is 24 in. square over all with twelve 1-in. square rods and $\frac{3}{8}$-in. round ties 16 in. c.c. similar to Fig. 6-1(d).

Problem 6-16. A spirally reinforced column is made 30 in. square for architectural reasons. The diameter of the core is 26 in.; the longitudinal reinforcement consists of 12 $1\frac{1}{8}$-in. square rods; the spiral is made of a $\frac{1}{2}$-in. round rod with a pitch of $2\frac{1}{2}$ in.; the length of the column is 18 ft.; and the allowable f'_c and f_s are 3,750 and 16,000 lb. per sq. in., respectively. The column is supposed to carry a load of 850,000 lb. Is it safe according to Eqs. (6-5) and (6-11)? If not, what can be done to make it safe?

Ans. Not safe because allowable P = 800,000 lb. Add 4 more $1\frac{1}{8}$-in. sq. rods or make section 2 in. larger.

CHAPTER 7

COMBINED BENDING AND DIRECT STRESSES

7-1. Introduction. In ordinary construction, there are many cases in which members are subjected to a combination of bending moments and direct, axial loads. Generally, in reinforced-concrete work, the direct load in such combinations is a compressive force. Lateral earth pressures which act upon subway and foundation walls, columns to which beams are connected eccentrically, frame action between beams and columns whereby the deflections of the former compel the latter to bend also because of the rigidity of the connections, wind loads which force the columns of a building to bend sidewise—all of these are ordinary causes of combined compressive and flexural stresses in the members that are affected by them.

In other cases, members are subjected to combined tension and bending. This is most frequently the result of a decrease in temperature that tends to shorten beams with fixed or restrained ends which resist such shortening. Occasionally, hangers and parts of reinforced-concrete trusses are designed to withstand tensile and flexural stresses.

The designer must not disregard these combined stresses. He must visualize what forces will exist and what deformations will occur in any proposed structure. He must find a way of designing the members to withstand the combined forces, doing so with reasonable accuracy and without undue labor.

Problems involving compression and bending generally come into one of two classes: the first includes those members which have compression upon all of their cross section; the second covers those which have compression upon part of the section and tension upon the remainder. The former case, being more simple, will be considered first, but, in all cases, it is the purpose of this chapter to explain and illustrate the fundamental theory rather than to devise short-cut methods.

7-2. Construction Procedure Affecting Columns. In attacking the problem of combined bending and compression, it is

advisable first to analyze the probable construction conditions in so far as they affect the design. Because of frame action, the maximum bending moment in a column will occur at the top of the floor or at the bottoms of the beams that frame into the column. In the case of lateral pressure against a column, there may be bending at the ends as well as near the middle of the member.

However, as pictured in Fig. 7-1, in ordinary building construction it is customary to place the concrete in the column forms

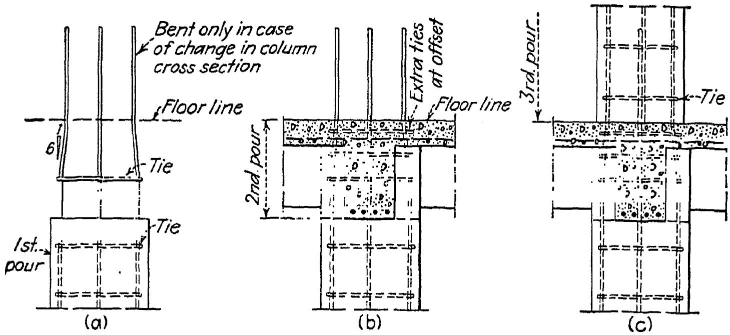


FIG. 7-1.—Sequence of construction of columns and floor system.

after the lower floor has been constructed. Then the concrete of the column is carried up to, or nearly to, the bottoms of the beams of the next higher floor. The concrete of the beams and the floor is poured next; then that of the column in the next story follows. In any case, each stopping point causes a plane of weakness because the concrete of one "pour" is permitted to set before that of the following one is deposited. The bond between the two pours is much weaker than the bond between portions of a monolithic mass of concrete. Such a plane of division between pours is called a "construction joint." Figure 7-1 shows that these joints generally occur at the points of maximum bending in the columns.

A construction joint may be roughened or keyed to lock the two sections together to resist shear. The full compressive strength can be relied upon, but the concrete at this point is much weaker than usual in its resistance to tension. Therefore, it seems reasonable to assume that the concrete of ordinary columns cannot withstand tensile stresses at the critical points.

7-3. Combined Compression and Bending without Resulting Tension upon the Section. Let $ABCD$ (Fig. 7-2) represent a short piece of a square column. Let P equal the direct load and M equal the bending moment, whatever its cause may be. Then $CEFD$ represents the pressure diagram for the direct load if it is uniformly distributed, A_t being the transformed area in terms of the concrete. $A_t = A_g + (n - 1)A_s$. The figure $CDHG$ pictures the diagram for the internal stresses which resist the bending moment M, where I_t is the moment of inertia of the transformed section. $I_t = I_c + (n - 1)I_s$. Then $CKLD$ is the diagram for the combined stresses. Of course there is no true neutral axis or point of zero stress within the section of the column.

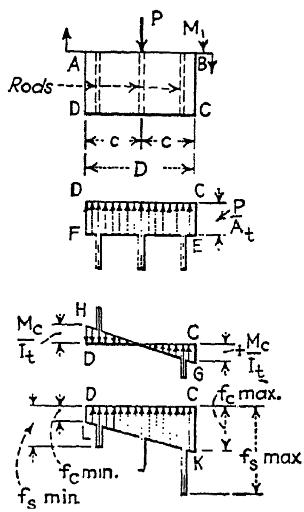


FIG. 7-2.

It must be realized that a tensile stress like HD in Fig. 7-2 annuls an equal compressive stress, but the effect is unchanged as far as bending is concerned. Furthermore, the rods are assumed to be conveniently symmetrical in this case, and plastic flow is neglected so that the diagrams may be assumed to represent the stresses in the concrete and in the steel in accordance with the elastic properties of the materials. Therefore, from Fig. 7-2,

$$CK = \max. f_c = \frac{P}{A_t} + \frac{Mc}{I_t} \tag{7-1}$$

where $c = D/2$—from the center of gravity axis.

This may be expressed in another form. Let R equal the radius of gyration of the transformed section. Then $R^2 = I_t/A_t$. Furthermore, let $M = Pe$, where e is the eccentricity of the direct load with respect to the axis of the column. In the cases where P acts at the center of the column but an external bending moment is applied, the latter can always be treated in the analysis as though $M = Pe$, so that $e = M/P$. Substituting $M = Pe$ and $I_t = R^2A_t$ in Eq. (7-1) yields the following:

$$\text{Max. } f_c = \frac{P}{A_t} + \frac{Pec}{I_t} = \frac{P}{A_t}\left(1 + \frac{ec}{R^2}\right).$$

However, $A_t = A_g[1 + (n - 1)p_g]$. Therefore,

$$\text{Max. } f_c = \frac{P\left(1 + \frac{ec}{R^2}\right)}{A_g[1 + (n - 1)p_g]}. \tag{7-2}$$

Correspondingly,

$$\text{Min. } f_c = \frac{P\left(1 - \frac{ec}{R^2}\right)}{A_g[1 + (n - 1)p_g]}. \tag{7-3}$$

Equations (7-2) and (7-3) are based upon the elastic theory. They can be used for preliminary design purposes. The simultaneous stresses in the steel are theoretically n times the stress in the concrete at the location of the steel. They can be found by proportion from the trapezoidal diagram $DCKL$ of Fig. 7-2.

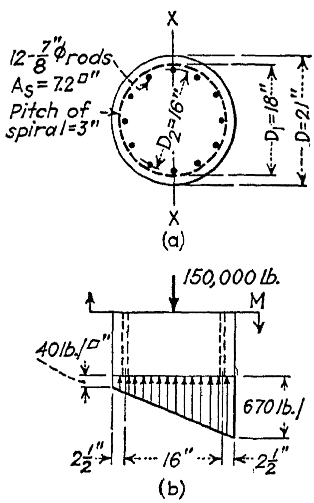


FIG. 7-3.

On the other hand, the Code limits the allowable compressive stress in such members to

$$f_c = f_a\left[\frac{1 + \frac{ec}{R^2}}{1 + C\left(\frac{ec}{R^2}\right)}\right] \tag{7-4}$$

where $C = f_a/0.45f'_c$. The term f_a is the average permissible unit stress on an equivalent axially loaded column:

For spiral columns:

$$f_a = \frac{0.225f'_c + f_s p_g}{1 + (n - 1)p_g}. \tag{7-5}$$

For tied columns:

$$f_a = \frac{0.18f'_c + 0.8f_s p_g}{1 + (n - 1)p_g}. \tag{7-5a}$$

The symbols have the same meanings as the similar ones in Eqs. (6-4) and (6-5).

Problem 7-1. If the spirally reinforced concrete column shown in Fig. 7-3 carries a centrally applied load of 150,000 lb. and a bending moment of 360,000 in.-lb., acting about the axis X-X, find the maximum and minimum values of f_c and f_s, using $n = 12$.

Use Eqs. (7-2) and (7-3). Also, in order to avoid undue labor in the calculations, it is satisfactory to replace the longitudinal rods with an annular ring having the same area as the entire group of rods and having a mean diameter equal to that of the circle in which the rods are set. Call this mean diameter D_2. This substitution of a ring of steel for the individual rods will cause no appreciable error unless there are only a very few rods in the column. Then

$$A_t = A_g[1 + (n - 1)p_g] = \frac{\pi \times D^2}{4} + (n - 1)A_s = \frac{\pi \times 21^2}{4} + 11 \times 7.2 =$$
$$346 + 79 = 425 \text{ sq. in.}$$

$$I_t = \frac{\pi D^4}{64} + \frac{(n - 1)A_s D_2^2}{8} = \frac{\pi \times 21^4}{64} + \frac{11 \times 7.2 \times 16^2}{8} = 9{,}540 + 2{,}530$$
$$= 12{,}070 \text{ in.}^4$$

since I for an annular ring is

$$\frac{\pi}{64}(D_a^4 - D_b^4) = \frac{\pi(D_a^2 - D_b^2)}{4}\frac{(D_a^2 + D_b^2)}{16} = \frac{\text{Area} \times D_2^2}{8} \text{ (approx.)}.$$

$$R^2 = \frac{I_t}{A_t} = \frac{12{,}070}{425} = 28.4$$

$$p_g = \frac{A_s}{A_g} = \frac{7.2}{346} = 0.0208$$

$$e = \frac{M}{P} = \frac{360{,}000}{150{,}000} = 2.4 \text{ in.}$$

Since $c = 10.5$ in.,

$$\text{Max. } f_c = \frac{P\left(1 + \frac{ec}{R^2}\right)}{A_g[1 + (n - 1)p_g]} = \frac{150{,}000\left(1 + \frac{2.4 \times 10.5}{28.4}\right)}{425} =$$
$$670 \text{ lb. per sq. in.}$$

Then, from Eq. (7-3),

$$\text{Min. } f_c = \frac{150{,}000\left(1 - \frac{2.4 \times 10.5}{28.4}\right)}{425} = 40 \text{ lb. per sq. in.}$$

The steel stress at the locations of the extreme rods is found from Fig. 7-3(b) as follows:

$$\text{Max. } f_s = nf_c = 12\left(40 + \frac{630 \times 18.5}{21}\right) = 7{,}140 \text{ lb. per sq. in.}$$

$$\text{Min. } f_s = nf_c = 12\left(40 + \frac{630 \times 2.5}{21}\right) = 1{,}380 \text{ lb. per sq. in.}$$

A checkup with Eqs. (7-4) and (7-5), using $f_c' = 2{,}500$ lb. per sq. in. and $f_s = 16{,}000$ lb. per sq. in., gives

$$f_a = \frac{0.225 \times 2{,}500 + 16{,}000 \times 0.0208}{1 + 11 \times 0.0208} = 730 \text{ lb. per sq. in.}$$

$$C = \frac{730}{0.45 \times 2{,}500} = 0.650$$

$$f_c = 730\left(\frac{1 + \dfrac{2.4 \times 10.5}{28.4}}{1 + 0.650 \times \dfrac{2.4 \times 10.5}{28.4}}\right) = 870 \text{ lb. per sq. in.}$$

Since the permissible value of f_c exceeds 670 lb. per sq. in., the latter will be called satisfactory.

7-4. Combined Compression and Bending with Resulting Tension upon the Section. This type of problem should be divided into several separate cases in order to illustrate each one clearly. However, each one will be based upon the following fundamental principles and assumptions:

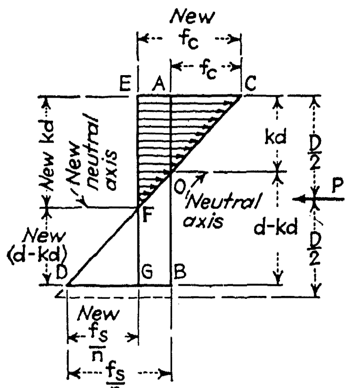


FIG. 7-4.

1. Regardless of the existence of the direct compressive load, the intensities of the stresses upon a cross section of the member are assumed to vary as the ordinates to a straight line.

2. The summation of all of the axial loads and the internal stresses upon the section must be zero in order to have equilibrium. Hence, $\Sigma V = 0$, or $\Sigma H = 0$.

3. The summation of the bending moments caused by the axial loads and by the internal stresses must be zero about any axis in order to have equilibrium. Thus, $\Sigma M = 0$.

Figure 7-4 illustrates the ordinary stress distribution in a beam by the triangles ACO and BDO. The neutral axis for a simple beam is usually above the middle of the beam's depth. However, when a direct load P is applied, causing a compressive stress EA, the rectangle $EABG$ is assumed to be superimposed upon the previous figure so that the new diagram is made up of the triangles

ECF and GDF. It is thus apparent that the neutral axis is shifted downward in the figure, and kd increases. Therefore, a column that carries bending and a beam that is subjected to a compressive force are the same thing in reality as far as analysis is concerned.

It is not always easy to determine by inspection whether or not there is tension upon any section of a column that is subjected to bending. If the factor ec/R^2 of Eq. (7-2) exceeds unity, then CK of Fig. 7-2 must be greater than twice the average stress CE. Also, f_c min. from Eq. (7-3) will have a minus sign. In order to expedite the work of making a preliminary test, it is usually sufficient to neglect the transformed section of the steel in computing ec/R^2 unless the percentage of steel is large. For a rectangular column, the approximate value of

$$\frac{ec}{R^2} = \frac{eDA}{2I} = \frac{eD \times b \times D \times 12}{2 \times bD^3} = \frac{6e}{D}.$$

For a circular column,

$$\frac{ec}{R^2} = \frac{eDA}{2I} = \frac{eD \times \pi D^2 \times 64}{2 \times 4 \times \pi D^4} = \frac{8e}{D}.$$

Therefore, using these approximate formulas, it is possible to determine whether or not the member in question is likely to have tension upon part of its cross section. The following articles are prepared to give the designer a reasonably accurate method for analyzing rectangular or circular columns when the tension in the concrete must be neglected.

If the direction of the applied bending moment is such that it acts diagonally with respect to the sides of a rectangular member, and if the direct compression is relatively small, approximate resultant unit stresses may be computed by resolving the bending moment into two proper components, calculating the stresses that result from each one, and then combining these partial values. However, one should notice that the compressive force is included twice so that the computed stresses are much too great when P is large. When a structure is sufficiently important, one can guess the location of the neutral axis (perpendicular to the bending tendency), divide the section into small imaginary parts, compute trial expressions for P and M, check the results

with the assumed axis, and then try another location for the axis if necessary.

Although it was stated in Chap. 6 that the boundary of the effective section of a column is limited to about $1\frac{1}{2}$ in. outside the hoops or spirals, it will not be restricted in this way in cases of combined bending and direct loads.

7-5. Rectangular Members with Reinforcement on Tensile Side and with Tension on Part of the Section. Figure 7-5 pictures a member that has tensile reinforcement only. It is acted upon by the force P and the bending moment M.

An analysis of the internal stresses shows that there is a compression in the concrete, denoted by C_c, that is equal to the volume of a triangular wedge. Therefore,

$$C_c = \frac{f_c}{2} bkd. \tag{7-6}$$

There is also a tension T_s in the reinforcement, and

$$T_s = A_s f_s.$$

By similar triangles,

$$\frac{f_s}{n} : f_c :: (d - kd) : kd$$

or

$$f_s = n f_c \frac{(1 - k)}{k}. \tag{7-7}$$

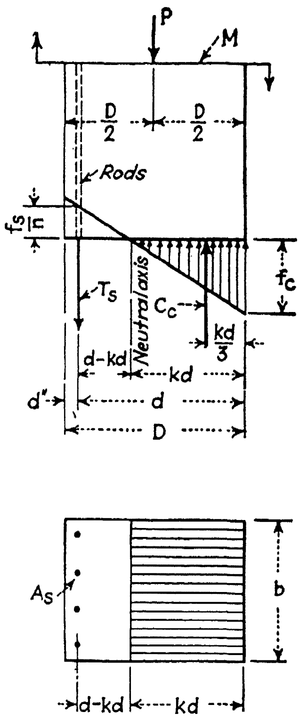


FIG. 7-5.

Therefore,

$$T_s = A_s n f_c \frac{(d - kd)}{kd} = A_s n f_c \frac{(1 - k)}{k}. \tag{7-8}$$

For any member with given dimensions and composed of known materials, b, d, A_s, and n are known. However, in Eqs. (7-6) and (7-8), f_c and k are unknown. Therefore, two equations are needed to find these unknowns. These can be obtained from the conditions for equilibrium; thus,

$$\Sigma V = 0, \qquad \text{or} \qquad P = C_c - T_s$$

$$P = \frac{f_c}{2}bkd - A_s n f_c \frac{(1-k)}{k}, \tag{7-9}$$

or, substituting pbd for A_s and solving for f_c,

$$f_c = \frac{Pk}{bd\left[\frac{k^2}{2} - pn(1-k)\right]}. \tag{7-10}$$

Next, using an axis at the center of the member, $\Sigma M = 0$, or

$$M = Pe = \frac{f_c}{2}bkd\left(\frac{D}{2} - \frac{kd}{3}\right) + A_s n f_c \frac{(1-k)}{k}\left(d - \frac{D}{2}\right). \tag{7-11}$$

It should be carefully noted that both Eqs. (7-9) and (7-11) must be satisfied. Dividing the latter by the former and canceling f_c gives

$$e = \frac{\frac{bk^2d^2}{2}\left(\frac{D}{2} - \frac{kd}{3}\right) + A_s nd(1-k)\left(d - \frac{D}{2}\right)}{\frac{bk^2d^2}{2} - A_s nd(1-k)}. \tag{7-12}$$

When this expression is simplified, it becomes

$$k^3 - \frac{3k^2}{d}\left(\frac{D}{2} - e\right) - 6pn(1-k)\frac{e'}{d} = 0 \tag{7-13}$$

where $e' = e + d - \frac{D}{2}$, the distance from the tensile steel to the real or equivalent eccentrically applied load. Thus, Eq. (7-13) takes both the ΣV and the ΣM conditions into account. Therefore, the remaining unknown k can be found by substituting the known data in Eq. (7-13) and solving it by cut-and-try methods. With this value of k, Eq. (7-10) can be solved for f_c, and thereafter f_s may be determined from Eq. (7-7).

A cubic equation such as Eq. (7-13) may be solved also as follows (calling the method "by the use of the differential")[1]:

1. Assume any reasonable value for the unknown k after the numerical values of the other terms have been substituted in the equation.

[1] I. S. Podolsky, "Reinforced Concrete Construction, Systematic Course," Part 1, Theory of Elements of Structures, Moscow Institute of Transportation, Moscow, U.S.S.R.

2. Solve the equation with this value of k, recording the result, with its sign.
3. Find the first differential of the original equation—Eq. (7-13) in this case.
4. Substitute the assumed value of k in the differential also, and find the remainder, with its sign.
5. Divide the remainder from 2 by that from 4, and subtract the quotient from the assumed value of k to find a second trial value.
6. Using the second trial value of k, repeat steps 2 to 5 inclusive. The second correction is usually all that is necessary, as will be seen in the solutions of subsequent problems.

Equation (7-13) contains the quantities d, D, p, and e' which are dependent upon the dimensions, the steel and the conditions of a particular case. Obviously, when the member does not exist, a tentative section must be chosen and then tested. It is laborious to make many assumptions and then to analyze each tentative member. A simple, approximate method must be used first. Such a method will be developed.

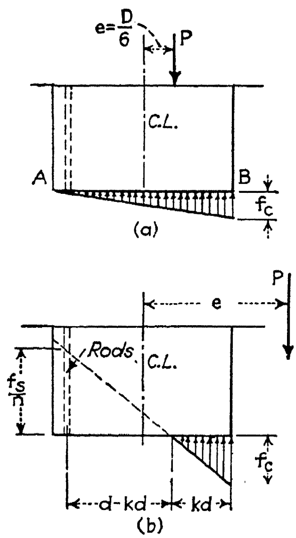


FIG. 7-6.

Figure 7-6(a) shows a column that has the load P applied at the edge of the middle third of the section; i.e., $e = D/6$. Neglecting the steel, the stress diagram will be a maximum at B and zero at A. Then, as the load P is shifted to the right so that e is increased, the bending moment becomes greater, and the neutral axis moves toward the compression side of the member. Furthermore, as this occurs, the relative effect of the direct load declines in comparison to that of the bending moment until, for practical purposes, the former can be neglected. During this process, the position of the neutral axis with respect to the compression side of the member changes from $D/6$ to about $\frac{3}{8}d$. Figure 7-7 has been prepared to show an empirical curve which may be assumed to approximate this variation in the value of k.

The approximate solution may now be outlined as follows:

1. Assume D; solve for e from M/P if it is not given directly; and find e/D. Next, scale the value of k from Fig. 7-7. Then $j = 1 - k/3$.

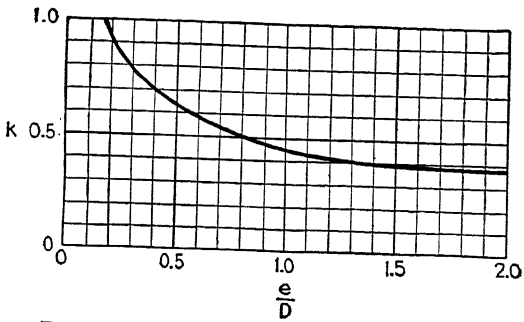


FIG. 7-7.—Diagram showing trial magnitude of k.

2. Referring to Fig. 7-8, write the equation for $\Sigma M = 0$ about the tensile steel.

$$Pe' = C_c jd = \left(\frac{f_c}{2}bkd\right)jd \tag{7-14}$$

where f_c = the allowable working stress. From this, the width b can be found. If it is unreasonably great or small, assume a new D, and begin the problem over again.

3. The tentative steel area A_s may be found by considering that $\Sigma V = 0$. Therefore, referring to Fig. 7-8,

$$T_s = C_c - P, \qquad \text{or} \qquad C_c - P = A_s f_s$$

$$\left(\frac{f_c}{2}bkd\right) - P = A_s \times nf_c\left(\frac{1-k}{k}\right). \tag{7-15}$$

$$p = \frac{A_s}{bd}.$$

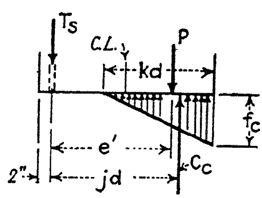


FIG. 7-8.

4. Having the tentative values of d, D, p, and also e', make a test analysis, using Eqs. (7-10) and (7-13).

In effect, this simply means that a member with a predetermined value of k is being proportioned to withstand the loads. However, the results obtained from the curve in Fig. 7-7 will give values that are reasonable for ordinary cases, at least for trial members.

Problem 7-2. Design a member with tensile steel only to withstand a direct load of 40,000 lb. and a bending moment of 280,000 in.-lb. Assume $n = 12$ and the allowable $f_c = 750$ lb. per sq. in.

1. Assume $D = 16$ in. and $d = 14$ in.

$$e = \frac{M}{P} = \frac{280{,}000}{40{,}000} = 7 \text{ in.}$$

$$\frac{e}{D} = \frac{7}{16} = 0.44.$$

From Fig. 7-7 for $e/D = 0.44$, $k = 0.67$. Therefore,

$$j = 1 - \frac{k}{3} = 0.78, \qquad kd = 9.4 \text{ in.,} \qquad \text{and} \qquad jd = 10.9 \text{ in.}$$

2. Referring to Fig. 7-8, $e' = e + \left(\frac{D}{2} - 2\right) = 13$ in. From Eq. (7-14),

$$40{,}000 \times 13 = \frac{750}{2} \times b \times 9.4 \times 10.9 \qquad \text{and} \qquad b = 13.5 \text{ in.}$$

3. From Eq. (7-15),

$$\frac{750}{2} \times 13.5 \times 9.4 - 40{,}000 = A_s \times 12 \times 750 \frac{(1 - 0.67)}{}$$

$$A_s = 1.69 \text{ sq. in. (four } \tfrac{3}{4}\text{-in. round rods} = 1.76 \text{ sq. in.)}$$

$$p = \frac{1.76}{13.5 \times 14} = 0.0093.$$

4. Substituting the preceding values of D, d, e, e', b, and p in Eq. (7-13) gives

$$k^3 - \frac{3k^2}{14}(8 - 7) - 6 \times 0.0093 \times 12(1 - k)\frac{13}{14} = 0.$$

$$k = 0.67 \qquad \text{(Too good a guess.)}$$

From Eq. (7-10),

$$f_c = \frac{40{,}000 \times 0.67}{13.5 \times 14\left[\frac{0.67^2}{2} - 0.0093 \times 12(1 - 0.67)\right]} = 755 \text{ lb. per sq. in.}$$

$$f_s = nf_c\frac{(1 - k)}{k} = 12 \times 755 \times \frac{0.33}{0.67} = 4{,}460 \text{ lb. per sq. in. (tension).}$$

The use of the curve in Fig. 7-7 may not always work so well as it did in this problem. Therefore, assume that the equation in step 4 is to be solved by the use of the differential. When the terms are rearranged,

$$k^3 - \frac{3}{14}k^2 + 0.622k - 0.622 = 0 \qquad (a)$$

$$d(fk) = 3k^2 - \frac{3}{7}k + 0.622 = 0 \qquad (b)$$

Try $k = 0.5$ as follows:

In (*a*), $0.125 - 0.054 + 0.311 - 0.622 = -0.240$.
In (*b*), $0.750 - 0.214 + 0.622 = +1.158$.

$$-\frac{0.240}{1.158} = -0.207 \text{ (correction).}$$

Therefore, corrected,

$$k = 0.50 - (-0.207) = 0.707 \text{ (new trial } k\text{).}$$

Next, try $k = 0.707$ as follows:
In (*a*), $0.353 - 0.107 + 0.440 - 0.622 = +0.064$.
In (*b*), $1.500 - 0.303 + 0.622 = +1.819$.

$$\frac{0.064}{1.819} = +0.035 \text{ (2d correction).}$$

Therefore, corrected,

$$k = 0.707 - 0.035 = 0.672 \text{ (final } k\text{).}$$

7-6. Rectangular Members with Reinforcement on Tensile and Compressive Sides with Tension on Part of the Section. A member that has both tensile and compressive reinforcement is pictured in Fig. 7-9. M is the bending moment, and P is the direct load. The compressive force in the steel will be called C_s.

The following can be found from Fig. 7-9:

$$C_c = \frac{f_c}{2}bkd. \tag{7-16}$$

$$C_s = A_s'(n-1)f_c\frac{(kd-d')}{kd}. \tag{7-17}$$

$$T_s = A_sf_s. \tag{7-18}$$

$$f_s = nf_c\frac{(d-kd)}{kd}. \tag{7-19}$$

$$f_s' = nf_c\frac{(kd-d')}{kd}. \tag{7-20}$$

The term kd will be used instead of k in the following work because it seems to facilitate the solution of practical problems.

Writing the equations for $\Sigma V = 0$ and $\Sigma M = 0$ about the center of the member,

$$P = C_c + C_s - T_s$$

$$P = \frac{f_c}{2}bkd + A_s'(n-1)\frac{(kd-d')}{kd}f_c - A_sn\frac{(d-kd)}{kd}f_c. \tag{7-21}$$

$$M = Pe = C_c\left(\frac{D}{2} - \frac{kd}{3}\right) + C_s\left(\frac{D}{2} - d'\right) + T_s\left(d - \frac{D}{2}\right)$$

$$Pe = \frac{f_c}{2}bkd\left(\frac{D}{2} - \frac{kd}{3}\right) + A_s'(n - 1)f_c\frac{(kd - d')}{kd}\left(\frac{D}{2} - d'\right) + A_s n f_c\frac{(d - kd)}{kd}\left(d - \frac{D}{2}\right). \quad (7\text{-}22)$$

Combining Eqs. (7-21) and (7-22) gives

$$e = \frac{\left[\frac{b}{2}(kd)^2\left(\frac{D}{2} - \frac{kd}{3}\right) + A_s'(n - 1)(kd - d')\left(\frac{D}{2} - d'\right) + A_s n(d - kd)\left(d - \frac{D}{2}\right)\right]}{\frac{b}{2}(kd)^2 + A_s'(n - 1)(kd - d') - A_s n(d - kd)}. \quad (7\text{-}23)$$

Equation (7-23) is in a convenient form for solution by trial. It is also easy to adapt it to the solution of problems that involve

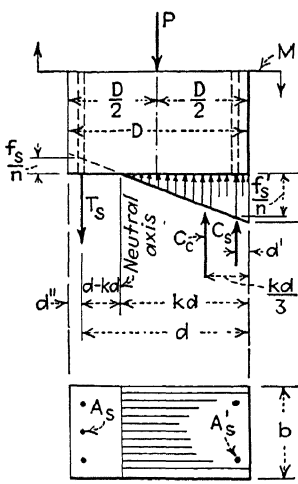


FIG. 7-9.

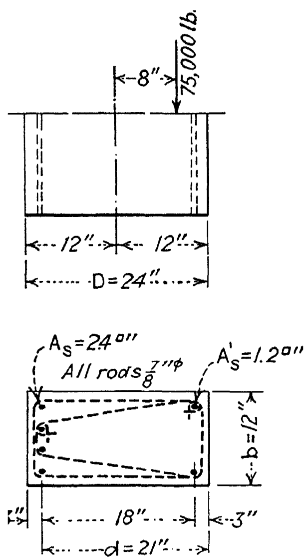


FIG. 7-10.

double rows of steel. For such a case, it is practicable to assume that the two rows of steel are combined as one at the center of gravity of the two rows.

Another convenient form for Eq. (7-23) is

$$\frac{b}{6}(kd)^3 + b\left(\frac{e}{2} - \frac{D}{4}\right)(kd)^2 + \left[A_s n\left(e + d - \frac{D}{2}\right) + A'_s(n-1)\left(e + d' - \frac{D}{2}\right)\right]kd - A_s n d\left(e + d - \frac{D}{2}\right) - A'_s(n-1)d'\left(e + d' - \frac{D}{2}\right) = 0. \quad (7\text{-}23a)$$

The solution of this expression by the use of the differential is generally much easier than it appears to be.

After kd has been determined, it can be substituted in Eq. (7-21) to find f_c and in Eqs. (7-19) and (7-20) to find f_s and f'_s, although f_c is generally the only unit stress whose magnitude is important.

If $A'_s = A_s$, $A_s = pbd$, $n - 1$ is replaced by n, $d' = d''$, and $a = \frac{D}{2} - d'$ or the distance from the center of the column to the steel, then Eq. (7-23a) becomes

$$k^3 + \frac{3}{2}\left(\frac{2e - D}{d}\right)k^2 + \frac{12pne}{d}k - \frac{6pn}{d^2}(eD + 2a^2) = 0. \quad (7\text{-}23b)$$

The same assumptions used in Eq. (7-21) give

$$f_c = \frac{2kP}{b[k^2d + 2pn(2kd - D)]}. \quad (7\text{-}21a)$$

These formulas are often sufficient for practical analysis. One should notice that p is not the same as p_g used in columns.

For the design of a member to withstand specified loads, it is possible, with good judgment, to design an approximate member in the same general manner as that which was outlined in Art. 7-5, then to add the compressive reinforcement, and finally to test the assumed section.

Problem 7-3. Find f_c, f_s, and f'_s for the member and the loading conditions that are shown in Fig. 7-10. Assume $n = 10$.

The first step in this problem is that of getting an idea of the approximate location of the neutral axis by using Eq. (7-2) in an abbreviated and approximate form. Thus,

$$\text{Max. } f_c = \frac{P\left(1 + \frac{6e}{D}\right)}{A_g + (n-1)(A_s + A'_s)} = \frac{75{,}000\left(1 + \frac{6 \times 8}{24}\right)}{12 \times 24 + (10 - 1)(2.4 + 1.2)}$$
$$= 702 \text{ lb. per sq. in.}$$

$$\text{1.} f_c = -234 \text{ lb. per sq. in.} \left(\text{using } -\frac{6e}{D}\right).$$

$$kd = \frac{D(\text{max.} f_c)}{\text{max.} f_c + \text{min.} f_c} = \frac{24 \times 702}{702 + 234} = 18 \text{ in.}$$

This value of kd exceeds the actual value because it is based upon tension in the concrete. When this tension is neglected, the neutral axis will shift toward the compression side, f_c will become larger, and kd will decrease. Therefore, assume $kd = 16.5$ in., and substitute this value in Eq. (7-23), which is reduced to the following by the insertion of the numerical values of b, A_s, A'_s, n, and D:

$$\frac{6(kd)^2(12 - 0.333kd) + 97.2(kd - 3) + 216(21 - kd)}{\qquad - 3) - 24(21 - kd)}$$

Then

$$e = \frac{6 \times 16.5^2(12 - 5.5) + 97.2(16.5 - 3) + 216(21 - 16.5)}{6 \times 16.5^2 + 10.8(16.5 - 3) - 24(21 - 16.5)} = 7.7 \text{ in}$$

Actually, $e = 8$ in. Therefore, assume $kd = 16$ in. For this, the right-hand side of the equation is found to be 8.1 in. The real value of kd, by proportion, will be 16.1 in. Then $k = 0.765$.

For purposes of comparison, use Eq. (7-23a) and the differential to find kd as follows, using the data from Fig. 7-10:

$$2(kd)^3 + 12(4 - 6)(kd)^2 + [24(8 + 21 - 12) + 10.8(-1)]kd - 24 \times 21 \times 17 - 10.8 \times 3(8 + 3 - 12) = 0$$

$$(kd)^3 - 12(kd)^2 + 198kd - 4{,}274 = 0. \qquad (a)$$

$$d[f(kd)] = 3(kd)^2 - 24kd + 198 = 0. \qquad (b)$$

Assume $kd = 15$
From (a), $15^3 - 12 \times 15^2 + 198 \times 15 - 4{,}274 = -624$
From (b), $3 \times 15^2 - 24 \times 15 + 198 = +513$

$$\frac{-624}{513} = -1.22 \text{ (correction).}$$

$$15 - (-1.22) = 16.22 \text{ (new trial value of } kd\text{).}$$

Using $kd = 16.22$, (a) gives +56, and (b) yields +599.

$$\frac{56}{599} = +0.09 \text{ (correction).}$$

$$\therefore kd = 16.22 - 0.09 = 16.1 \text{ in.}$$

Substituting kd and d in Eqs. (7-21), (7-19), and (7-20) gives

$$75{,}000 = \frac{f_c}{2} \times 12 \times 16.1 + 1.2 \times 9\frac{(16.1 - 3)}{16.1}f_c - 2.4 \times 10\frac{(21 - 16.1)}{16.1}f$$

765 lb. per sq. in.

$$f_s = nf_c\frac{(d - kd)}{kd} = 10 \times 765\frac{(21 - 16.1)}{16.1} = 2{,}330 \text{ lb. per sq. in.}$$

$$f_s' = nf_c\frac{(kd - d')}{kd} = 10 \times 765\frac{(16.1 - 3)}{16.1} = 6{,}220 \text{ lb. per sq. in.}$$

7-7. Rectangular Members with Reinforcement near All Sides and with Tension on Part of the Section. The ordinary rectangular column will have longitudinal reinforcement near all four faces of the member as illustrated in Fig. 7-11(a). Obviously, it will be tedious to consider each set of rods independently in writing the equations for ΣV and ΣM. Therefore, when the number of rods is great, they may be replaced by an imaginary steel shell which has the same area as the rods. Its thickness may be called t.

Writing the equation for ΣV as usual and that for ΣM about the neutral axis O, instead of the center of the member, gives

$$P = \frac{bf_c}{2}kd + A_s'(n - 1)f_c\frac{kd - d'}{kd} + t(n - 1)\frac{(kd - d')^2}{kd}f_c - A_snf_c\frac{(d - kd)}{kd} - tn\frac{(d - kd)^2}{kd}f_c \quad (7\text{-}24)$$

$$M = P\left(kd - \frac{D}{2} + e\right) = bf_c\frac{(kd)^2}{3} + A_s'(n - 1)f_c\frac{(kd - d')^2}{kd} + t(n - 1)\frac{(kd - d')^3}{kd}f_c \times \frac{2}{3} + A_snf_c\frac{(d - kd)^2}{kd} + tn\frac{(d - kd)^3}{kd}f_c \times \frac{2}{3}.$$

Combining these yields

$$kd - \frac{D}{2} + e = \left\{\frac{b(kd)^3}{3} + A_s'(n - 1)(kd - d')^2 + 0.67t(n - 1)(kd - d')^3 + A_sn(d - kd)^2 + 0.67tn(d - kd)^3\right\} \div \left\{\frac{b}{2}(kd)^2 + A_s'(n - 1)(kd - d') + t(n - 1)(kd - d')^2 - A_sn(d - kd) - tn(d - kd)^2\right\}. \quad (7\text{-}25)$$

Equations (7-24) and (7-25) are purposely left in a general form so as to be clearly applicable to almost any problem. In many cases, such as square columns, it is sufficiently accurate to omit the terms involving t for the side rods for preliminary work.

Problem 7-4. Find f_c, f_s, and f'_s for the column that is pictured in Fig. 7-12, assuming $P = 250{,}000$ lb., $e = 11$ in., and $n = 8$.

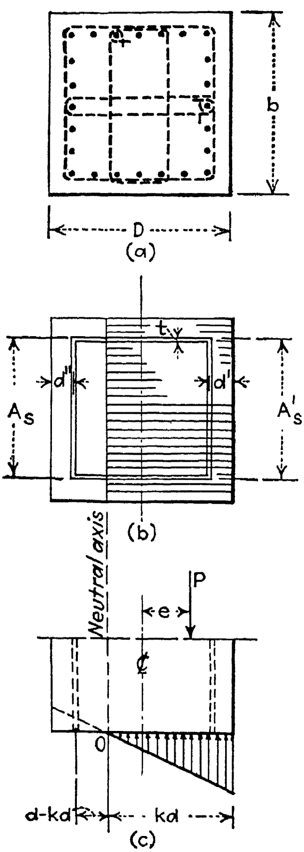


FIG. 7-11.

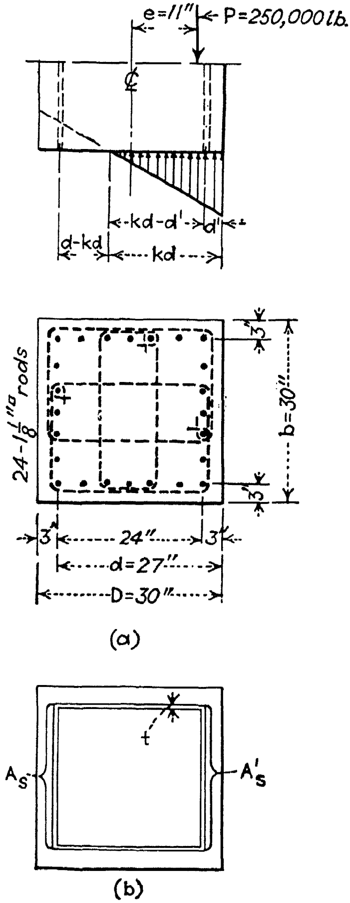


FIG. 7-12.

The total area of steel is

$$A_{sg} = 24 \times 1.27 = 30.5 \text{ sq. in.}$$

$$A_s = A'_s = \frac{30.5}{4} = 7.6 \text{ sq. in.}$$

$$t \quad \frac{30.5}{96} = 0.318 \text{ in.}$$

$$\text{Approx. max. } f_c = \frac{P\left(1 + \frac{6e}{D}\right)}{A_g + (n-1)A_{sg}} = \frac{250{,}000\left(1 + \frac{6 \times 11}{30}\right)}{30 \times 30 + 7 \times 30.5} = 718 \text{ lb. per sq. in.}$$

$$\text{Min. } f_c = -269 \text{ lb. per sq. in.}$$

$$\text{Approx. } kd = \frac{30 \times 718}{718 + 269} = 21.8 \text{ in.}$$

When the known data for this problem are inserted in Eq. (7-25), it becomes

$$kd - 4 = \{10(kd)^3 + 53.2(kd - 3)^2 + 1.49(kd - 3)^3 + 60.8(27 - kd)^2 + 1.70(27 - kd)^3\} \div \{15(kd)^2 + 53.2(kd - 3) + 2.23(kd - 3)^2 - 60.8(27 - kd) - 2.54(27 - kd)^2\}.$$

Then, assuming $kd = 19$ in., so as to be less than the approximate 21.8 in., this equation gives $19 - 4 = 15.04$ in., instead of 15 in., which is close enough. From Eq. (7-24),

$$250{,}000 = 15 \times 19 f_c + 7.6 \times 7 \times \frac{16}{19} f_c + 2.23 \times \frac{16^2}{19} f_c - 7.6 \times 8 \times \frac{8}{19} f_c - 2.54 \times \frac{8^2}{19} f_c$$

or

$$f_c = 768 \text{ lb. per sq. in.}$$

$$f_s = nf_c \frac{(d - kd)}{kd} = 8 \times 768 \frac{(27 - 19)}{19} = 2{,}590 \text{ lb. per sq. in.}$$

$$f_s' = nf_c \frac{(kd - d')}{kd} = 8 \times 768 \frac{(19 - 3)}{19} = 5{,}170 \text{ lb. per sq. in.}$$

7-8. Circular Columns Having Combined Compression and Bending with Tension on Part of the Section. In the previous article, rectangular columns were considered. The same principles apply to circular columns that have tension upon part of their cross section. However, the shape of these members causes a great deal of difficulty in the numerical work. Tables and diagrams have been prepared by others to facilitate the solution of such problems, but their usefulness is restricted to cases that are controlled by the same fundamental data as those upon which the charts are based. However, it is the purpose of this text to develop a general method of analysis which is founded upon the basic theory involved, so that any special problem that differs from those which may be solved by such tables can be worked out.

Figure 7-13(a) pictures a circular, spirally reinforced column with a diameter equal to D and with the diameter of the circle

of rods equal to D_2. This member is acted upon by a direct load P and by a bending moment M, as shown in Fig. 7-13(*b*).

An analysis of the stress diagram in Sketch (*b*) will show that the imaginary solid that represents the compression upon the

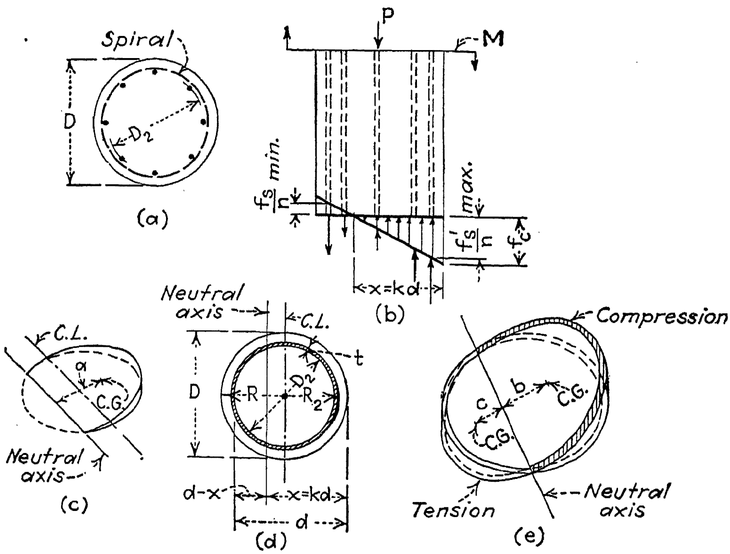


FIG. 7-13.

concrete is an ungula of a circular cylinder [a wedge-shaped solid as pictured in Fig. 7-13(*c*)].

In order to simplify the calculations, the longitudinal rods in the column will be assumed to be replaced by a hollow steel shell or pipe which has the same cross-sectional area as that of all of the longitudinal reinforcement. Then, if A_{sg} = the area of these rods, and if R_2 = the radius of the circle in which the rods are located,

$$t = \frac{A_{sg}}{2\pi R_2} \tag{7-26}$$

where t = the width or thickness of the shell. Figure 7-13(*d*) pictures the steel ring.

Figure 7-13(*b*) indicates that this steel shell is subjected to compression upon part of its area and to tension upon the rest of its cross section. Therefore, the diagrams that represent the

stresses upon these two parts of the ring will be like hollow ungulas or like wedges cut from a pipe, as pictured in Fig. 7-13(e).

In order to show the situation more clearly, let Fig. 7-14 picture the cross section of a circular column and the diagram of the internal stresses. The applied bending moment, whatever may be its cause, can be represented by the direct load P acting with an eccentricity e. In order to have equilibrium, it is obvious that $\Sigma V = 0$ and $\Sigma M = 0$.

The following symbols will be adopted for brevity in the writing of the foregoing equations for equilibrium:

$x = kd$
C_c = total compressive stress in the concrete (ungula)
C_s = total compressive stress in the steel (hollow wedge)
T_s = total tensile stress in the steel (hollow wedge)
a = distance from neutral axis to line of action of C_c
b = distance from neutral axis to line of action of C_s
c = distance from neutral axis to line of action of T_s.

Therefore, using these symbols, and taking moments about the neutral axis,

$$P = C_c + C_s - T_s. \tag{7-27}$$

$$P(e + x - R) = C_ca + C_sb + T_sc. \tag{7-28}$$

By combining these equations,

$$(e + x - R) = \frac{C_ca + C_sb + T_sc}{C_c + C_s - T_s}. \tag{7-29}$$

In order to assist in the writing of the equations for equilibrium, Fig. 7-15 has been prepared to give the volume of an ungula in terms of R and f_c for any value of x. It also gives the moment of this solid about its "cutting edge," the neutral axis. In a similar manner, Fig. 7-16 gives the volume of a hollow wedge in terms of t, z_1 or z_2, and R_2, also the moment of such a wedge about the neutral axis.

In using these curves, the following relationship must be remembered:

$C_s = (n - 1) \times$ volume of the hollow wedge where $z_1 = f_c\dfrac{(x - d')}{x}$.

$T_s = n \times$ volume of the hollow wedge where $z_2 = f_c\dfrac{(d - x)}{x}$.

Equations (7-27), (7-28), and (7-29) can now be rewritten in more convenient form. They become

$$P = K_1R^2f_c + (n-1)K_2tf_c\frac{(x-d')}{x}R_2 - nK_3tf_c\frac{(d-x)}{x}R_2. \quad (7\text{-}30)$$

$$P(e+x-R) = K_4R^3f_c + (n-1)K_5tf_c\frac{(x-d')}{x}R_2^2 + nK_6tf_c\frac{(d-x)}{x}R_2^2. \quad (7\text{-}31)$$

$$(e+x-R) = \frac{K_4R^3 + (n-1)K_5t\frac{(x-d')}{x}R_2^2 + nK_6t\frac{(d-x)}{x}R_2^2}{K_1R^2 + (n-1)K_2t\frac{(x-d')}{x}R_2 - nK_3t\frac{(d-x)}{x}R_2}. \quad (7\text{-}32)$$

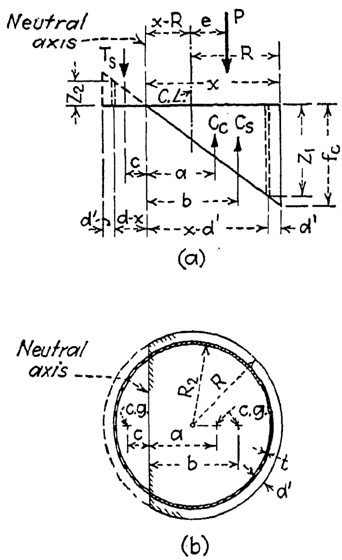


FIG. 7-14.

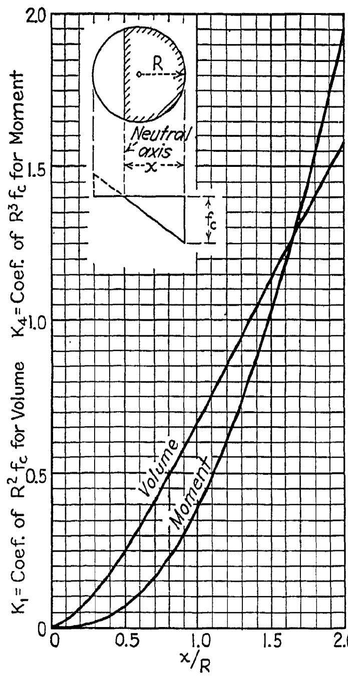


FIG. 7-15.—Volume and moment of ungula about its cutting edge.

The symbols K_1, K_2, and K_3 are the coefficients given by the curves for volumes in Figs. 7-15 and 7-16. K_4, K_5, and K_6 are

the corresponding coefficients which can be obtained from the curves for moments in these same figures.

The procedure as applied to problems may be outlined as follows:

1. Find the eccentricity, and see if $8e/D$ exceeds 1. If so, there will be tension on part of the section. Make an approximate solution for maximum f_c and minimum f_c by using Eq. (7-2) which is based upon the concrete withstanding tension. Let $8e/D$ be substituted for ec/R^2.

2. From these limiting values of f_c find the approximate location of the neutral axis by means of similar triangles.

3. Assume a value of x somewhat less than that which is given by the preceding step, and substitute it in Eq. (7-32). If the first trial is not successful, a second attempt will usually locate the neutral axis.

4. With the proper value of $x = kd$, solve Eq. (7-30) for f_c.

5. If desired, find f_s' and f_s from the expressions for z_1 and z_2, multiplying both by n.

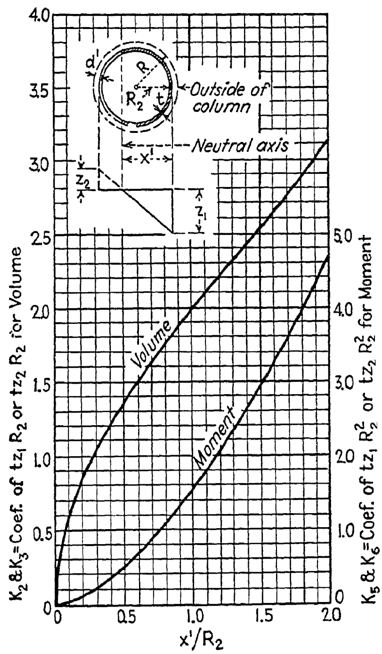


FIG. 7-16.—Volume and moment of wedge of hollow cylinder about its cutting edge.

In the special case of a rectangular column that has the reinforcement placed in a circle, the methods of this article will apply

with the substitution of a rectangular wedge for the diagram of stress in the concrete.

Problem 7-5. Find the stresses in a circular column for which the data are as follows: $P = 100{,}000$ lb., $M = 600{,}000$ in.-lb., $D = 22$ in., $D_2 = 17$ in., $d' = 2.5$ in., reinforcement = ten $\frac{7}{8}$-in. round rods, $n = 10$, $d = 19.5$ in. Adequate spiral reinforcement is assumed.

$$(1)\ e = \frac{M}{P} = \frac{600{,}000}{100{,}000} = 6 \text{ in.}$$

$$\frac{8e}{D} = \frac{8 \times 6}{22} = 2.18$$

$$\text{Max. } f_c = \frac{P\left(1 + \frac{ec}{R^2}\right)}{A_g[1 + (n - 1)p_g]} = \frac{P\left(1 + \frac{8e}{D}\right)}{A_g[1 + (n - 1)p_g]}$$

$$A_g = \frac{\pi 22^2}{4} = 380 \text{ sq. in.}$$

$$p_g = \frac{A_s}{A_g} = \frac{10 \times 0.6}{380} = 0.0158$$

$$\text{Max. } f_c = \frac{100{,}000\left(1 + \frac{8 \times 6}{22}\right)}{380(1 + 9 \times 0.0158)} = 732 \text{ lb. per sq. in.}$$

$$\text{Min. } f_c = \frac{100{,}000\left(1 - \frac{8 \times 6}{22}\right)}{380(1 + 9 \times 0.0158)} = -272 \text{ lb. per sq. in.}$$

(2) From Fig. 7-17,

$$\frac{x}{22} = \frac{732}{1{,}004} \qquad \text{or} \qquad x = 16 \text{ in.} \pm.$$

(3) Assume $x = 15$ in.
The value of x/R for the concrete (Fig. 7-15) is $\frac{15}{11} = 1.36$.
The value of x'/R_2 for the steel in compression (Fig. 7-16) is

$$\frac{(15 - d')}{R_2} = \frac{(15 - 2.5)}{8.5} = 1.47.$$

The value of x'/R_2 for the steel in tension is

$$2 - 1.47 = 0.53.$$

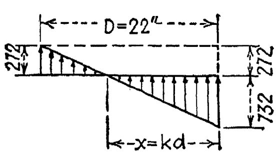


FIG. 7-17.

$$z_1 = f_c\frac{(x - d')}{x} = \frac{(15 - 2.5)}{15}f_c = 0.833f_c.$$

$$z_2 = f_c\frac{(d - x)}{x} = \frac{(19.5 - 15)}{15}f_c = 0.3f_c.$$

$$t = \frac{A_s}{\pi D_2} = \frac{6}{\pi \times 17} = 0.112 \text{ in.}$$

Substituting these values in Eq. (7-32) with the values of K_1, etc., as found in Figs. 7-15 and 7-16, gives

$$(6 + x - 11) = \frac{0.8 \times 11^3 + 9 \times 2.83 \times 0.112 \times 0.833 \times 8.5^2 + 10 \times 0.58 \times 0.112 \times 0.3 \times 8.5^2}{0.99 \times 11^2 + 9 \times 2.52 \times 0.112 \times 0.833 \times 8.5 - 10 \times 1.4 \times 0.112 \times 0.3 \times 8.5}$$

$$(6 + x - 11) = \frac{1,251}{134} = 9.35.$$

Then $x = 14.35$ in. compared to the 15 in. assumed.

Next, try $x = 14$ in.

$$x/\mathrm{R} \text{ for the concrete } = \frac{14}{11} = 1.27.$$

$$x'/\mathrm{R}_2 \text{ for the steel in compression } = \frac{(14 - 2.5)}{8.5} = 1.35.$$

$$x'/\mathrm{R}_2 \text{ for the steel in tension } = 0.65.$$

$$z_1 = \frac{(14 - 2.5)}{14} f_c = 0.82 f_c.$$

$$z_2 = \frac{(19.5 - 14)}{14} f_c = 0.393 f_c.$$

Solving Eq. (7-32) again gives

$$6 + x - 11 = \frac{0.69 \times 11^3 + 9 \times 2.45 \times 0.112 \times 0.82 \times 8.5^2 + 10 \times 0.79 \times 0.112 \times 0.393 \times 8.5^2}{0.92 \times 11^2 + 9 \times 2.38 \times 0.112 \times 0.82 \times 8.5 - 10 \times 1.57 \times 0.112 \times 0.393 \times 8.5}.$$

$$x - 5 = 8.97 \qquad \text{or} \qquad x = 13.97 \text{ in.} \qquad \text{(Close enough.)}$$

If the second guess had not been so close, the first and second values could be plotted as in Fig. 7-18, and the value of x would be given by the intersection of the line that joins them with the zero line. The values of z_1 and z_2 should be changed to agree with this scaled value of x. Also, the new values of x/R and x'/R_2 should be determined for use with Figs. 7-15 and 7-16.

(4) Substituting in Eq. (7-30) with x/R for concrete $= 1.27$, x'/R_2 for steel in compression $= 1.35$, x'/R_2 for steel in tension $= 0.65$, $z_1 = 0.82f_c$, and $z_2 = 0.393f_c$ gives

$$P = 0.92 \times 11^2 f_c + 9 \times 2.38 \times 0.112 \times 0.82 \times 8.5 f_c - 10 \times 1.57 \times 0.112 \times 0.393 \times 8.5 f_c.$$

$$f_c = \frac{100,000}{122} = 820 \text{ lb. per sq. in.}$$

(5) f'_s compression $= \dfrac{(x - d')}{x} f_c n = 0.82 \times 820 \times 10 = 6{,}720$ lb. per sq. in.

$$f_s \text{ tension } = \frac{(d - x)}{x} f_c n = 0.393 \times 820 \times 10 = 3{,}220 \text{ lb. per sq. in.}$$

7-9. Approximate Method for the Design of Rectangular Columns with Tension on Part of the Section. The previously

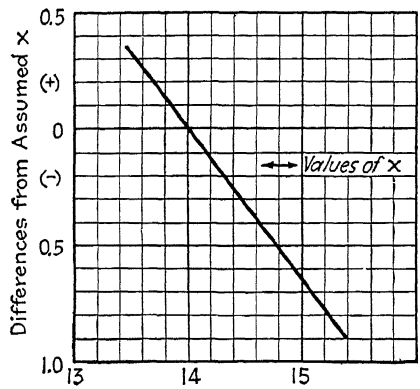


FIG. 7-18.

explained methods of design and analysis of members having combined direct loads and bending moments require considerable labor in their application. As a practical matter, the actual conditions do not justify such a degree of accuracy in many cases because there are so many unknown factors or empirical assumptions in the initial determination of the loads and their distribution that the subsequent calculations need not be more exact than the fundamental data upon which they are based. It is therefore desirable to use a reasonably accurate, approximate method of design and analysis which is easy to apply. Such a method will now be explained. It was originally developed and used by Frederick C. Lowy[1] and Erick M. Black.[2] It has been tested in many cases in the office of The Port of New York Authority.

[1] Formerly designer, The Port of New York Authority, 111 Eighth Ave., New York City.

[2] Formerly assistant designing engineer, City Railway, Newark, N. J.

Consider the distribution of the stresses in a column for an eccentrically applied load. Figure 7-19(a) shows a column with the load P which has an eccentricity of sufficient magnitude to make the stress upon the section vary from zero at A to a maximum at B. If there were no steel, e would equal $D/6$. In other words,

$$e = r_k = \frac{D}{6} = \frac{bD^2}{6} \div bD = \frac{\text{section modulus}}{\text{area}}$$

where r_k will be called the "kern radius." Similarly, for the reinforced-concrete member, let

$$r_k = \frac{\text{section modulus of the entire transformed section}}{\text{area of the entire transformed section}}.$$

Let I'_c equal the moment of inertia of the complete section includ-

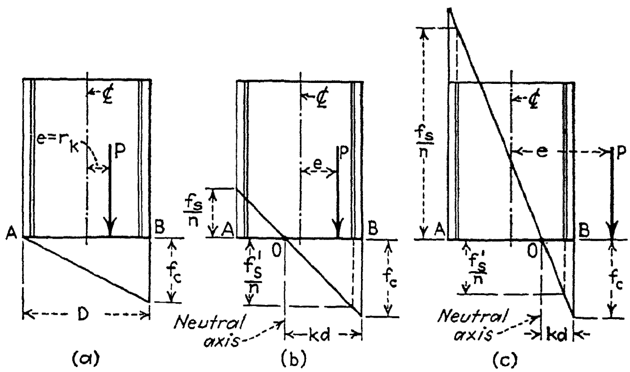


FIG. 7-19.

ing $(n - 1)$ times the area of the steel. The whole section must be used in this calculation. Therefore,

$$r_k = \frac{I'_c}{0.5D[A_g + (n - 1)A_{sg}]} \tag{7-33}$$

where A_{sg} = the total area of steel in the entire cross section. Thus, the value of r_k may be considered as the limit of the central portion, or kern (core), of the member. If the load acts within this kern, there will be no tension upon the section; if it is applied at the limit of r_k, the stress at A will be zero, but at B, f_c = twice the average stress; if e exceeds r_k, the neutral axis O will move from A toward B, as shown in Figs. 7-19(b) and (c), causing tension near A and making f_c greater than twice the average

stress in direct compression. However, when the magnitude of e becomes large compared to the depth of the section D, the member becomes primarily a beam, the direct stress being insignificant. Usually, if $e = 2D$, the direct load may be neglected; and for such a case, kd will approximate $D/3$.

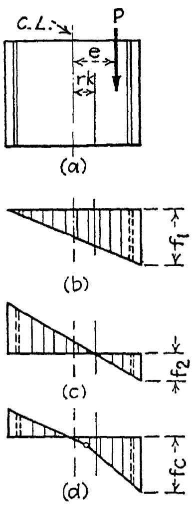


FIG. 7-20.

Considering the fact that r_k is about equal to $D/6$, it is seen that the kern radius comes approximately at the neutral axis of the section when the latter is acting as an ordinary beam. Therefore, it will be assumed that the stress diagram for any position of P, where e exceeds r_k, is made up of one part which is the triangle of Fig. 7-20(b), where P is assumed to be at r_k, plus a second diagram as shown in Fig. 7-20(c), the latter being caused by the load P acting, with a lever arm of $e - r_k$, upon the section as a pure beam. The resultant stress diagram is shown in Fig. 7-20(d), being indicated in the picture as bounded by a broken line merely for the reasons which will be explained later.

In the practical design of columns, it will be found that the stress in the concrete f_c is the critical one. Since f'_s, the compressive stress in the longitudinal reinforcement, is only n times the stress in the concrete at the same point, it is usually low except for the indefinite increase which is caused by plastic flow. Also, f_s is usually unimportant unless the bending moment is very severe. Therefore, it is satisfactory to solve for f_c alone in most cases, it being equal to $f_1 + f_2$ (Fig. 7-20).

In any such problem, the procedure is simply as follows:

(1) Assume a section to be analyzed.

(2) Find $r_k = \dfrac{I'_c}{0.5D[A_g + (n-1)A_{sg}]}$. (7-34)

(3) Find $f_1 = \dfrac{2P}{A_g + (n-1)A_{sg}}$. (7-35)

(4) Find $f_2 = \dfrac{P(e - r_k)kd}{I_c}$. (7-36)

(5) $f_c = f_1 + f_2$. (7-37)

It must be noted that I_c in step (4) is the moment of inertia of the transformed section of the member as an ordinary beam, allowing no tension upon the concrete.

This method still appears to require a great deal of work. However, in any structure that has a large number of columns, it will be found advisable to simplify the work by using a moderate variety of sections. Therefore, the tentative members can be chosen; their areas, section moduli, and kern radii can be computed; and then the members can be used for any combination of P and M or P and Pe for which they are safe. Having the properties of the sections, the labor of finding f_1 and f_2 is very small.

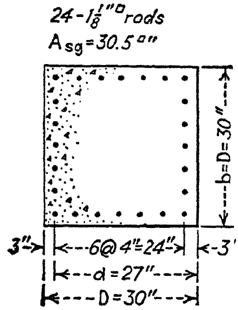


FIG. 7-21.

If it is necessary to find the stresses in the steel, it is not advisable to use the results of this approximate method because the errors which are introduced by the approximations often multiply considerably in any attempt to use them in Eqs. (7-19) and (7-20). That is why Fig. 7-20(d) is shown with a broken line so as to warn the reader not to attempt such a calculation.

For important structures it is advisable to design the members by this approximate method and then to check them by more exact means.

Problem 7-6. Find f_c for the column shown in Fig. 7-21 if $P = 250{,}000$ lb., $e = 11$ in., and $n = 8$. (This is the same as Problem 7-4 of Art. 7-7.)

The first step is the calculation of I'_c:

$$I'_c = \frac{bD^3}{12} + (n - 1)I_s$$

$$I'_c = \frac{30 \times 30^3}{12} + (8 - 1)(14 \times 1.27 \times 12^2 + 4 \times 1.27 \times 8^2 + 4 \times 1.27 \times 4^2) = 88{,}260 \text{ in.}^4$$

Then, from Eq. (7-34),

$$r_k = \frac{I'_c}{0.5D[A_g + (n - 1)A_{sg}]} = \frac{88{,}260}{15[900 + 7 \times 30.5]} = 5.28 \text{ in.}$$

The next step is the calculation of I_c. For a square member with the rods symmetrically placed, neglecting the fact that the area of the steel in compression should be multiplied by $(n - 1)$ rather than by n, and taking

moments about the neutral axis,

$$\frac{D(kd)^2}{2} = nA_{sg}\left(\frac{D}{2} - kd\right),$$

$$kd = -\frac{nA_{sg}}{D} + \sqrt{\frac{(nA_{sg})^2}{D^2} + nA_{sg}}. \tag{7-38}$$

$$kd = -\frac{8 \times 30.5}{30} + \sqrt{\frac{(8 \times 30.5)^2}{30^2} + 8 \times 30.5} = 9.5 \text{ in.}$$

$$I_c = \frac{30 \times 9.5^3}{3} + 8(14 \times 1.27 \times 12^2 + 4 \times 1.27 \times 8^2 + 4 \times 1.27 \times 4^2) + 8 \times 30.5(15 - 9.5)^2 = 39{,}700 \text{ in.}^4$$

$$f_1 = \frac{2P}{(n-1)A_{sg}} = \frac{2 \times 250{,}000}{900 + 7 \times 30.5}$$

$$f_2 = \frac{[illegible]}{I_c} = \frac{250{,}000(11 - 5.28)9.5}{39{,}700} \quad 342 \text{ lb. per sq. in.}$$

$$f_c = f_1 + f_2 = 449 + 342 = 791 \text{ lb. per sq. in.}$$

This value of f_c differs only about 2 per cent from the value, 810 lb. per sq. in., which was found in Problem 7-4.

Problem 7-7. Assume the same column as that of Fig. 7-21, and test it for the following combinations of direct load and bending, assuming $n = 8$ and the allowable $f_c = 800$ lb. per sq. in.

(1) $P = 200{,}000$ lb., $M = 2{,}800{,}000$ in.-lb.
(2) $P = 300{,}000$ lb., $M = 2{,}000{,}000$ in.-lb.
(3) $P = 175{,}000$ lb., $M = 3{,}000{,}000$ in.-lb.
(4) $P = 350{,}000$ lb., $M = 2{,}400{,}000$ in.-lb.

For all of these cases $[A_g + (n - 1)A_{sg}] = 1{,}114$ sq. in., $kd = 9.5$ in., $I_c = 39{,}700$ in.^{4} and $r_k = 5.28$ in. as for Problem 7-6.

(1) $$e = \frac{2{,}800{,}000}{200{,}000} = 14 \text{ in.}$$

$$f_1 = \frac{2 \times 200{,}000}{1{,}114} = 359 \text{ lb. per sq. in.}$$

$$f_2 = \frac{200{,}000(14 - 5.28)9.5}{39{,}700} = 417 \text{ lb. per sq. in.}$$

$$f_c = 359 + 417 = 776 \text{ lb. per sq. in.}$$

(2) $$e = \frac{2{,}000}{300} = 6.67 \text{ in.}$$

$$f_1 = \frac{2 \times 300{,}000}{1{,}114} = 539 \text{ lb. per sq. in.}$$

$$f_2 = \frac{300{,}000(6.67 - 5.28)9.5}{39.700} = 100 \text{ lb. per sq. in.}$$

$f_c = 539 + 100 = 639$ lb. per sq. in. (Too low for economy.)

(3) $e = \dfrac{3{,}000}{175} = 17.1$ in.

$f_1 = \dfrac{2 \times 175{,}000}{1{,}114} = 314$ lb. per sq. in.

$f_2 = \dfrac{175{,}000(17.1 - 5.28)9.5}{39{,}700} = 494$ lb. per sq.

$f_c = 314 + 494 = 808$ lb. per sq. in.

(4) $e = \dfrac{2{,}400}{350} = 6.86$ in.

$f_1 = \dfrac{2 \times 350{,}000}{1{,}114} = 628$ lb. per sq. in.

$f_2 = \dfrac{350{,}000(6.86 - 5.28)9.5}{39{,}700} = 132$ lb. per sq. in.

$f_c = 628 + 132 = 760$ lb. per sq. in.

These calculations illustrate the ease with which any given member can be tested for its stresses when it is subjected to various combinations of loading.

7-10. Combined Bending and Tension. There are some cases in which reinforced-concrete members are subjected to bending moments and direct tensile forces. One of the most common instances of such a combination occurs when a drop in temperature affects a member the ends of which are not free to move.

To illustrate the internal resisting stress when bending and tension are combined, let Fig. 7-22(*a*) represent a beam that has unsymmetrical reinforcement. Sketch (*b*) shows the assumed stress distribution when the bending moment M is applied. If the external tensile force P is added at the center of the member, it will cause an additional elongation of the tensile reinforcement and a decrease of the compressive stresses. Up to a certain extent, the concrete may resist tension, but such action will not be relied upon. If the resistance of the member was symmetrical, the stress diagram would be a rectangle like $ABCD$ of Fig. 7-22(*c*). However, the unequal resistance due to relieving of the compressive stresses and the increase of the tensile ones may cause another bending moment which will make the diagram become somewhat as shown by $AEFD$. When the two sets of stresses are combined as shown in Sketch (*d*), the resultant diagram may be represented by the triangles GHO and OLM of Fig. 7-22(*e*).

The general result of all of this is a decrease in the compressive stresses, an increase in the tensile ones, and a raising of the neutral axis, i.e., a decrease of kd. However, the same principles that were utilized in the previous articles must apply to this case also. Therefore, using H instead of V, $\Sigma H = 0$, and $\Sigma M = 0$.

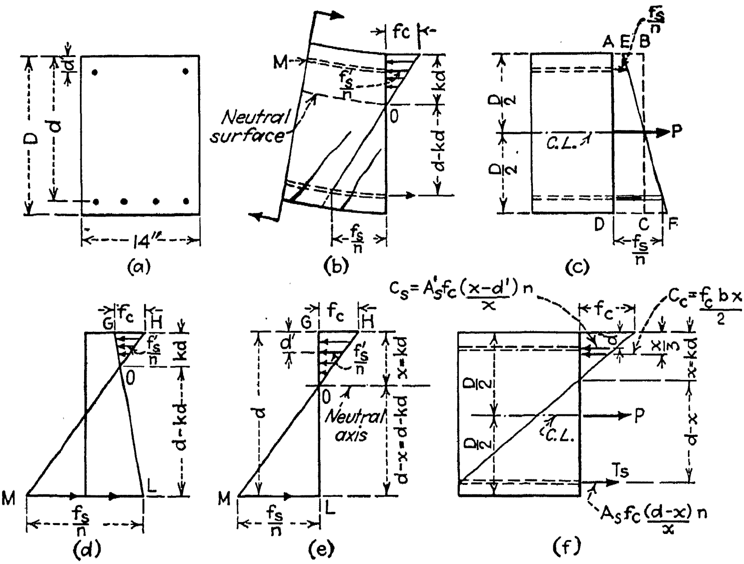


FIG. 7-22.

From Fig. 7-22(f) these equations can be written and then solved in the same manner as for the preceding cases.

Practice Problems

Problem 7-8. Assume a short, circular column like that of Fig. 7-3(a). Let $D = 22$ in., $D_1 = 19$ in., $D_2 = 17.5$ in., the rods $= 12$ equally spaced 1-in. rounds ($A_s = 9.48$ sq. in.), and $n = 10$. Find the stresses in the steel and the concrete for a centrally applied load of 200,000 lb. and a bending moment of 250,000 in.-lb. about the axis X-X.

Discussion. Follow the same procedure as in Problem 7-1.

Problem 7-9. Recompute Problem 7-8 with all data remaining the same except for the use of twelve $\frac{3}{4}$-in. round rods.

Problem 7-10. Assume a rectangular member like that of Fig. 7-5. Let $b = 16$ in., $d = 18$ in., the total depth $D = 20$ in., the longitudinal reinforcement $=$ five $\frac{7}{8}$-in. round rods, $n = 12$, and the allowable $f_c = 800$ lb. per sq. in. Find the stresses in the member if the direct load $= 50{,}000$ lb.

and the bending moment about an axis parallel to the row of rods = 500,000 in.-lb. Is the member safe?

Discussion. Use Eqs. (7-13), (7-10), and (7-7).

Problem 7-11. Find the stresses in the member of Problem 7-10 if the direct load = 75,000 lb., the bending moment = 450,000 in.-lb., the allowable f_c = 900 lb. per sq. in., and $n = 10$. Is the member safe?

Problem 7-12. Assume a rectangular member like that of Fig. 7-10. Let $b = 16$ in., $d = 18$ in., the total depth = 21 in., the longitudinal reinforcement = five $\frac{7}{8}$-in. round rods 3 in. from the left edge and another equal set 3 in. from the right edge, and $n = 12$. Find the stresses in the member if the direct load = 80,000 lb. and the bending moment about an axis parallel to the rows of rods = 560,000 in.-lb.

Discussion. Use Eq. (7-23*a*) to find kd, Eq. (7-21) to get f_c, and Eqs. (7-19) and (7-20) to find f_s and f_s'.

Ans. kd = 14.3 in., f_c = 610 lb. per sq. in., f_s' = 5,800 lb. per sq. in., f_s = 1,900 lb. per sq. in.

Problem 7-13. Find the stresses in the member of Problem 7-12 if $n = 8$.

Problem 7-14. Assume a square column like that of Fig. 7-12(*a*). Let $b = D = 28$ in., $d = 25$ in., the longitudinal reinforcement = twenty $1\frac{1}{8}$-in. square rods spaced uniformly, the cover over the rods = 3 in., and $n = 8$. Find the stresses if the direct load = 175,000 lb. and the bending moment = 2,100,000 in.-lb.

Discussion. Solve the problem in the same manner as Problem 7-4.

Problem 7-15. Find the stresses in the column of Problem 7-14 if $n = 10$, the direct load = 160,000 lb., and the bending moment = 2,000,000 in.-lb.

Problem 7-16. Find the stresses in a circular column like that of Fig. 7-13(*a*) if P = 120,000 lb., M = 840,000 in.-lb., D = 24 in., d' = 3 in., D_2 = 18 in., d = 21 in., $n = 10$, and the reinforcement is twelve 1-in. round rods.

Discussion. Follow the procedure used in Problem 7-5, using the curves of Figs. 7-15 and 7-16.

Problem 7-17. Recompute the stresses in the column of Problem 7-8 if $n = 8$, the direct load = 90,000 lb., and the bending moment = 450,000 in.-lb.

CHAPTER 8

RETAINING WALLS

8-1. Introduction. Retaining walls are used to provide lateral support for a mass of earth or other material the top of which is at a higher elevation than the earth or rock in front of the wall, as shown in Fig. 8-1.

Gravity retaining walls such as that in Sketch (*a*) depend mostly upon their own weight for stability. They are usually

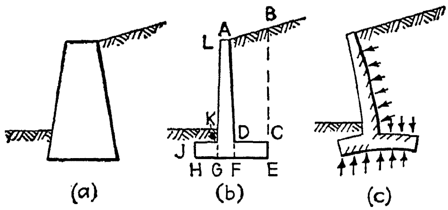


FIG. 8-1.

low in height. They are expensive because of their inefficient use of materials; sometimes they can be cheapened by using cyclopean concrete—concrete in which fairly large rocks are buried.

In contrast to them, Fig. 8-1(*b*) pictures an ordinary "cantilever" retaining wall. Part of its stability is obtained from the weight of the earth mass *ABCD*, but the wall's resistance to collapse depends upon the strength of its individual parts as cantilever beams. This action is pictured in Fig. 8-1(*c*).

Figure 8-2 shows some of this type of work as it looks in a large construction job. It is part of the Exit Plaza of the South Tube of the Lincoln Tunnel at New York City. In the background is part of a wall for which the concrete of the lower half has been placed and from which the forms have been stripped. Beside it is another portion for which the reinforcement has been placed and the forms are being built. The next part is a combined wall and pump room with the inside forms and part of the reinforcement clearly visible. The nearest portion shows the forms in place, braced, and ready for the pouring of the concrete.

The design of retaining walls requires a combination of theory and practical engineering sense. The designer must think of

them in terms of the procedures that are involved in their construction. Therefore, in order to show these things clearly and to illustrate all of the principles that may be involved in such construction, the problems that are used in this chapter are practical cases which are taken directly from such work as that

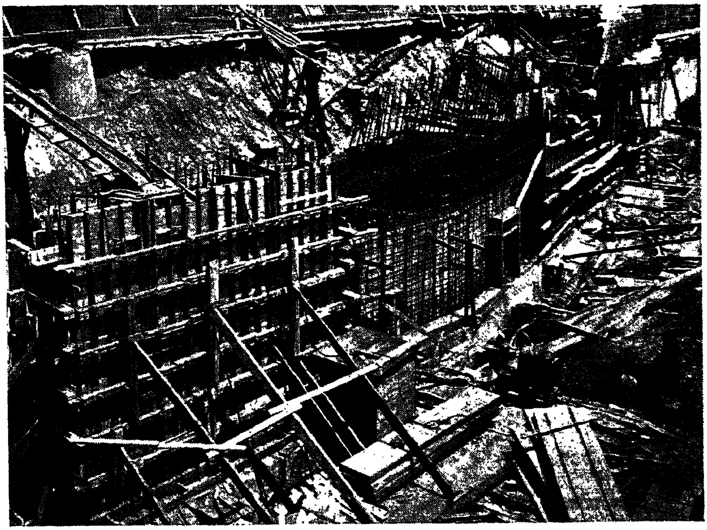


FIG. 8-2.—Construction of retaining walls, Exit Plaza of the Lincoln Tunnel, New York City.

in Fig. 8-2. The solutions of more simple problems will be relatively easy.

8-2. Definitions of Parts. The various portions of a typical reinforced-concrete retaining wall are defined as follows, using Fig. 8-3(*a*) for reference:

1. Stem—the portion *ADKL*.
2. Footing, or base—the part *JCEH*.
3. Toe—the projecting part of the footing on the side toward which the wall tends to tip *JKGH*.
4. Heel—the projecting portion of the footing on the side from which the wall tends to tip *CDFE*.
5. Back—the surface *AD*.
6. Front—the surface *LK*.
7. Foundation—the material under the footing, below *HE*.

8-3. Types of Reinforced-concrete Retaining Walls. *T-shaped Wall.* A "T-shaped" retaining wall is shown in Fig. 8-3(*a*). This is the most simple and common type of cantilever wall. The base should be from 0.4 to 0.6 times the total height *AH*, but it will vary somewhat with the position of the stem along the base and with the strength of the foundation. The length of the toe *JK* should be about one-fourth to one-half of the base, the stem being located nearer the rear when it is desired to obtain foundation pressures that are as small and as nearly uniformly distributed as it is possible to have them—as for

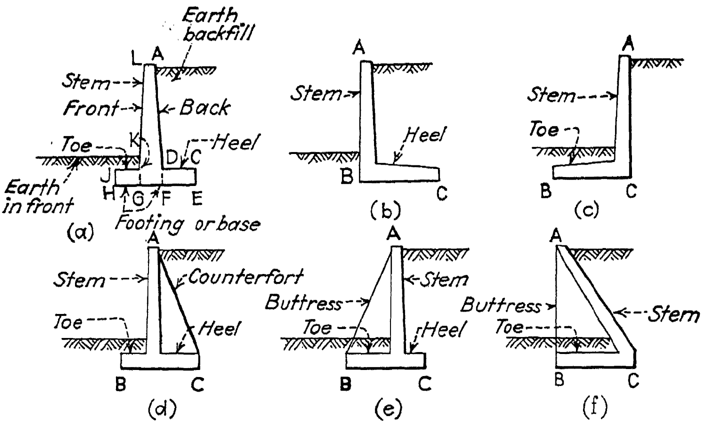


FIG. 8-3.

foundations upon clay. Possible troubles from sliding because of the decrease in the dead load will be discussed later.

L-shaped Wall. An "L-shaped" wall such as that in Fig. 8-3(*b*) is used when the wall is along a property line or in other situations where a toe cannot be provided. Its disadvantages are excessive pressure at the front edge *B* and difficulty in resisting the bending moment at the junction between the stem and the heel. The base should be about 0.5 to 0.55 times the height *AB*.

Reversed L-shaped Wall. A "reversed L-shaped" wall is illustrated in Fig. 8-3(*c*). It is usually difficult to make such a wall stable and to keep it from sliding if the height is great, because of the fact that the dead load is relatively small. However, such walls are useful in cases where it is expensive or impossible to provide a heel. The base *BC* should be about 0.5 to 0.6 times the height *AC*.

Counterforted Wall. A "counterforted" wall such as that of Fig. 8-3(*d*) is a modification of the T- or L-shaped ones. It has intermittent vertical ribs called counterforts. This is advantageous for very high walls because the counterforts can be heavily reinforced so as to act as ties to connect the stem and the heel, really transforming the last two parts into continuous slabs which are supported by the counterforts. Although the stem and the heel can be relatively thin, the extra formwork and details may offset the economy in materials. The base should be about the same width as for a T- or an L-shaped wall.

Buttressed Wall. A "buttressed" wall such as that in Fig. 8-3(*e*) is like a reversed counterfort type with ribs or walls that serve the same general functions as the corresponding parts in a counterforted wall except that they are compression members instead of ties. The toe and the stem are continuous slabs. Such a wall may be built with an inclined stem as shown in Fig. 8-3(*f*). The base *BC* should ordinarily be about 0.5 to 0.6 times the height *AB*, depending upon the size of the heel.

8-4. Stability and Safety Factor. The stability of a retaining wall is its ability to hold its position and to perform its function safely. The safety factor is a measure of the magnitudes of the forces that are required to cause failure of the structure, compared to the forces that are really acting upon it. Thus, if the safety factor is 1, the wall will be upon the point of failure. If, for any given design, it is 2, then the overturning moment or the horizontal forces may be doubled before the wall will fail. The magnitude of the safety factor to be used in a design will depend upon the engineer's judgment, the specifications, or the building code that is to be followed. In general, it may vary from 1.5 to 2.

A retaining wall may fail in one of four ways: by the collapse of its component parts, by overturning about the front of its toe, by excessive pressure upon its foundation, or by sliding upon its foundation. In a well-balanced design, the wall should be equally safe in all respects.

The bending of the individual parts and the pressures that act upon them are illustrated in Fig. 8-1(*c*). Each part must act as a cantilevered beam.

In order to illustrate overturning, let *W* of Fig. 8-4 represent the resultant of all of the vertical forces, including the weight

of the wall and the vertical component of the lateral earth pressure (if any) and of the earth on top of the base; let H = the resultant of all of the horizontal forces. Then Hn represents a moment that tends to overturn the wall as an entity, rotating it about the point A, but Wm is the "righting," or "stabilizing," moment which resists this overturning. If Hn exceeds Wm, the wall will tip over because the resultant of H and W will pass outside the base (beyond A).

Failure due to excessive pressure on the foundation will result in the tipping or overturning of the structure. When the wall is founded upon rock, it may actually rotate about the corner A, because the rock is very strong, but when it is on earth, the latter will settle, and the wall will tilt about a point to the right of A if the concentration of pressure becomes too great. The effective value of m will be decreased. If the overturning moment exceeds the reduced righting moment, the wall will overturn.

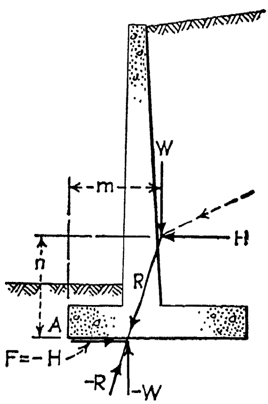


FIG. 8-4.

Sliding on the foundation may be demonstrated by considering Fig. 8-4. The resultant of W and H is represented by R. This, in turn, must be resisted by an equal and opposite reaction which may be called $-R$ and which will have components equal to $-W$ and $-H$. The latter force is caused by the friction of the base upon the foundation so that $F = -H$. Then, if f = the coefficient of sliding friction, $F = Wf$. When F can actually counteract H, the wall is said to be "stable against sliding." Of course, when the wall is built upon an irregular rock surface, there is no difficulty about sliding.

A retaining wall needs weight in order to resist overturning and sliding. Therefore, it is not usually advisable to use high-strength concrete and excessively thin sections for ordinary walls because they will be too light in weight. Web reinforcement in retaining walls is very troublesome. It should be avoided by keeping the shearing stresses low.

It is often difficult to secure the desired safety factor against sliding in the case of earth-borne walls. The ground in front

of the structure may have considerable abutting power or passive resistance to being shoved away, but the wall should stand without depending upon this force. Absence of the earth in front of the wall when the backfill is placed behind the structure, thoughtless excavation of the earth along the toe by someone in the future, possible scouring or washing away of this material—all of these are reasons for this statement. Sometimes this passive resistance is relied upon, but this should be done with caution.

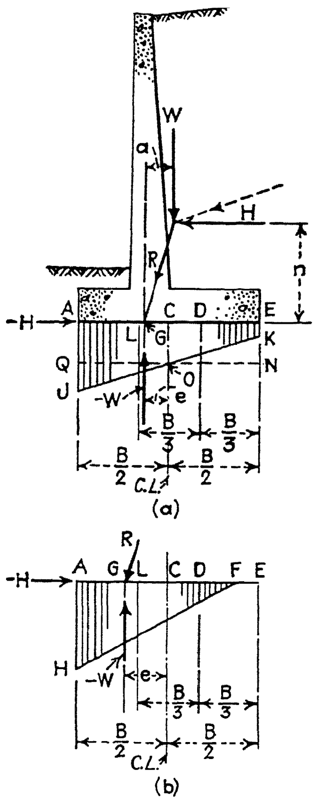


FIG. 8-5.

8-5. Foundations. Retaining walls that are founded upon earth present more of a problem than do those which are supported upon rock, because the high pressures that can be applied to the rock are not permissible upon the earth. Those pressures are the result of the combined action of the direct vertical load W and the overturning moment Hn.

Two different sets of pressure distributions are shown in Fig. 8-5. These differences are caused by the relative effects of the direct load and the overturning moment, but it is convenient to look upon them in terms of the position of the point at which the resultant of W and H intersects the bottom of the footing. If G is this point, its location with respect to W can be found by taking moments about G itself. Therefore,

$$a = \frac{Hn}{W}. \tag{8-1}$$

This enables one to compute e, the eccentricity of the resultant with respect to C, the center of the base.

When G is at the right of L, it is inside the middle third of the footing. Therefore, the pressure diagram $AEKJ$ of Sketch (a)

is made up of a uniformly distributed pressure equal to EN which equals W/B for a 1-ft. strip of wall, and a uniformly varying pressure which is caused by the moment

$$M = We.$$

$$QJ \qquad \text{or} \qquad KN = \frac{Mc}{I} = We \times \frac{6}{1 \times B^2}.$$

Therefore,

$$p = \frac{W}{B} \pm 6\frac{We}{B^2} = \frac{W}{B}\left(1 \pm \frac{6e}{B}\right) \tag{8-3}$$

where p is the intensity of the greatest or the least pressures, and B is the width of the footing. The positive sign is to be used to find the maximum pressure AJ; the negative sign will give the minimum pressure EK. For earth-borne walls, the pressure diagram should be as nearly rectangular as it is practicable to make it.

When the resultant falls outside the middle third (left of L), the pressure diagram in Fig. 8-5(b) results. Since there can be no tension upon the base near E, Eq. (8-3) cannot be used. The pressure diagram is assumed to be the triangle AFH. Its total area must equal the direct load W, and its center of gravity must lie vertically below G. Therefore, for a strip 1 ft. wide, $AH \times AF/2 = W$. Therefore, the maximum pressure AH is

$$AH = p = \frac{2W}{AF} = \frac{2W}{3AG}. \tag{8-4}$$

The exact distribution of the resisting pressure of the foundation may not vary as a straight line, but it is sufficient to assume that it does so.

The safe bearing value of any given soil and its probable deformation are so problematical and so dependent upon the qualities of the material itself that it is unwise to set any exact magnitudes for them, but the data of Table 8-1 may be used as a general guide. However, the conditions at the site must be examined and the bearing value of the soil should be tested before one designs an important retaining wall.

The frictional resistance of the foundation when a wall tends to slide upon it is also uncertain. For some soils, the resistance of the material against sliding upon itself is greater than that for the concrete sliding upon it. In order to take advantage of this

greater frictional resistance, the footings of walls are often made with projections on the bottom, called "cutoff walls," as shown in Figs. 8-6(*a*) and (*b*). Sometimes the bottom of the toe is

TABLE 8-1.—GENERAL DATA REGARDING FOUNDATION MATERIALS

Material	Safe bearing capacity, kips per square foot	Angle of repose, degrees	Maximum coefficient of friction of concrete on foundation
Sound rock................	80		
Poor rock..................	30		
Gravel and coarse sand......	10 to 12	37	0.6 to 0.7
Sand (dry).................	6 to 8	33	0.4 to 0.6
Fine sand (wet but confined)	4	25	0.3 to 0.4
Clay and sand mixed........	4 to 5	36	0.4 to 0.5
Hard clay..................	5 to 6	36	0.4 to 0.6
Soft clay..................	2	26	0.3

sloped upward, as in Fig. 8-6(*c*). The first method is likely to cause disturbance of the earth as the result of digging the trench in it. Probably the real value of these measures comes from the

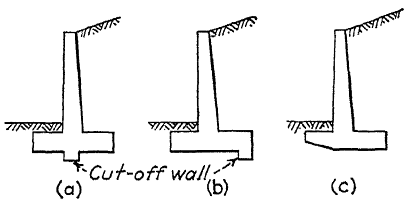


FIG. 8-6.

increase of the shearing and abutting values of the confined earth when it is subjected to the large pressures that exist under the forward portion of the footing.

It is very important to found all walls upon undisturbed material. The consolidation that has been produced by Nature has probably given the soil the best treatment that it can have for use as an ordinary foundation. In no case should a wall be placed upon newly deposited fill when avoidance of settlement is important. Furthermore, the excavation for the footing should be sufficient to remove organic matter and to get below the frost line—about 4 to 5 ft. in cold climates.

Another foundation problem which must not be overlooked in long walls is that of founding them partly upon rock and partly upon earth. If such conditions cannot be avoided, the structure should be designed so that the last portion that rests upon rock has a seat to receive the footing of the adjacent earth-borne structure, the latter being a rather long section which can act

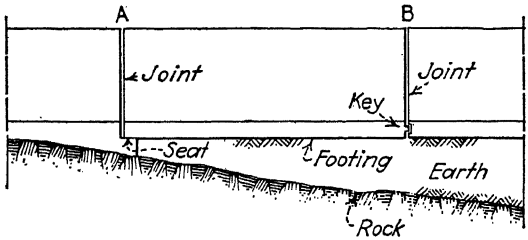


FIG. 8-7.

like a vertical beam as shown in Fig. 8-7. The compaction of the earth will then merely open the joint at A slightly, and it will close the one at B somewhat, but the motions will not be very apparent. On the other hand, if the rock has only local

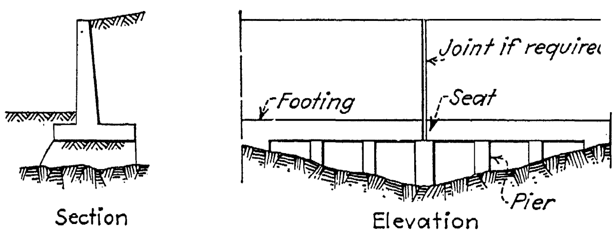


FIG. 8-8.

gullies of shallow depth, concrete piers without reinforcement may be built to carry the loads to the surface of the rock as shown in Fig. 8-8.

8-6. Lateral Earth Pressure. When an excavation is made in earth or when earth is piled up, the soil tends to slump and to move sidewise as shown in Fig. 8-9. Force is required to prevent this motion. The force exerted by the earth against any opposing structure is called the "active" earth pressure in order to differentiate it from the "passive" pressure—the resistance

of the earth to being shoved aside by an outside force. The greatest angle at which the earth slope will remain in equilibrium is called the "angle of repose," and it is usually denoted by ϕ. It ordinarily varies from 30 to 40° from the horizontal.

The magnitude of the active earth pressure—called "earth pressure" hereafter—is rather indeterminate. The soil behind any wall may vary greatly in its characteristics from place to place; when the backfill is deposited behind the wall in Fig. 8-10, it may exert a certain lateral pressure upon the structure; but when the wall deforms slightly, as shown by the dotted lines, it may tend to relieve itself of some of the pressure because of the cohesion or friction of the earth upon itself; if traffic or some other force causes vibrations that break down this internal frictional resistance, it may cause an increase of pressure upon the wall; when the earth dries out, it may shrink and settle as it dries; and when the soil is saturated by a rain, it may expand again.

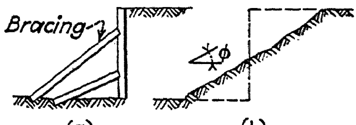


Fig. 8-9.

It is not possible to discuss in this text the various theories of earth pressures that have been developed by others. However, it is necessary to adopt one theory of earth pressures for use as a basis of design. Coulomb's theory[1] will be used in this text. This theory is based upon the assumption that the wedge of earth that lies above the plane of rupture—a plane at or above the one that is established by the angle of repose—will tend to slide downward and will shove the wall before it or will tip it over. A general case is pictured in Fig. 8-11(a), where ϕ is the angle of repose of the soil and P is the total earth pressure per linear foot of wall, P being parallel to the plane of rupture. The meaning of each of the other symbols is obvious.

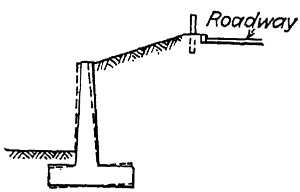


Fig. 8-10.

Coulomb's general formula for the total thrust of the earth per foot of the length of the wall is

[1] For brief explanation of both Coulomb's and Rankine's theories, see Milo S. Ketchum, "Structural Engineers' Handbook."

$$P = \frac{1}{2}wh^2 \frac{\sin^2(\theta - \phi)}{\sin^2\theta \sin(\theta + z)\left(1 + \sqrt{\frac{\sin(z + \phi)\sin(\phi - \delta)}{\sin(\theta + z)\sin(\theta - \delta)}}\right)^2} \tag{8-5}$$

where the meanings of all of the terms are indicated in Fig. 8-11(*a*). When applied to a reinforced-concrete cantilever wall, the conditions become as shown in Fig. 8-11(*b*). The vertical line *BE*—labeled h_1—through the end of the heel can be taken

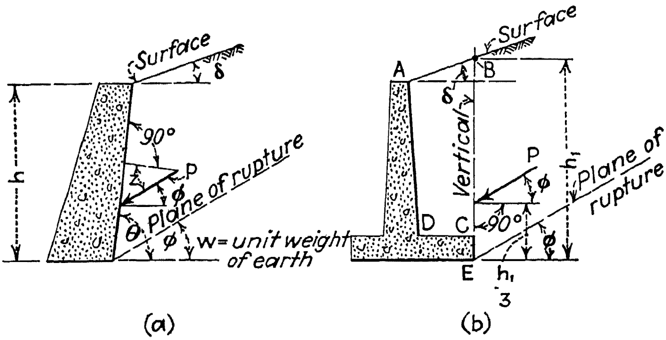


FIG. 8-11.

as the effective back of the wall unit, of which *ABCD* is earth. Then $z = \phi$ and $\theta = 90°$. Therefore, Eq. (8-5) becomes

$$P = \frac{1}{2}wh_1^2 \frac{\cos\phi}{(1 + \sqrt{2\sin^2\phi - 2\sin\phi\cos\phi\tan\delta})^2}. \tag{8-6}$$

The point of application of this force is $h_1/3$ above the bottom of the footing, assuming a triangular pressure diagram. Roughly, this is about equivalent to the fluid pressure caused by a liquid weighing 30 to 35 lb. per cu. ft.

If the angle of repose $\phi = 30°$, then Eq. (8-6) becomes

$$P = \frac{1}{2}wh_1^2 \frac{0.87}{(1 + \sqrt{0.5 - 0.87\tan\delta})^2}. \tag{8-7}$$

It has been stated that the force *P* is assumed to be parallel to the plane of rupture. This is a debatable question when the embankment or fill is subjected to vibrations due to trains, trucks, etc., which tend to break down the plane of rupture. In these cases, it is safer to assume that the earth pressure acts horizontally.

8-7. Surcharge. "Surcharge" generally denotes a temporary or live load which is applied on top of the earth behind a retaining wall, tending to increase the earth pressure. These loads may be caused by trains, vehicles, or even piles of materials.

Surcharge diagrams are shown in Fig. 8-12. Sketch (*a*) pictures a highway upon an embankment; W' is the wheel load of a truck. Obviously, the earth spreads the load W' over increasingly large areas of soil as the depth is increased. The pressure diagram may be assumed to be a cone. The horizontal

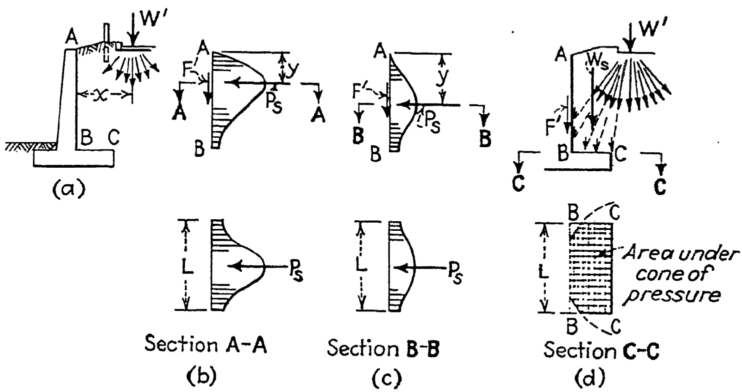


FIG. 8-12.

components of the pressure lines upon a vertical plane like AB of Fig. 8-12(*a*) must vary with the pressures and their directions. Experiments that have been made by M. G. Spangler[1] indicate that the diagram of horizontal pressures on AB which are due to the surcharge is somewhat as shown in Fig. 8-12(*b*)—a protuberance. They also indicate that the distance x has a great effect upon this diagram. When x is small, the magnitude of the resultant horizontal surcharge P_s is large, and its distance from the top of the wall is small. As x increases, the maximum intensity of pressure decreases as in Fig. 8-12(*c*); the forces are spread over a greater area; and y increases.

[1] Associate structural engineer, Iowa Engineering Experiment Station, Iowa State College, Ames, Iowa. The data have been published in *Paper* J-1, Vol. 1, p. 200, of *Proceedings* of the International Conference on Soil Mechanics and Foundation Engineering; also in Iowa Engineering Experiment Station, Iowa State College, *Bull.* 140.

At the same time, the pressures from the surcharge have vertical components as shown in Fig. 8-12(d). Part of these cause downward, stabilizing forces upon the heel BC; others may cause a downward frictional force F'. However, the latter does not seem to be reliable because of the effect of vibrations. Therefore, it will be allocated to those things which may increase the safety of the structure but are not included in the design.

With Spangler's report as a starting point, curves have been drawn as shown in Figs. 8-13, 8-14, and 8-15. The first two

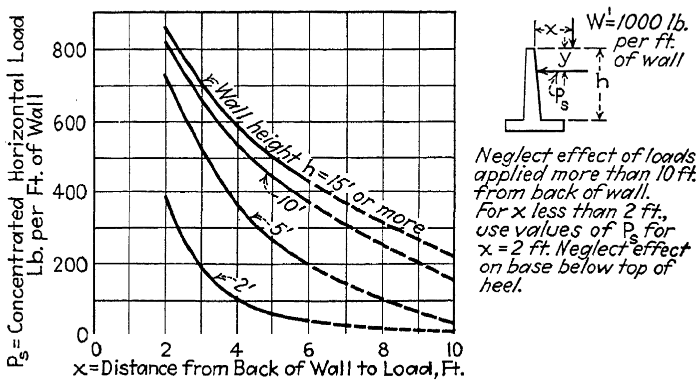


FIG. 8-13.—Lateral pressure due to concentrated surcharge loads.

enable one to determine and to locate a single horizontal force which should be included in the equations for moment and for sliding. On the other hand, Fig. 8-15 is prepared in order to give an approximate uniform load for the downward pressure on the heel as shown in Fig. 8-12(d). In preparing these surcharge diagrams,[1] W' was assumed to be 18,000 lb. The resultant of the horizontal components of the surcharge forces was used as a concentrated load. A piece of wall 18 ft. long was assumed to act as a unit, and the total force was then divided by 18 to get its average magnitude per foot of wall. When x exceeds 10 ft., the effect of the surcharge is neglected; when h is greater than 15 ft., the effect of the surcharge is assumed to be constant in magnitude and position because the effect of the bottom of the diagram of Fig. 8-12(b) is negligible. However if the load W' differs from that assumed herein—1,000 lb. per ft. over 18 ft.—the values in Fig. 8-13 may be modified in proportion but Fig.

[1] The curves have been prepared by Dr. A. H. Baker, formerly designer, The Port of New York Authority.

8-14 should not be greatly affected. It is unnecessary to include impact in these loads. Of course, the real downward pressure at any point on the heel (Fig. 8-15) varies with the location of the load, the height of the wall, and the relative position of the point on the heel. However, great refinement is not justified.

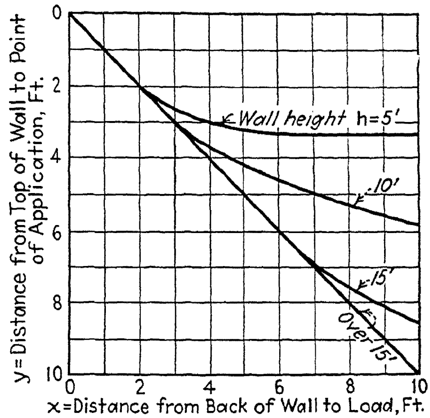


FIG. 8-14.—Location of point of application of resultant lateral pressure due to concentrated surcharge loads.

Therefore, a conical distribution at a maximum of 45° is assumed, and the curves have been made accordingly. This downward pressure becomes relatively so small when a wall is more than 15 ft. high that one may neglect it, but it is included in problems here for purposes of illustration.

If the surcharge is a uniform load—say 300 lb. per sq. ft.—it may be converted into an equivalent depth of earth. Using 100 lb. per cu. ft. as the weight of earth, this surcharge is similar to an extra fill 3 ft. deep. The wall can be designed as though the top of the earth really came to a point 3 ft. above its actual surface—as far as the earth pressure alone is concerned.

8-8. Water Pressure. Although a wall may have longitudinal drains behind it or small holes called "weepers" through the stem, as pictured in Fig. 8-16(*a*), the ground water behind the wall may become impounded if the outlets clog up or become filled with ice. If the wall is built upon solid rock, the water may be almost entirely held back. If the foundation is composed of earth, water may be able to escape under the footing, but it is likely to require some head to force it through the soil.

The presence of water in the soil increases the lateral pressure on the wall. If the weight of saturated earth is assumed to be 120 lb. per cu. ft., and if the lateral pressure that it exerts is 60 per cent[1] of this weight, then this pressure equals 72 lb. per sq. ft. per ft. of depth. However, using Eq. (8-7) for dry earth,

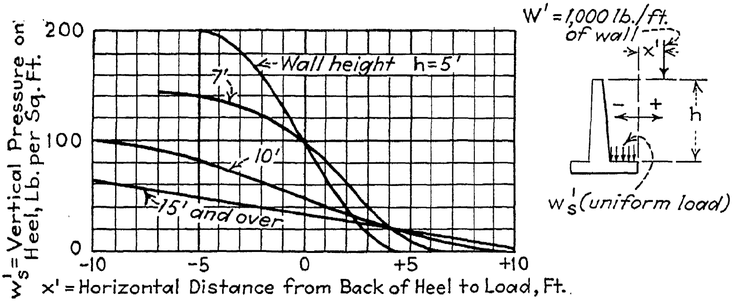


FIG. 8-15.—Downward pressure due to concentrated surcharge loads.

assuming $\delta = 0$, and using $w = 100$ lb. per cu. ft., $P =$ If $h_1 = 1$ ft., $P = 0.15w = 0.15 \times 100 = 15$ lb. This shows that the pressure for the dry earth is 30 lb. per sq. ft. per ft. of depth, since it varies uniformly from zero to a maximum. ($P =$ area of pressure triangle $= p \times 1/2$). The saturated earth exerts a pressure that is 42 lb. per sq. ft. larger than that of the

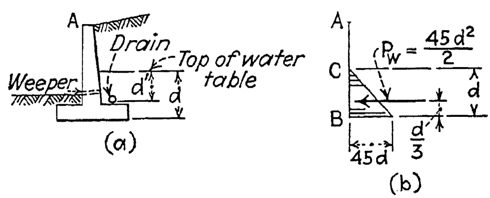


FIG. 8-16.

dry earth, but this excess will be called 45 lb. per sq. ft. per ft. of depth. However, the top of the "water table" must be ascertained before one can determine the total pressure caused by the water.

Theoretically, the existence of water pressure on the back of the wall would be accompanied by hydrostatic uplift under the footing and downward pressure on top of it, all but the last of these pressures tending to tip over the wall. However, these

[1] Karl Terzaghi, Pressure of Saturated Sand, *Eng. News-Record*, Feb. 22, 1934.

forces are rather indeterminate, and the inclusion of them causes needless complication of the calculations. On this account, the horizontal hydrostatic pressure will be used alone; the maximum distance from the invert of the drain to the top of the water table —d' in Fig. 8-16(a)—will be assumed to have an arbitrary magnitude of 8 ft. (For small walls, the water will be assumed never to be less than 5 ft. below the top of the earth even when it reduces d'.) The diagram of horizontal hydrostatic pressures will therefore be taken as shown in Fig. 8-16(b), the resultant being called P_w. If the back of the wall is sloped, the water pressure also has a vertical component, but it is usually negligible because of the steepness of the back.

8-9. Local Thrust at the Top of a Wall Due to Temperature. Some retaining walls are likely to have a localized thrust (P_T of Fig. 8-17) which will act near their tops. One cause of such a horizontal thrust is the action of frost. If the soil were saturated and then it froze solid, no ordinary wall could withstand the resultant pressure. With properly drained soil, the pressure from freezing must be greatly reduced from that which would be caused by solid ice. However, it seems that the expansion of backfill upon freezing and then its contraction and settlement upon thawing may account for the failure of some retaining walls after years of service, particularly small ones.

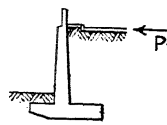


FIG. 8-17.

In order to make some arbitrary allowance for this pressure from ice, 700 lb. per lin. ft. of wall, applied at the top of the ground and parallel to the surface of the earth, will be assumed. This figure has no experimental verification. It is the product of experience and judgment, and is being used as indicated because of its relatively large effect upon small walls for which the thrust of the earth alone is small. If the ground is nearly level and the conditions facilitate the formation and collection of ice, this pressure will be included in the calculations; if the earth slopes and is thoroughly drained, it may be omitted. It should also be omitted from all calculations for sliding because the ground in front of the wall will be frozen also and it will resist the shear caused by ice pressure behind the structure.

8-10. Preliminary Steps in the Design of a Cantilever Retaining Wall. In order to illustrate the theory of the design of a reinforced-concrete retaining wall, a practical example will be

worked out in detail, using a problem that will illustrate most of the forces to be encountered in such work. For this, assume that the wall is to support a roadway with a 4-ft. sidewalk; it is to have a concrete parapet to protect traffic; the top of the pavement is to be 20 ft. above the adjacent ground. Assume also that the following data are specified:

w = weight of earth = 100 lb. per cu. ft.
Safe bearing value of earth = 7,000 lb. per sq. ft.; its ultimate value = 14,000 lb. per sq. ft.
Angle of repose of earth = ϕ = 30°.
Maximum coefficient of sliding friction of concrete on earth = 0.45.
Maximum coefficient of shearing friction of earth on earth = 0.55.

$$f'_c = 2{,}500 \text{ lb. per sq. in.,} \qquad \text{and} \qquad n = \frac{30{,}000}{2.500} = 12.$$

f_c = 750 lb. per sq. in.
f_s = 18,000 lb. per sq. in.; elastic-limit stress = 36,000 lb. per sq. in.
$v_L = 0.02f'_c$ = 50 lb. per sq. in. without special anchorage, $0.03f'_c$ with special anchorage.
$u = 0.05f'_c$ = 125 lb. per sq. in.

The maximum stresses in shear and bond shall not exceed two times those given above when testing for the safety factor.

Safety factor = 2, because the wall supports an important highway, and this value is very conservative. Its use will emphasize the relative seriousness of failure due to sliding, also the difficulty of securing the necessary safety factor against such action.

Equation (8-6) will be used to calculate the earth pressure.

The surcharge force and its position will be determined from Figs. 8-13 and 8-14; its downward pressure on the heel, from Fig. 8-15.

Water pressure will be 45 lb. per sq. ft. per ft. of depth, using a height of 8 ft. above the weepers.

Ice pressure will be 700 lb. per lin. ft. at the top of the roadway.

In any problem like this, there generally are certain fundamentals and details that are desired by the engineer who is in charge. These automatically influence the design. In this

case, some such points will be explained for the purpose of illustration, because these practical things are important and because they should be decided upon before the calculations are made. They are shown in Fig. 8-18.

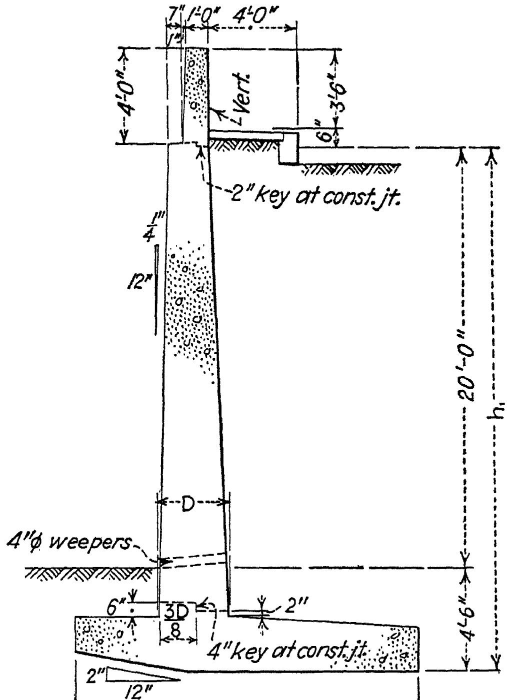


FIG. 8-18.

1. The wall is to be the T-type with a large heel and short toe—the latter being about $0.25B$—so as to utilize the dead load of the earth.

2. The bottom of the toe is to be sloped up 2 in. per ft. to help resist sliding.

3. The front face is to be battered $\frac{1}{4}$ in. per ft. for appearance.

4. The parapet is used as a guard rail. It is to be 12 in. thick at the top, and it is to be 3 ft. 6 in. high.

5. The top of the main wall is to be at least 20 in. thick. This provides a shelf which may improve the appearance of the

wall by reducing its apparent height. It also enables men to work inside the reinforcement and the forms during the placing of the concrete. (It is very difficult to avoid honeycombing when concrete is poured into forms for high, thin walls, but it can be avoided if portions of the upper part of the forms are

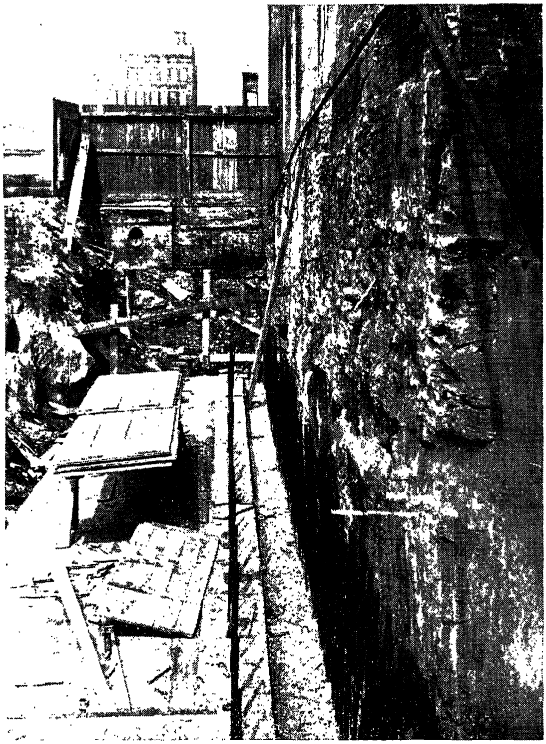


FIG. 8-19.—Footing of reversed L-shaped retaining wall, New York approach to Lincoln Tunnel.

left off until the concrete is placed in the lower part of the structure.)

6. The footing is to be poured in sections about 30 to 40 ft. long with a construction joint for the stem as shown in Figs. 8-18 and 8-19, the latter being a picture of a footing for a heelless wall. The projections above the toe and heel are important as stops against which the forms for the stem can be held so as to keep them from being displaced or becoming wavy during the pouring of the concrete. The 4-in. key is made so as to resist the

shear. The width of the forward, raised portion of it should be from one-fourth to one-half of the thickness of the stem, because the concrete that will be in the tensile side of the stem is not so effective in resisting shear as that which is in compression.

7. The stem is to be poured in one operation so as to avoid the marks and the change of appearance that may occur at horizontal construction joints. (Such deep lifts are difficult to make, and they are likely to have detrimental effects upon the quality of the concrete.)

8. The parapet is to be placed flush with the back of the wall with a 7-in. step at the outside as shown in Fig. 8-20(*a*).

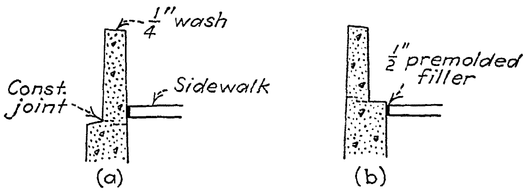


FIG. 8-20.

The step is placed below the sidewalk so that the top construction joint will not be visible. This arrangement with the offset on the outside face of the wall avoids the likelihood of settlement of the sidewalk away from the ledge, which might occur if it were constructed as in Fig. 8-20(*b*). On the other hand, if the sidewalk in the latter case is supported upon a shelf on the stem, settlement of the fill may tilt the slab or even crack it.

9. The bottom of the footing is to be 4 ft. 6 in. below the finished grade in front of the wall so as to be below the frost line.

10. Weepers 4 in. in diameter and 10 ft. c.c. are to be used at the level of the surface of the ground in front of the wall.

8-11. Design of the Stem. The effective height of the stem will be assumed to be 22 ft. The surcharge for trucks will be assumed to be about 1 ft. from the curb. The loading diagrams are then as pictured in Fig. 8-21.[1]

From Eq. (8-7), with $\delta = 0$ and $w = 100$,

$$P = \frac{1}{2}wh_1^2\frac{0.87}{(1 + \sqrt{0.5 - 0.87\tan\delta})^2} = 15h_1^2. \tag{8-8}$$

$$P = 15 \times 22^2 = 7{,}260 \text{ lb.},$$

[1] Note that the answers in such problems as these are not carried out to more than three significant figures in most cases.

where h is used instead of h_1. The force P is assumed to be horizontal because of the vibrations caused by traffic. When there are no such vibrations, the horizontal force may be assumed to equal $P \cos \phi$ (see Problem 8-1).

Writing the equation for moments about C, the top of the toe in Fig. 8-21, since the stem is a vertical cantilever beam,

$$M = P_T h + P\frac{h}{3} + P_w\frac{(8 + 2)}{3} + P_s(h - y)$$

$$M = 700 \times 22 + 7{,}260 \times \frac{22}{3} + 10 \times \frac{450}{2} \times \frac{10}{3} + 500 \times 17$$

$$M = 84{,}600 \text{ ft.-lb.}$$

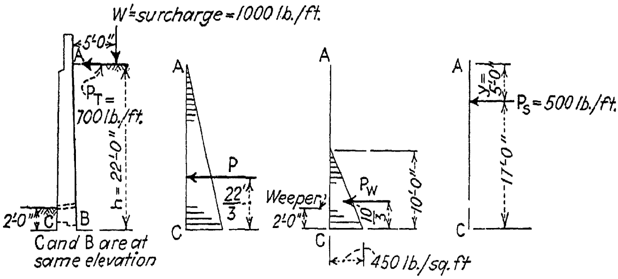


FIG. 8-21.

From Table 5 in the Appendix, with f_s, f'_c, and f_c equal to 18,000, 2,500, and 750 lb. per sq. in., respectively, K for a balanced design is 111, and $j = 0.889$. Then, using Eq. (2-7),

$$bd^2 = \frac{M}{K}, \qquad \text{or} \qquad = \sqrt{\frac{84{,}600 \times 12}{12 \times 111}} = 27.6 \text{ in.}$$

Assume $d = 28$ in. and the cover of the rods $= 3$ in. (to provide thorough protection against the moisture in the adjacent earth). $D = 31$ in., giving a batter of $\frac{1}{4}$ in. in 12 in. for the back below the sidewalk.

$$A_s = \frac{M}{f_s jd} = \frac{84{,}600 \times 12}{18{,}000 \times 0.889 \times 28} = 2.27 \text{ sq. in. per ft. (approx.)}$$

If 1-in. round rods are used at 4 in. c.c., $A_s = 2.37$ sq. in.; if $4\frac{1}{2}$ in. c.c., $A_s = 2.11$ sq. in. Inasmuch as the direct compressive load due to the weight of the stem will decrease the tension in the bars, the latter will be assumed.

The intensities of the shear and the bond stresses at the base of the stem are

$$v_L = \frac{H}{bjd} = \frac{P_T + P + P_w + P_s}{bjd} = \frac{700 + 7{,}260 + 2{,}250 + 500}{12 \times 0.889 \times 28} = 36 \text{ lb. per sq. in.}$$

$$u = \frac{H}{(\Sigma o)jd} = \frac{10{,}710}{3.14 \times \frac{12}{4.5} \times 0.889 \times 28} = 51 \text{ lb. per sq. in.}$$

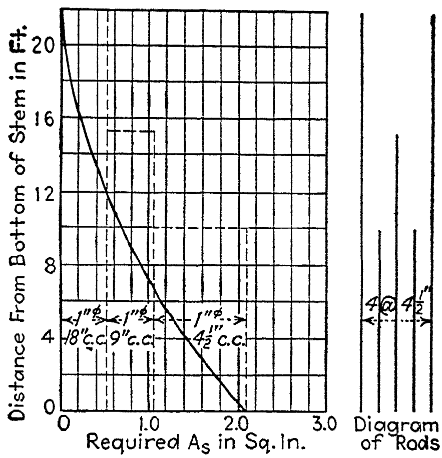


Fig. 8-22.—Diagram for determining the required lengths of reinforcement in stem.

It is unnecessary to extend all of these heavy rods for the full height of the stem. The required areas of steel at intermediate heights can be found with sufficient accuracy by assuming $j = 0.88$ and designing the wall for bending alone, neglecting any compressive steel. At mid-height,

$$M = P_T\frac{h}{2} + P\frac{h}{2 \times 3} + P_s\left(\frac{h}{2} - y\right)$$

$$M = 700 \times 11 + (15 \times 11^2) \times \tfrac{11}{3} + 500 \times 6 = 17{,}400 \text{ ft.-lb.}$$

$$A_s = \frac{M}{f_s jd} = \frac{17{,}400 \times 12}{18{,}000 \times 0.88 \times 22.5} = 0.59 \text{ sq. in.}$$

From such calculations, the curve of Fig. 8-22 can be drawn, showing the permissible cutoff of the rods. It is customary to extend them beyond the

theoretical points a distance that need not exceed what is required to develop them through bond. In this case,

$$L_s = \frac{A_s f_s}{\Sigma o \times u} = \frac{0.79 \times 18{,}000}{3.14 \times 125} = 36 \text{ in.}$$

The rods should not be much farther apart at the top than the thickness of the wall at this point.

The stem has a safety factor of 2 because, if M is doubled, the stresses in the steel and the concrete will not exceed the elastic limit of the rods or the ultimate strength of the concrete.

8-12. Stability and Foundation Pressure. Before the base of the wall is designed in detail, it is advisable to assume its size, to test the retaining wall for stability, and to see that the foundation pressure is satisfactory. To do so for this case, assume that the thickness of the footing is about equal to that of the base of the stem, that $B = 0.6h_1$, and that the toe is about $0.25B$.

Thus, let $D = 30$ in.;

$B = 0.6(22 + 2.5) = 14.7$ ft. Assume $B = 14$ ft. 9 in. and the toe $= 3$ ft. 6 in.

Figure 8-23(a) diagrammatically pictures the forces that are to be considered.[1] The stabilizing effect of the surcharge W_s is found from Fig. 8-15 for $-8.67 - 0.46 + 5 = -4.13$. It is 45 lb. per sq. ft. Also, from Eq. (8-8),

$$P = 15 \times 24.5^2 = 9{,}000 \text{ lb.}$$

$$P_w = \frac{12.5^2}{2} \times 45 = 3{,}500 \text{ lb.}$$

The weight of the earth on the toe will be neglected.

The magnitudes and lines of action of the resultant of the vertical forces W and the resultant of the horizontal forces H are found as follows, taking moments about E because of convenience in finding the proper lever arms:

Vertical forces		*Weight*	*Lever arm*		*Moment*
Stem W_c	=	7,640	$\times$ 9.93	=	75,900
Footing $W_F = 2.5 \times 14.75 \times 150$	=	5,530	$\times$ 7.38	=	40,800
Earth $W_e = 8.67 \times 22 \times 100$	=	19,100	$\times$ 4.34	=	82,900
Earth $W'_e = 0.46 \times 22 \times 0.5 \times 100$	=	500	$\times$ 8.82	=	4,400
Surcharge $W_s = 45 \times 8.67$	=	390	$\times$ 4.34	=	1,700
W	=	33,160 lb.	ΣM	=	205,700 ft.-lb

$$x = \frac{\Sigma M}{W} = \frac{205{,}700}{33{,}160} = 6.2 \text{ ft.}$$

[1] Notice that P is assumed to be horizontal so that $P_v = 0$. This is conservative, but there is much difference of opinion among engineers as to whether or not P may be assumed to slope at the angle ϕ under such conditions. If P is sloped, P_v is its vertical component.

Horizontal forces		*Force*		*Lever arm*	*Moment*
Temperature P_T	=	700	×	24.5	17,200
Surcharge P_s	=	500	×	19.5	9,800
Earth P	=	9,000		8.17	73,500
Water P_w				4.17	14,600
	H =	13,700 lb.		ΣM =	115,100 ft.-lb.

$$n = \frac{\Sigma M}{H} = \frac{115,100}{13,700} = 8.4 \text{ ft.}$$

These two forces are shown in Fig. 8-23(*b*).

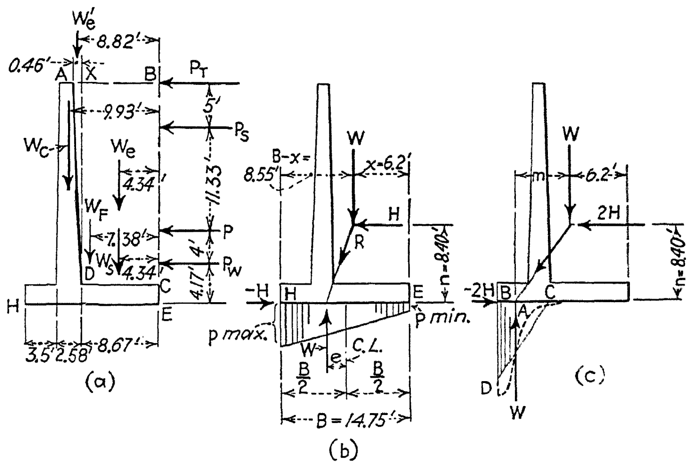


FIG. 8-23.

The next step is the determination of the foundation pressures. The eccentricity of the resultant of W and H with respect to the center of the base is found by taking moments about H of Fig. 8-23(*b*). Thus,

$$e = \frac{B}{2} - \left[\frac{W(B - x) - Hn}{W}\right]$$

$$e = 7.38 - \frac{283,500 - 115,100}{33,160} = 2.3 \text{ ft.}$$

This is less than $B/6 = 2.46$ ft. Therefore, the pressures on the foundation are

$$\text{Max. } p = \frac{W}{B}\left(1 + \frac{6e}{B}\right) = \frac{33,160}{14.75}\left(1 + \frac{6 \times 2.30}{14.75}\right) = 4,350 \text{ lb. per sq. ft.}$$

$$\text{Min. } p = \frac{W}{B}\left(1 - \frac{6e}{B}\right) = 144 \text{ lb. per sq. ft.}$$

The maximum is quite conservative.

The safety factor against overturning must now be tested. Assume the overturning moment Hn to be doubled. In this case, let n remain

unchanged, and replace H by $2H$ as in Fig. 8-23(c). Therefore $\Sigma M = 13{,}700 \times 2 \times 8.40 = 230{,}200$ ft.-lb. Take moments about A [Fig. 8-23(c)].

$$m = \frac{2Hn}{W} = \frac{230{,}200}{33{,}160} \quad 6.94 \text{ ft.}$$

$BA = 14.75 - 6.2 - 6.94 = 1.61$ ft. The point A is thus inside the toe B, but the pressure upon the foundation must be investigated. In Fig. 8-23(c), assume a triangular pressure diagram with the resultant W at its center of gravity. Therefore, $AC = 2(BA) = 3.22$ ft., and the maximum pressure is

$$W = \frac{BD \times BC}{2}$$

or

$$BD = \frac{2W}{BC} = \frac{2 \times 33{,}160}{4.83} = 13{,}700 \text{ lb.}$$

This value is less than twice the permissible pressure of 7,000 lb. per sq. ft. which was one of the conditions of the problem. If BD had exceeded the capacity of the earth, the latter would yield, and the pressure diagram might look like the dotted line in Fig. 8-23(c).

Remembering that the thrust due to the effects of temperature P_T is to be omitted when one computes the forces that cause sliding, the required coefficient of friction is

$$f = \frac{H - P_T}{W} = \frac{13{,}000}{33{,}160} = 0.39.$$

This seems to give a safety factor of the permissible coefficient divided by the required one, $0.45 \div 0.39 = 1.15$, which is far below the desired value. However, assume that the passive resistance of the soil in front of the wall is 10 times its active pressure so that $P = 10 \times 15h_1^2$.* Then, for a depth of 4.5 ft., this gives 3,040 lb. extra resistance per ft. of wall which increases the safety factor to $0.45 \div \dfrac{13{,}000 - 3{,}040}{33{,}160}$ = about 1.5. The bottom of the toe will be sloped upward as pictured in Fig. 8-18 in order to take advantage of the fact that the coefficient of shearing friction of earth on earth was assumed to equal 0.55 at the start of the problem. Since Fig. 8-23(c) shows that, before failure, most of the pressure diagram will be acting upon the toe, this higher coefficient will be used. Then the new safety factor is $0.55 \div \dfrac{13{,}000 - 3{,}040}{33{,}160} = 1.83$. This is not quite up to the original requirements, but it will be accepted.[1]

* An assumption that approximates Rankine's theory.

[1] Earth-borne retaining walls generally must be heavier than those which are on rock in order to prevent sliding. However, when one considers all probable forces, as has been done here, a safety factor of 2 is very conservative. If P_v is added to the vertical forces, the situation is improved.

8-13. Design of the Heel. The footing, as it has been assumed in the preceding article ($D = 30$ in.), will now be designed for strength.

From Fig. 8-23(a), it is apparent that the weight of the earth and the heel and also the pressure from the surcharge are the forces that will affect the heel when the wall tends to tip over, causing the heel to bend as shown in Fig. 8-24(a). The pressure upon the foundation tends to relieve this bending. Figure 8-24(b) indicates the portion of the pressure diagram that acts in this way.

The heel is a cantilever beam which projects from the line of resistance, the tensile steel in the stem. Therefore, taking moments about O,

$M = (W_e + W_s) \times$ lever arm + weight of heel $\times$ lever arm $-$ area of pressure diagram $\times$ lever arm.

$$M = (19{,}100 + 390)(4.33 + 0.25) + \frac{2.5 \times 8.92^2}{2} \times 150 - \frac{144 \times 8.92^2}{2}$$
$$- \frac{2{,}544 \times 8.92}{2} \times \frac{8.92}{3} = 64{,}700 \text{ ft.-lb.}$$

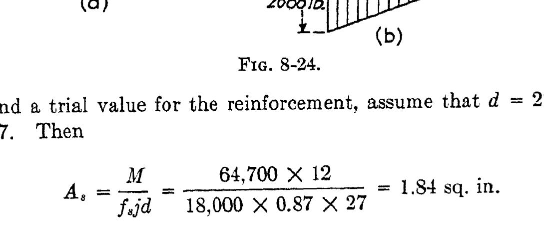


FIG. 8-24.

To find a trial value for the reinforcement, assume that $d = 27$ in. and $j = 0.87$. Then

$$A_s = \frac{M}{f_s jd} = \frac{64{,}700 \times 12}{18{,}000 \times 0.87 \times 27} = 1.84 \text{ sq. in.}$$

The rods that are used should be spaced in multiples of $4\frac{1}{2}$ in. so as to match those coming down from the stem. Using 1-in. round rods at $4\frac{1}{2}$ in. c.c. gives $A_s = \dfrac{0.79 \times 12}{4.5} = 2.11$ sq. in. Therefore,

$$p = \frac{A_s}{bd} = \frac{2.11}{12 \times 27} = 0.0065, \qquad \text{and} \qquad pn = 0.0065 \times 12 = 0.078$$

$$k = \sqrt{2pn + (pn)^2} - pn = 0.324 \text{ (see Fig. 12, Appendix).}$$

$$j = 1 - \frac{0.324}{3} = 0.89$$

$$f_s = \frac{M}{A_s jd} = \frac{64{,}700 \times 12}{2.11 \times 0.89 \times 27} = 15{,}300 \text{ lb. per sq. in.}$$

$$f_c = \frac{2M}{kjbd^2} = \frac{2 \times 64{,}700 \times 12}{0.324 \times 0.89 \times 12 \times 27^2} = 616 \text{ lb. per sq. in.}$$

$$V = W_e + W_s + \text{heel} - \text{pressure diagram.}$$

Therefore,

$$V = 19{,}100 + 390 + 2.5 \times 8.92 \times 150 - 144 \times 8.92 - \frac{2{,}544 \times 8.92}{2}$$
$$= 10{,}200 \text{ lb.}$$

$$v_L = \frac{V}{bjd} = \frac{10{,}200}{12 \times 0.89 \times 27} = 35 \text{ lb. per sq. in.}$$

$$u = \frac{V}{(\Sigma o)jd} = \frac{10{,}200}{3.14 \times \frac{12}{4.5} \times 0.89 \times 27} = 51 \text{ lb. per sq. in.}$$

Before calling these results satisfactory, the heel must be checked to see that, if the wall tips so as to relieve the upward foundation pressures under the heel, the resultant stresses will not exceed the elastic limit of the steel—$f_s = 36{,}000$ lb. per sq. in.—or the ultimate strength of the concrete. For this case,

$$M = (W_e + W_s) \times \text{lever arm} + \text{weight of heel} \times \text{lever arm.}$$

Therefore,

$$M = (19{,}100 + 390)(4.33 + 0.25) + \frac{2.5 \times 8.92^2 \times 150}{2} = 104{,}200 \text{ ft.-lb.}$$

Since this does not exceed twice the 64,700 ft.-lb. previously calculated, the rods in the heel will not be overstressed.

The heel will be sloped on the top from $D = 30$ in. at the stem to 18 in. at the back edge so as to save concrete, but the stresses will not be recomputed. Alternate rods may be cut off at about 7 ft. from the stem, but the others will be extended full length and will be hooked as pictured in Fig. 8-29.

8-14. Design of the Toe. Part of the pressure diagram of Fig. 8-23(b) is reproduced in Fig. 8-25(a). Under its influence the toe will bend as shown in Fig. 8-25(b), but the weight of the concrete of the toe will be counted upon to counteract some of the bending moment. Technically, the frictional resistance along the base and under the toe will also tend to annul part

of the bending moment, but it is too uncertain and too theoretical to rely upon. Then

$$M = (3,352 - 2.0 \times 150)\frac{3.5^2}{2} + 0.67 \times 998 \times \frac{3.5^2}{2} = 22,800 \text{ ft.-lb.}$$

A trial area of the rods is, assuming $d = 24$ in., because of the sloping bottom of the toe, and $j = 0.87$,

$$A_s = \frac{M}{f_s jd} = \frac{22,800 \times 12}{18,000 \times 0.87 \times 24} = 0.73 \text{ sq. in.}$$

Since the stem reinforcement is 1-in. round bars $4\frac{1}{2}$ in. c.c., assume that

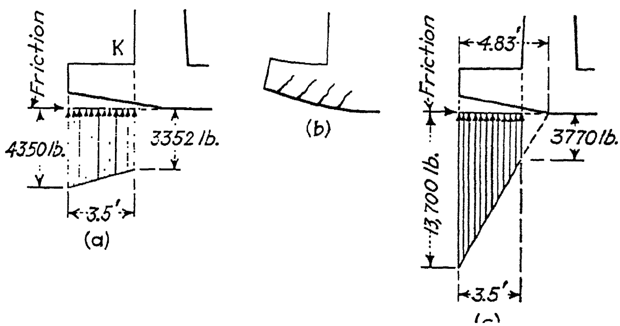


FIG. 8-25.

every other rod from the stem is bent around into the bottom of the toe. Then

$$A_s = 0.79 \times \frac{12}{9} = 1.05 \text{ sq. in.}$$

Neglect the rods in the top of the toe.

$$p = \frac{A_s}{bd} = \frac{1.05}{12 \times 24} = 0.0036, \qquad pn = 0.0036 \times 12 = 0.043$$

$$k = \sqrt{2pn + (pn)^2} - pn = 0.253, \qquad j = 0.92$$

$$f_s = \frac{M}{A_s jd} = \frac{22,800 \times 12}{1.05 \times 0.92 \times 24} = 11,800 \text{ lb. per sq. in.}$$

$$f_c = \frac{2M}{kjbd^2} = \frac{2 \times 22,800 \times 12}{0.253 \times 0.92 \times 12 \times 24^2} = 340 \text{ lb. per sq. in.}$$

$$v_L = \frac{V}{bjd} = \frac{0.5(4,350 + 3,352)3.5 - 300 \times 3.5}{12 \times 0.92 \times 24} = 47 \text{ lb. per sq. in.}$$

$$u = \frac{V}{(\Sigma o)jd} = \frac{12,400}{3.14 \times \frac{12}{9} \times 0.92 \times 24} = 134 \text{ lb. per sq. in.}$$

These stresses will be accepted.

The magnitudes of v_L and u are close to the allowable values that were assigned in the problem. Therefore, the rods should be hooked to improve the condition. The toe will be shaped, and the reinforcement will be arranged, as shown in Fig. 8-29.

If the safety factor is tested by doubling H, and if the resultant pressure diagram of Fig. 8-23(c) is applied to the toe, the upward pressures will be as shown in Fig. 8-25(c). Then

$$M = (3{,}770 - 300) \times \frac{3.5^2}{2} + 0.67 \times 9{,}930 \times \frac{3.5^2}{2} = 62{,}000 \text{ ft.-lb.}$$

$$f_s = \frac{62{,}000 \times 12}{1.05 \times 0.92 \times 24} = 32{,}000 \text{ lb. per sq. in.}$$

$$f_c = \frac{2 \times 62{,}000 \times 12}{0.253 \times 0.92 \times 12 \times 24^2} = 930 \text{ lb. per sq. in.}$$

These two stresses are within the allowable values.

$$v_L = \left(\frac{13{,}700 + 3{,}770}{2} - 300\right)\frac{3.5}{12 \times 0.92 \times 24} = 111 \text{ lb. per sq. in.}$$

$$u = \frac{29{,}500}{3.14 \times \frac{12}{9} \times 0.92 \times 24} = 319 \text{ lb. per sq. in.}$$

The shearing stress is satisfactory as a maximum value, since v_L can be $0.03 \times 2{,}500 \times 2 = 150$ lb. per sq. in., but the magnitude of u should not exceed $0.05 \times 2{,}500 \times 2 = 250$ lb. per sq. in. However, rods E of Fig. 8-29, which are added for the reasons that are explained in Art. 8-16, will make up the deficiency.

8-15. Design of the Parapet. The parapet at the top of the wall must be made sufficiently strong to withstand the effect of a vehicle colliding with it. The force of such a blow is problematical but, since it may be spread over a considerable length of the parapet, it will be assumed to be 500 lb. per ft. of wall, applied as shown in Fig. 8-26. It will not be considered in the design of the main wall because of the latter's safety factor and inertia.

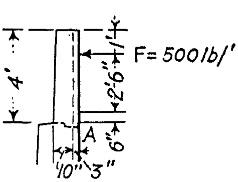


Fig. 8-26.

Assuming $j = 0.9$,

$$A_s = \frac{M}{f_s jd} = \frac{500 \times 36}{18{,}000 \times 0.9 \times 10} = 0.11 \text{ sq. in.}$$

Using $\frac{1}{2}$-in. round rods 12 in. c.c., $A_s = 0.2$ sq. in. These rods will be adopted and no further analysis is necessary.

8-16. Arrangement of Reinforcement and Other Practical Details. Figure 8-27 is a photograph of the lower portion of the stem of a 35-ft. retaining wall which is located on the south side of one of the plazas of the Lincoln Tunnel at New York City. The work has been stopped at a horizontal construction joint. It shows many of the features that have been or will be discussed, such as the division into sections by expansion joints, keyways,

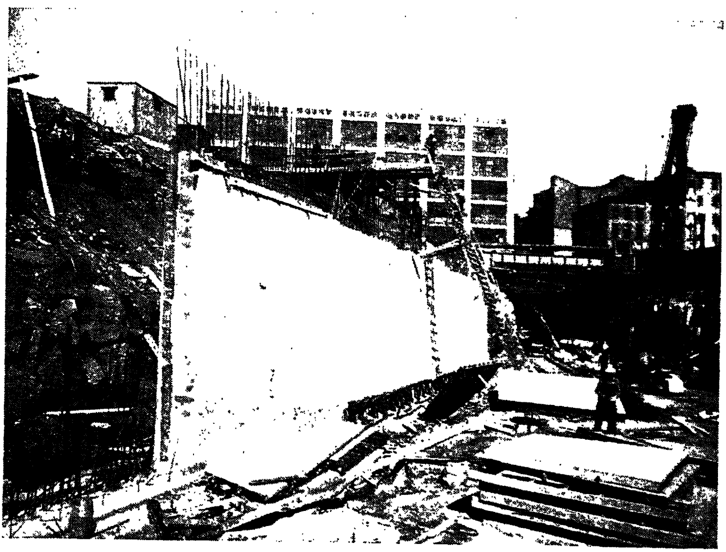


FIG. 8-27.—Partially completed retaining wall, Exit Plaza of the Lincoln Tunnel, New York City.

form ledges, dowels, flashing at expansion joints, and extensions of rods for the next pour.

Expansion (or contraction) joints should be located about 30 to 40 ft. apart so as to eliminate shrinkage and temperature cracks. Special horizontal reinforcement will be placed in the stem of each section to hold it together as a unit. For this purpose, an area of steel equal to 0.2 per cent of the cross section of the stem will be used with about two-thirds of the steel near the front face because of its greater exposure. The parapet will have reinforcement equal to 0.3 per cent. Then

$$A_s = 0.002\frac{(20 + 31)}{2} \times 12 = 0.61 \text{ sq. in. per ft. of height of stem.}$$

Use $\frac{3}{4}$-in. round rods 12 in. c.c. at the front and 24 in. c.c. at the back. ($A_s = 0.66$ sq. in.)

$$A_s = 0.003 \times 12.5 \times 12 = 0.45 \text{ sq. in. per ft. of height of parapet.}$$

Use $\frac{1}{2}$-in. round rods 10 in. c.c. front and back. ($A_s = 0.48$ sq. in.)

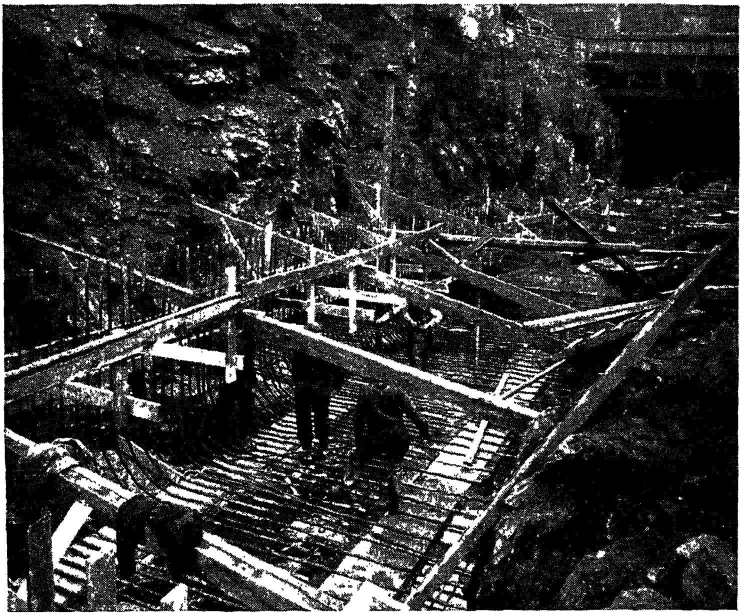
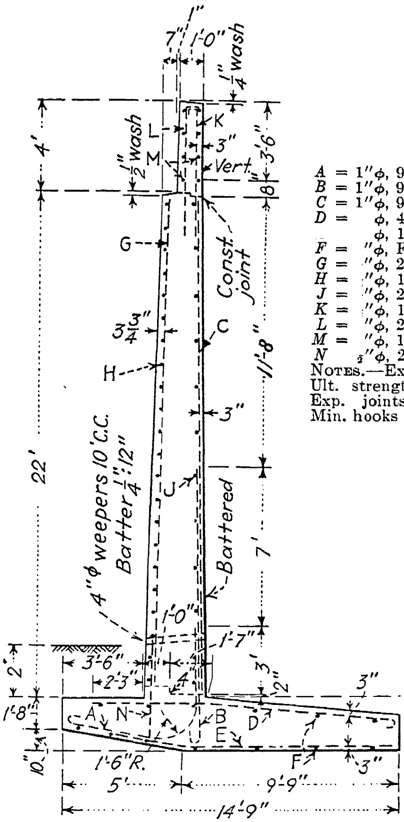


FIG. 8-28.—Placing reinforcement in the footing of a retaining wall, Lincoln Tunnel, New York City.

The longitudinal rods in the footing itself will be designed upon the basis of their use as ties; for a wall that is on earth, they should be 0.1 to 0.2 per cent of the cross section of the base in order to serve as adequate temperature reinforcement. Of course, when a footing is on rock, these rods are needed only as ties because the keying effect of the rough rock will not let the wall move.

Figure 8-28 is a picture at the base of the wall of Fig. 8-27—the small-heel and large-toe type. Note the way the toe rein-

forcement is bent up to serve as dowels for the stem, the cutting off of alternate rods in the toe, the tie rods, the heel reinforcement which the men are placing, the strips of concrete which will support the future chairs or spacers to be placed under the rods, and the timbers in the background for making the keys



Schedule of reinforcement

A = 1"ϕ, 9" c.c.
B = 1"ϕ, 9" c.c.
C = 1"ϕ, 9" c.c., alt. rods 15'-3" lg.
D = ϕ, 4½" c.c., alt. rods 11'-10" lg.
ϕ, 18" c.c.
F = "ϕ, F.L. (Full length)
G = "ϕ, 27" c.c.
H = "ϕ, 12" c.c., F.L.
J = "ϕ, 24" c.c., F.L.
K = "ϕ, 12" c.c.
L = "ϕ, 24" c.c.
M = "ϕ, 10" c.c., F.L.
N ½"ϕ, 27" c.c., 5'-0" lg.

NOTES.—Exposed surfaces to be smooth finish. Ult. strength of concrete = 2,500 lb. per sq. in. Exp. joints 30'-0" ± c.c. Min. lap = 36 diam. Min. hooks = 8 diam.

FIG. 8-29.—Dimensions and reinforcement.

and form-supporting strips which are shown completed in Fig. 8-27.

A sketch of the wall that has been designed here, as it would be made for contract purposes, is shown in Fig. 8-29. The explanations for certain practical details that have been shown in the sketch are as follows:

1. Rods A are extended as dowels for the stem, splicing rods C which are placed on the concrete of the construction joint after the base is set. (Rods should not be "hung in the air" if it is avoidable.) If the spacing of the bars is small, rods A should be placed farther from the back so that rods C can be set at their rear instead of beside them, thus avoiding a "screening effect" which might seriously injure the development of bond on the backs of these rods. The radius of the bend in rods A should be reasonable but not excessive.

2. Rods B are hooked into the footing and extended into the stem as far as they are required in order to avoid unnecessary laps. However, these rods must not be too long, or they will become difficult to hold in place. They are also useful in resisting the effect of the downward shear which comes from the loads on the heel. If the heel is large and all of the rods from the stem are bent like rods A, there is no reinforcement that extends directly into the region of the compressive forces caused by the cantilever action of the heel.

3. Rods D are extended into the toe to develop them beyond the neutral axis of the stem. The designer should notice that there is considerable question regarding the bond stresses along rods D in front of the stem. A glance at Fig. 8-1(c) shows the fundamental actions of the parts. Both the heel and the toe tend to rotate in the same direction. A substantial fillet at the junction of the toe and the stem might help to develop the bond under the compressive side of the stem, but it would be somewhat troublesome to build. In fact, rods A are also a possible source of weakness in bond, but one may look upon them as "cables" carried through the concrete to prevent closing of the angle between the top of the toe and the front of the stem.

4. Rods E are added arbitrarily in the footing to care for the bending stresses caused by the weight of the stem before the backfill is placed.

5. Rods F are below the main rods so as to simplify the supporting of the latter.

6. Rods G are set on the construction joint so as to support rods H which serve as temperature reinforcement.

7. Rods J are in front of A, B, and C because it is assumed that the back form for the stem will be erected first; then C will be placed. Rods G will be set next; then rods H will be wired to them; after which the front forms will be erected.

8. Rods K are bent to knit the top of the parapet together and to develop them.

9. Rods L are stuck into the wet concrete and serve as ties for rods M, the latter being on the outside because they are placed later. They might be placed inside of K and L.

10. Rods N are anchors to hold rods G and H so that they will not be displaced during the concreting.

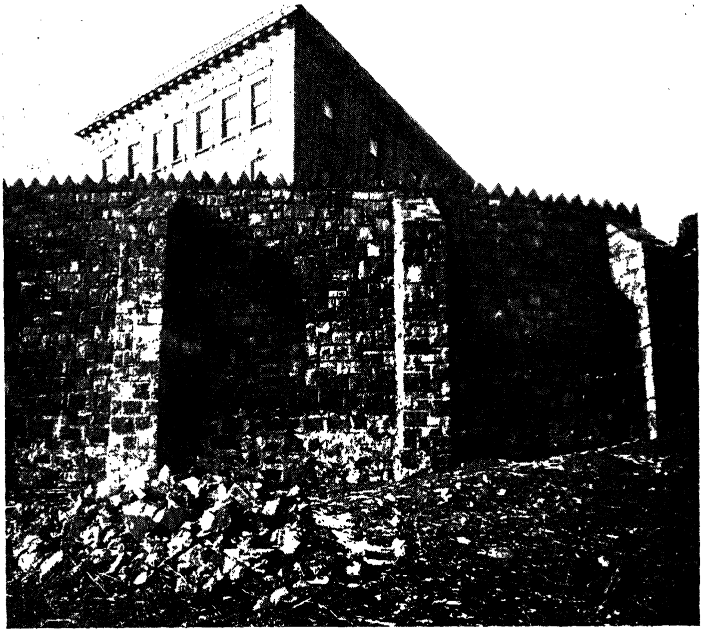


FIG. 8-30.—A buttressed masonry retaining wall, Weehawken, N. J.

8-17. Stone-faced Retaining Walls. A buttressed masonry retaining wall is pictured in Fig. 8-30. Such a wall must be more or less of the gravity type. However, walls of similar appearance may be built with stone facing which is backed by reinforced concrete as shown in Fig. 8-31.

If the masonry is applied on the face of the concrete and anchored thereto after the latter is set, the wall should be designed as an ordinary reinforced-concrete structure, neglecting the facing. However, if the stone facing is laid first, as it should be, a few feet at a time, then backed up with concrete which bonds

to it thoroughly, the stones should not be disregarded in the design. Bearing in mind that stress and strain are coexistent, it is not correct to assume that the stones carry no load. Therefore, some fundamental principles will be explained, and a method for the design of such walls will be outlined.

What is the distribution of the stresses upon a cross section of the wall of Fig. 8-32(*a*)? If the section is taken through the mortar—E = about 1,500,000 lb. per sq. in., for cement and

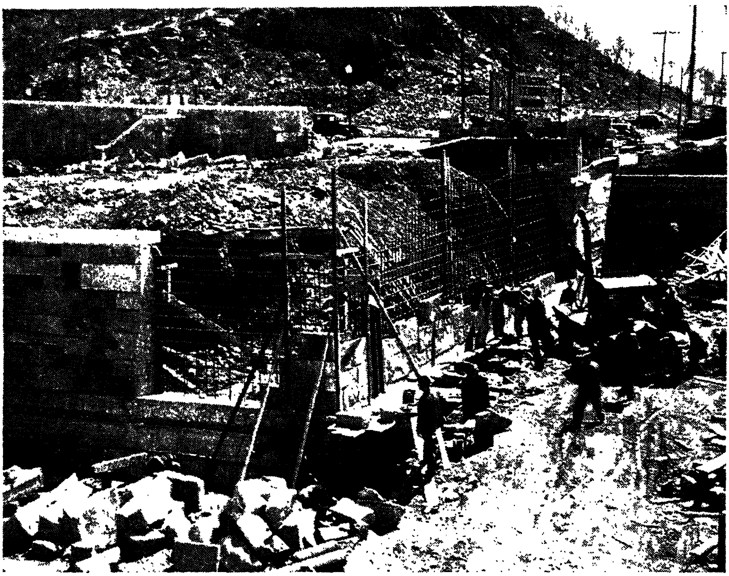


FIG. 8-31.—Construction of a stone-faced retaining wall, New Jersey Plaza of the Lincoln Tunnel.

lime, f'_c = about 1,500 lb. per sq. in.—the location of the neutral axis will depend partly upon n, which may be 20. Theoretically, the situation will be as shown by the solid lines in Fig. 8-32(*b*). If the section passes through the granite—E = say, 6,000,000 lb. per sq. in., and for solid granite f'_c = about 12,000 lb. per sq. in.—the theoretical position of the neutral axis will be very different because n may be 5. The dotted lines of Fig. 8-32(*b*) show this latter condition. Obviously, the neutral axis will not jump from O to O' and back again at each stone; the unit stress in the mortar cannot be one thing and that in the stone something else; the

stress in the steel opposite the mortar cannot be thousands of pounds different from what it is an inch or two away, opposite the stone. The stresses must be consistent and reasonable.

If the wall is assumed to be a set of blocks as shown in Fig. 8-32(c), with a rubber band tied to their backs and with shear dowels near their fronts, and if a moment is applied to the set, they will distort as pictured. The compressive stresses will be concentrated at the front edges of the blocks. The bands will stretch. Then $k = 0$, and $j = 1$. This will approximate the

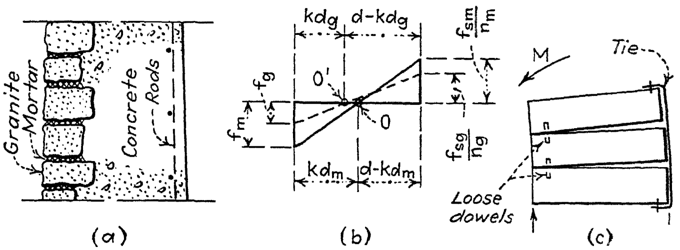


FIG. 8-32.

action of the granite and the steel alone if the wall is entirely granite and steel with no concrete. On the other hand, if the wall is composed of concrete only, k may be somewhere around 0.25 to 0.4, and j may be 0.92 to 0.87. Furthermore, the mortar in the joints, being weaker than the granite, will control the allowable magnitudes of the compressive stresses.

In general, the values of Table 8-2 may be used, assuming the best mortar and good workmanship:

TABLE 8-2.—PROPERTIES OF STONE MASONRY

Kind of masonry	Weight, lb. per cu. ft.	E, lb. per sq. in.	n	Allowable compressive stress, lb. per sq. in., f_c	
				Cement-lime mortar	Cement mortar
Best granite ashlar . . .	165	4,000,000	7.5	650	800
Medium granite ashlar	160	4,000,000	7.5	500	700
Rubble	150	2,000,000	15	250	350

The last column in this table may seem to indicate high values for f_c, but they will almost surely occur unless the wall is made unreasonably thick or the unit stress in the steel is kept very small. Unless the facing is relatively thin, the fact that some of the concrete is in compression may be neglected.

Therefore, in the design of masonry-faced walls, the following procedure is recommended:

1. Deduct at least 1 in. from the front face because of the weathering of joints. Then find d accordingly.
2. Use f_c and n as given in Table 8-2 for the materials and workmanship that are applicable.
3. Allow the usual maximum stress in the steel—18,000 lb. per sq. in.—but the designer should notice that f_s will be low when a poor quality of masonry is used, because f_c must be low.
4. Design the wall as though it were concrete, but use the value of n from Table 8-2.
5. Be sure that the stones are well bonded into the concrete so as to resist the longitudinal shear.
6. Use expansion joints as for concrete walls.
7. Generally, unless the wall is very thick, use no front layer of steel because it will interfere with the work. Place all of the temperature steel near the back.

A stone-faced wall supporting a side hill above a roadway cut is pictured in Fig. 8-33. The following features should be noticed:

1. The back of the stem is covered with membrane and asphalt waterproofing in order to avoid staining of the front by the leakage of water. This waterproofing is protected from injury by covering it with a 4-in. layer of concrete. (Sometimes bricks or 2-in. precast concrete blocks set in mortar are used.)
2. Since the wall rests upon rock, seepage water cannot readily pass under it. Weepers discharging onto the sidewalk will be objectionable, therefore a longitudinal drainage pipe with loose joints and surrounding gravel cover is placed on the heel. It has occasional outlets through the wall to a main drain under the roadway.
3. The toe is large and the heel is short so as to minimize excavation, but the latter must provide sufficient space for men to work in—about 21 in. clear.
4. The term "net line" means the theoretical line inside which no rock will be allowed to project. These lines are used as the theoretical dimensions of the base. The bottoms of the heel and toe are sloped upward in order to reduce excavation costs and because it is impossible and useless to require anyone to blast out rock so as to provide sharp, reentrant angles or

corners. A trench may look very nice on a drawing, but when the designer looks for it in the field he will probably see just a sort of gully. That is why it is usually customary to establish "payment lines" for excavation and concrete, these lines being beyond the net lines. However, the foundation must be clean and free from loose material. The concrete must be poured to sound bed rock as indicated in Fig. 8-33.

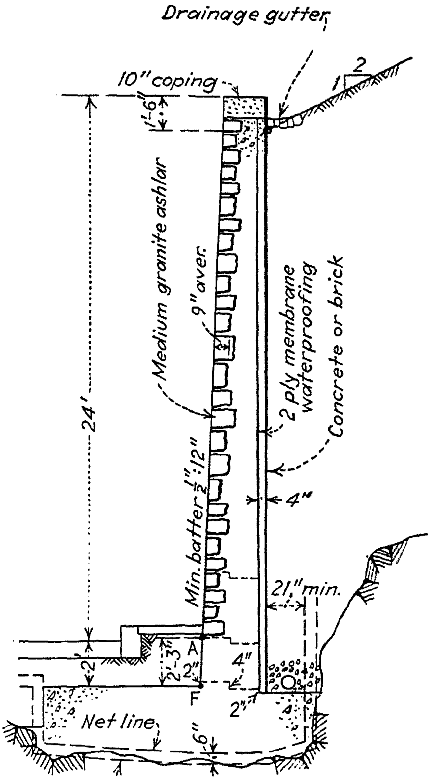


FIG. 8-33.—A masonry-faced wall with membrane waterproofing.

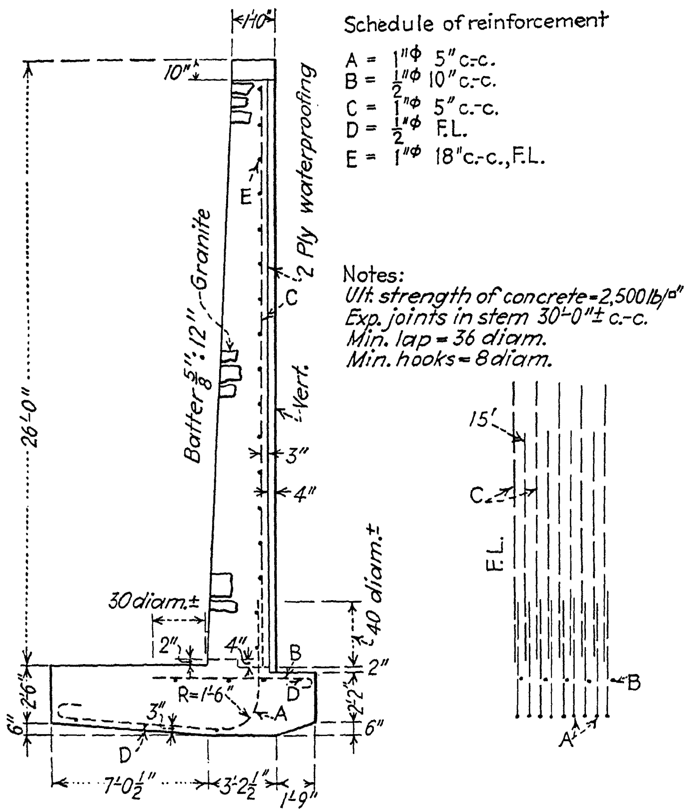
5. The batter on the front face of the wall is used to avoid the optical illusion of leaning forward—an effect that may result if a vertical face is viewed against the background of the sloping hillside behind it.

6. The toe is depressed below the curb and sidewalk in order to provide space for ducts and pipes and to get below frost depth.

7. The coping seals over the top of the masonry facing and the membrane waterproofing. Both are thus protected against the penetration of water and loosening of the materials.

The reinforcement used in this wall is shown in Fig. 8-34.

8-18. Miscellaneous Data. The wall in Fig. 8-27 appears to be tapered—and it really is—becoming shallower in the background. This varying of height is a common occurrence, and



Dimensions and Reinforcement
FIG. 8-34.

it introduces some problems in detail. These problems vary with the particular situation, but some suggestions that may be helpful are the following:

1. Design the wall for a series of heights varying by 4 or 5 ft., then interpolate for intermediate heights if necessary.

2. Fix the alignment of the wall with the front edge of the top or coping as the controlling line.

3. Use a constant width of coping and top thickness of the stem unless the range of variation of height is too great. In such a case, make one or two definite breaks in the back of the coping and wall at contraction or expansion joints in order to decrease the thickness of the top.

4. Maintain a constant depth of coping or uniform top marking arrangement as shown in Fig. 15-6.

5. If the top of the wall is level whereas the bottom varies, keep intermediate markings and joints level also; if the top slopes and the bottom is level, keep them parallel to the bottom. In neither case should markings or joints die out to feather edges. It is better to make distinct breaks at vertical construction joints. In the case of Fig. 15-6, when the top and

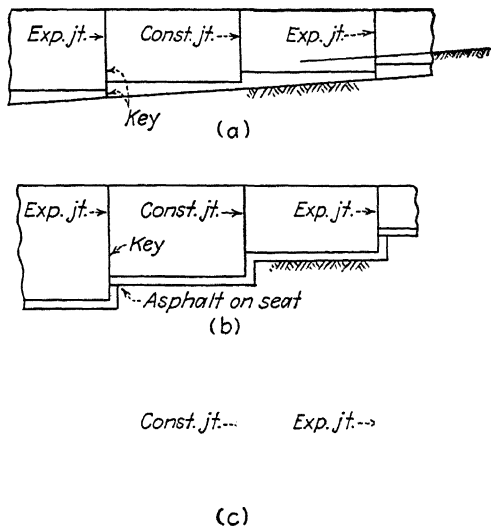


FIG. 8-35.—Construction of footings for tapered retaining walls.

lower V-cuts approach within one or two feet of each other, omit the lower one of them beyond the vertical joint; if they are separating and exceed the adopted spacing by more than one or two feet, add another one. In any such case, draw a perspective view as well as an elevation before accepting any pattern of markings.

6. Use a constant batter for the front and rear faces, starting from the top, and let the width of the stem vary at the bottom.

7. If the rate of slope of the bottom is small, the lower side of the footing may be set parallel to the grade whereas its top, if buried, may be in level steps as pictured in Fig. 8-35(*a*). This uses more concrete but it simplifies the formwork and the reinforcing.

8. If the rate of slope of the bottom is large, the lower side of the footing may have to be stepped as in Fig. 8-35(*b*). Note particularly the filler wall at the offset and the relation to the vertical joint. The footings should not

be made as shown in Fig. 8-35(c) because the differences in action at the junction of the higher and lower footings will almost certainly crack the wall. Furthermore, be sure that the excavation can be stepped safely without weakening the bearing value of the soil too much. It may be advisable to excavate as near the desired shapes as possible, then pour the footing directly on the soil, wherever it is.

9. Endeavor to use a uniform style of reinforcement and size of rods in the wall, varying the spacing in sections in order to simplify the work. Do not attempt to vary the spaces by less than one inch. However, make a complete change when necessary; e.g., do not try to use 1¼-in. rods in 6-ft. walls.

Practice Problems

Problem 8-1. Assume a concrete retaining wall like that in Fig. 8-3(a). LK is vertical, $LA = 1$ ft., $LK = 15$ ft., $JK = 3$ ft., $KD = 2$ ft., $DC = 4$ ft., $KG = 2$ ft. 2 in., $DF = 2$ ft., HE is level. Let w, the weight of earth, $=$ 100 lb. per cu. ft.; ϕ, the angle of repose, $= 34°$. The soil will withstand an allowable working pressure of 6,000 lb. per sq. ft.; its ultimate strength is 15,000 lb. per sq. ft. The maximum coefficient of friction is 0.5. Using these data, find the following:

a. If the earth is level and flush with point A, find the bending moment at the level of K for the earth pressure alone.

Discussion. Use Eq. (8-6) with $\delta = 0$ to find P, which is inclined at the angle ϕ; then find the horizontal component of P in order to compute the bending moment.

b. Repeat Part a, but include the effect of water pressure for an 8-ft. head above K at 45 lb. per sq. ft. per ft. of depth, plus ice pressure of 700 lb. per lin. ft. at A.

c. How much bending moment is caused at K by the surcharge effect of 1,000 lb. per ft. of wall at a point 5 ft. behind A? Combine this with b and with a, but assume that the full magnitude of P is to act horizontally.

Discussion. Use the data in Figs. 8-13 and 8-14 to find the magnitude and position of the lateral thrust.

d. Draw the pressure diagram for Part a. Test for sliding and for a safety factor of 1.75 for overturning.

e. Draw the pressure diagram for Part b. Test for sliding and for a safety factor of 1.75 for overturning.

f. Draw the pressure diagram for Part c. Test for sliding and for a safety factor of 1.75 for overturning. Consider the surcharge as given in Figs. 8-13, 8-14, and 8-15. In this case, neglect the vertical component of the earth pressure in computing stability. (This will show that the footing should be larger.)

g. Find the stresses in the concrete and the steel at K for Part c. Neglect the effect of the weight of the stem. Use $n = 12$, the steel $=$ ⅞-in. round rods 6 in. c.c., and $d = 21$ in.

h. Find the stresses in the heel and toe for Part f if $n = 12$, A_s and d for the toe $= 1.2$ sq. in. and 23 in., respectively, and A_s and d for the heel $=$ 0.88 sq. in. and 21 in., respectively.

i. If the earth is sloped, as in Fig. 8-1(*b*), at 1 vertically to 2 horizontally, starting 6 in. below *A*, find the bending moment at the level of *K* due to earth pressure and a head of water 8 ft. above *K* as for Part *b*.

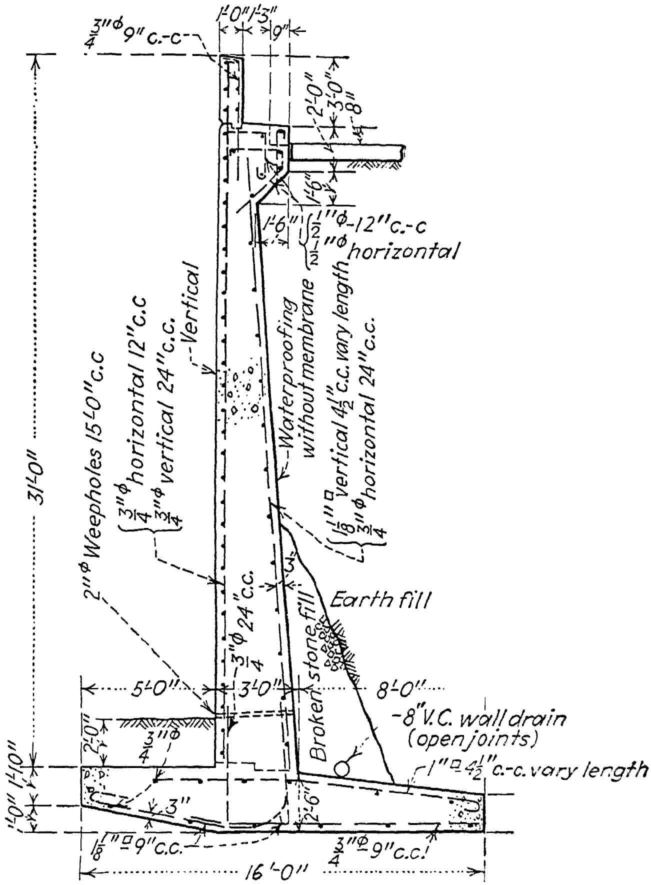


FIG. 8-36.—Retaining wall designed for the connection to the Harlem River Speedway, Highbridge Park, New York City.

Discussion. Use Eq. (8-6) with tan $\delta = \frac{1}{2}$ to find P; then use $P_H = P \cos \phi$.

j. Draw the pressure diagram for Part *i*. Test for sliding and for a safety factor of 1.75 for overturning. The vertical component of the earth pressure is to be included in computing the stability.

k. Find the stresses in the concrete and the steel in the stem at *K*, also in the heel and the toe, if the earth is sloped as for Part *j*.

Problem 8-2. Assume the heavy concrete retaining wall of Fig. 8-36. Analyze it completely if the following data are to be used: $w = 100$ lb. per cu. ft., $\phi = 34°$, $\delta = 0$; P is to act horizontally with its full magnitude (because of vibrations, its vertical component being assumed $= 0$); surcharge is to be as given in Figs. 8-13, 8-14, and 8-15 for $W' = 1{,}000$ lb. per ft. of wall, the wheel load being applied 1 ft. from the curb; ice pressure $= 500$ lb. per lin. ft.; water pressure $= 45$ lb. per sq. ft. per ft. of depth with a head of 8 ft. above the top of the toe; $f'_c = 3{,}000$ lb. per sq. in., $n = 10$; the maximum coefficient of friction $= 0.6$; the safety factor $= 1.75$.

CHAPTER 9

FOUNDATIONS

9-1. Introduction. A building or a bridge is generally considered to have two main portions—the superstructure and the substructure. The latter is often called the "foundation." It supports the superstructure, but it may contain various parts or units of its own. There are many special kinds of foundations for which concrete is used, but this chapter will be confined mostly to general types of construction which utilize reinforced concrete in bending.

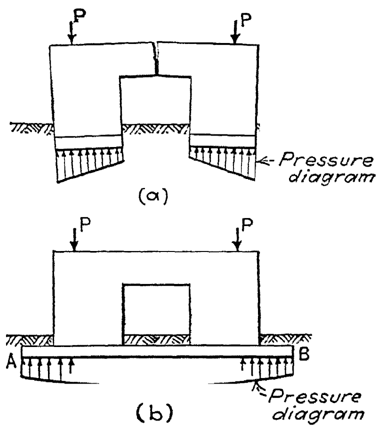


FIG. 9-1.

When concrete rests directly upon sound rock, the latter often has a strength equal to or greater than that of the concrete itself so that there is no difficulty in obtaining sufficient support for the superstructure. Sometimes, concrete caissons or piles are used to carry the loads through inadequate material to rock or some other suitable stratum at a lower level. These are special cases. Most of the illustrations herein will be confined to structures that are supported upon the earth which is directly under them.

The term foundation generally includes the entire supporting structure. Sometimes, as in the discussion of retaining walls, it is used to designate the material upon which the wall is supported. It must not be confused with the word "footing," which generally applies only to that portion of the structure which delivers the load to the earth, as illustrated by AB of Fig. 9-1(*b*). These are called "spread footings" because they distribute the concentrated load over a large area which has a low intensity of pressure. When a soil under a building is so poor that the footings are all joined together in one large slab or floor under the whole structure, the resultant footing is sometimes called a "reinforced-concrete mat."

9-2. Fundamental Principles. There are certain fundamental principles to be considered in the design of earth-borne foundations. When a load is placed upon earth, the latter is compressed somewhat. The amount of this deformation depends upon the intensity of the load, the loaded area, the nature of the soil, the depth at which the load is applied, and similar matters. These settlements will not be investigated here, but they must be kept in mind because it is essential to plan any foundation so that the entire structure will settle equally. If the unit bearing pressures vary greatly, the settlements of different parts of the structure are likely to vary also, causing cracks to appear. For instance, if a wall is loaded as shown in Fig. 9-1(*a*), the ends tend to settle and to cause a crack in the top. It is best to have the pressures uniformly distributed; but if this cannot be done, it is desirable to have the greater deformations near the center so that there will be a tendency for the structure to compress near the top rather than to open up. Of course, cracks at the bottom should also be prevented.

No foundation should tilt. The resultant of the resisting pressures under the foundation will pass through the center of gravity of the applied loads, including those of the foundation itself. However, if the resultant is offside the center of gravity of the bearing area, the side having the higher pressures may settle more and cause the structure to lean in that direction.

The effects of live loads and impact are usually omitted in the study of the settlements of foundations unless the structure is a warehouse or a building which may be subjected to large live

loads for long periods of time. The reason for this is the fact that the soil will not move or squeeze out quickly. One-half or more of the live load may be considered in the case of warehouses, but the uniformity of the distribution of the foundation pressures for dead load alone must be investigated also. However, the live load and impact must be included in the loads for which the members of the foundations are designed.

If the foundation of a building is partly upon rock and partly upon earth, the situation is dangerous. This is obvious. In such a case, one should use caissons or piles down to the rock under the portion of the structure that does not rest directly upon it, isolate the two portions of the structure so that the settlement will not cause trouble, or, if the condition is unavoidable, use a very low intensity of pressure upon the soil.

It is advisable to place the bottoms of all spread footings below the frost line, usually 4 to 5 ft. in cold climates.

The safe bearing value of any soil is a difficult thing to ascertain. Borings and loading tests should be made at the site before any important structure is built. However, Table 8-1 of the preceding chapter may be used as a general guide. When a foundation goes to a depth of 10 ft. or more below the surface or any adjacent excavation in natural, undisturbed soil, these specified soil pressures may be increased to some extent.

The vertical component of hydrostatic pressure under a foundation is not considered as an additional load in the design of the footings. Since $\Sigma V = 0$, it makes no difference if the structure tends to float like a boat as a result of hydrostatic uplift, because the total pressure is dependent upon the weight of the entire structure. However, the side walls of basements in such special cases must be designed to withstand lateral pressures, and the basement floor will have to be of the mat type —or partially so.

As stated in connection with retaining walls, the pressure upon the soil must be computed by including the weight of the footing, but the loads for which the footing itself must be designed may not include this weight because the wet concrete is already supported by the earth before it sets. This reduced intensity of load may be called the "net" pressure upon the footing.

9-3. Plain Concrete Pedestal. It is customary to use plain concrete pedestals under columns even when the structure rests

upon rock. Such pedestals can be used also in case of earth-borne foundations when the loads are small.

It must be remembered that the reinforcement in concrete cannot be effective in its ordinary function unless the concrete can elongate or bend. Some footings are primarily blocks rather than members which act as beams. For instance, suppose that a column is supported by a concrete pedestal which rests upon rock as pictured in Fig. 9-2(*a*). The reinforcement which is shown at the bottom is almost perfectly useless. If the rock is properly prepared and roughened, the footing cannot possibly spread sidewise so as to open up cracks and to stretch the reinforcement because of the shearing resistance of the rock. Then, since there is no strain in the direction of the rods, there can be no stress in them.

The lines of the forces in a pedestal are somewhat as pictured in Fig. 9-2(*b*), a combination of direct compression and shear. The only reinforcement that will do much good in such a case is that which will prevent the formation of cracks at the top of the pedestal as shown in Fig. 9-2(*b*). These cracks are not likely to occur if the pier has sufficient area. The Code states that the stress at the bottom of a column shall not exceed

$$r_a = 0.25f'_c\sqrt[3]{\frac{A}{A'}} \qquad (9\text{-}1)$$

where r_a = the compressive unit stress in the pedestal directly below the column, A' = this loaded area, and A = the cross-sectional area of the pedestal itself. Otherwise, the pedestal must be reinforced as a column.

Figures 9-2(*c*), (*d*), and (*e*) show some ways of arranging the reinforcement. The first two picture hoops or bands with a few supporting ties, alternate bands being reversed in direction. The last shows a two-way mat with the rods bent down. Both schemes are to prevent bulging and failure because of shear.

Whenever a heavily reinforced column rests upon a concrete pedestal (or footing), the stresses in the longitudinal rods cannot suddenly vanish. If the steel is needed in the column shaft, it, or its equivalent, is also needed where the column joins the pedestal. It is therefore necessary to provide dowels that have the same area as the main rods and that extend up into the column and down into the pedestal sufficiently to develop them

by means of bond. Such dowels are shown in Fig. 9-2(*c*). Spirals are not often used in the pedestal, but sometimes the ends of the longitudinal rods in the column can be squared off and made to bear directly upon a steel slab which rests upon the concrete.

When a steel column and a billet are supported upon a concrete pedestal, the allowable stress under the steel slab may be $0.375f'_c$, according to the Code, if the cross-sectional area of the pedestal is three times the area of the bottom of the billet, $0.25f'_c$ if the

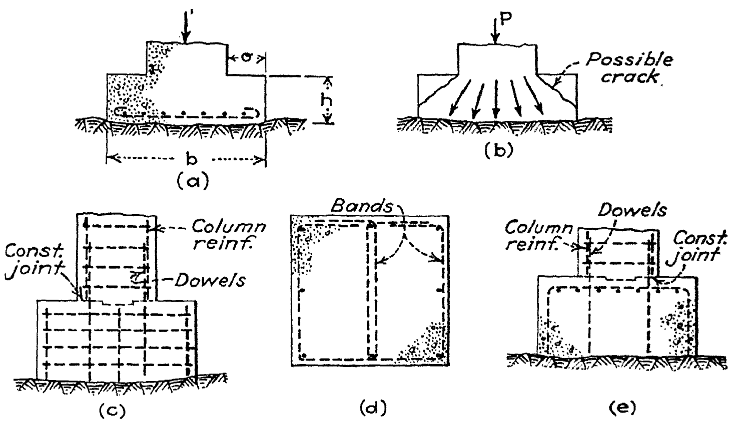


FIG. 9-2.

areas are equal, or it may vary proportionately between these limits.

The dimensions of a pedestal like that of Fig. 9-2(*a*) are largely a matter of judgment. If founded upon rock, the distance *o* should not be less than 4 or 6 in. so that the forms for the column can be supported upon the pedestal. The depth *h* must be sufficient to develop the dowels. In no case should an unreinforced-concrete pedestal have a depth that is less than the offset *o*.

When a plain concrete pedestal is on earth, it must be large enough to spread the load without exceeding the allowable pressure upon the soil. In such a case, too, a certain amount of bending may be set up in the pedestal itself which then becomes a short, thick, cantilever beam acting in two directions. It can be

designed as a beam of homogeneous material, using a limiting tensile stress of $0.03f'_c$ and a maximum shearing stress of $0.02f'_c$, according to the Code. The principles will be illustrated by direct application to a problem.

Problem 9-1. Find the stresses in the plain concrete pedestal of Fig. 9-3(a) if $f'_c = 2{,}500$ lb. per sq. in.

Although the intensities of the soil pressures may vary somewhat as shown by the dotted diagram in Fig. 9-3(a), it is customary and satisfactory to assume that they are practically equal at all points. The rectangle $SVXW$ will therefore represent these pressures. Since the weight of the footing itself is excluded, the upward pressure

$$p = \frac{P}{B^2} = \frac{30{,}000}{2.5^2} = 4{,}800 \text{ lb. per sq. ft.}$$

At NQ,

$$f_c = \frac{P}{A} = \frac{30{,}000}{144} = 208 \text{ lb. per sq. in.}$$

This is much less than Eq. (9-1), which gives

$$r_a = 0.25f'_c \qquad 0.25 \times \qquad 1{,}150 \text{ lb. per sq. in.}$$

Therefore, no reinforcement is needed as far as the pressures at the top of the pedestal are concerned.

The upward pressures upon the bottom of the pedestal tend to cause compression in its top and tension at the bottom so as to split it apart, as shown in exaggerated fashion in Fig. 9-3(c). The concrete, since it is unreinforced, must act as a beam of homogeneous material in resisting this tension. The Code recommends disregarding the bottom 2 in. of the pedestal.

The footing under a continuous wall acts as a pair of cantilever beams which bend one way, but this pedestal must bend in two directions. Therefore, assume that the portion of the base that is outside the column area [$AFHG$ of Fig. 9-3(b)] may be divided into quarters, like the trapezoid $ABEF$.[1] This acts as a cantilever beam about the face AF. The bending moment is

$$M = p \times AC \times AF \times \frac{AC}{2} + p \times 2 \times BC \times \frac{AC}{2} \times 0.6AC$$

$$M = 4{,}800\left(\frac{0.75^2}{2} \times 1 + 0.6 \times 0.75^3 \qquad 2{,}560 \text{ ft.-lb.} = 30{,}700 \text{ in.-lb.}\right.$$

[1] The 1940 Code recommends assuming the bending moment in an individual spread footing to be 85 per cent of the static moment on the entire portion of the footing on side of the section considered, the allowable bond stress also being reduced 25 per cent. However, many engineers use the trapezoidal-area assumption shown herein.

The lever arm $0.6AC$ is used instead of $0.67AC$, in accordance with the earlier Codes. When AC is less than the depth of the footing, the full width JK is assumed to be effective in resisting the bending. Therefore, the tensile stress in the bottom of the pedestal is

$$f_t = M \div \frac{Bd^2}{6} = \frac{30{,}700 \times 6}{30 \times 12^2}$$
$$= 43 \text{ lb. per sq. in.}$$

In calculating the shearing stresses in a pedestal or footing, the punching shear around the perimeter of the column is sometimes considered. This is supposed to be a pure shear without any combination with tension. However, it is difficult to see how the pedestal can fail in this way. If the angle α of Fig. 9-3(c) is less than 45°, a part of the effect of the upward pressures p will be a diagonal compression D_c which transmits them directly to the column. There will be also a resisting tension T. It is apparent that this condition is not one of pure shear or pure flexure. Therefore, in this case, no shearing stress needs to be computed, since α is considerably less than 45°.

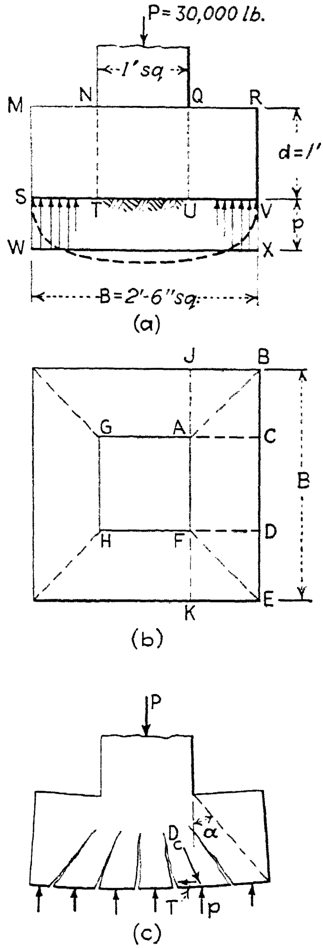


FIG. 9-3.

If the pedestal is so wide that 45° slope lines from N and Q of Fig. 9-3(a) fall inside S and V, the footing generally ought to be reinforced.

A plain concrete footing may be used under a wall if its depth exceeds twice the projection beyond the face of the wall. It can be analyzed as previously described by considering a rectangular piece 1 ft. wide. However, one should question its use if the loads are very heavy and the structure is important.

9-4. Reinforced-concrete Footing for a Column. When a column load is large and it must be spread over a considerable area of earth, a plain concrete pedestal will not be strong enough unless it is very thick. It is then economical to use a reinforced-concrete footing. Such a footing may be made as one large,

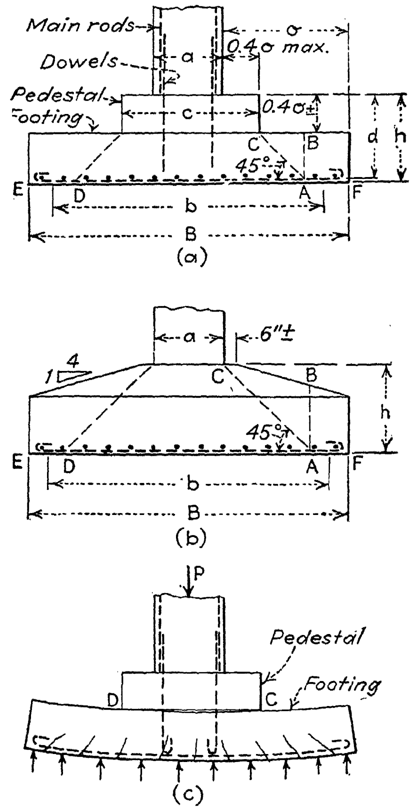


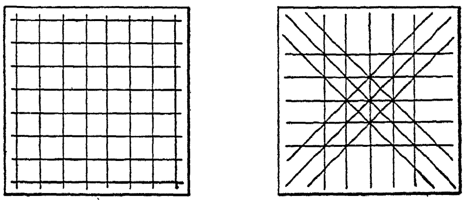
FIG. 9-4.

flat slab; it may be stepped as in Fig. 9-4(*a*) in order to save concrete; or it may be sloped on the top as pictured in Fig. 9-4(*b*).

The design of a footing is not a very exact affair. The members are so short and thick that they are not free to curve so as to make the common theory of flexure accurately applicable. Shearing and bond stresses are usually high.

The upper part, or pedestal, of a stepped footing may be poured monolithically with the main footing so as to avoid failure through longitudinal shear at the junction between the two. This causes trouble in the form work and sometimes in slumping of the concrete from the higher to the lower level. However, the pedestal is useful in reducing the stresses in the footing and in providing embedment to develop the dowels. It should not project too much—possibly not more than 0.3 or 0.4 of the offset o—and its depth need not exceed about $0.4o$ [Fig. 9-4(a)] for proper proportioning.

The action of the pedestal of a stepped footing will be more clearly understood by examining Fig. 9-4(c). If the footing



(a) (b)

FIG. 9-5.

bends under the load, there will be compression in the concrete between D and C. The curvature of the footing, if it could be so extreme, would cause a crack between the footing and the pedestal, with the latter receiving a concentrated compressive load near D and C. These loads, in turn, might tend to cause tension in the bottom of the pedestal, but it is impossible to have the footing shorten from D to C while the pedestal elongates at the same point unless the concrete fails along the junction. However, it is sufficient to assume that the pedestal is an unyielding support for the footing and that the critical bending moments will be in the cantilevered portions of the footing.

The reinforcement for the footing may be arranged in two layers as shown in Fig. 9-5(a), or it can be placed in two normal and two diagonal bands as pictured in Fig. 9-5(b). The latter is not worth while in ordinary simple footings because of the complication of the details of the reinforcement and the packing up of the four layers at the center.

For both types of footing shown in Figs. 9-4(a) and (b), it is customary to assume that there will be a sort of cone of compres-

sion which will act downward from the pedestal or column, the outer slope being inclined at 45° and shown by CA in the figures. This cone intersects the rods at A and D. The pressure upon the portion AF will cause shear along AB as for any other beam. As far as the steel is concerned, all of the rods within DA can be counted in resisting bending. Those between A and F, normal to the picture, are not so effective as those between A and D. It is sufficient to assume that the real effective breadth of the beam b is the average of DA and EF. It is often advisable to alter the spacing of the rods in Figs. 9-4(a) and (b)

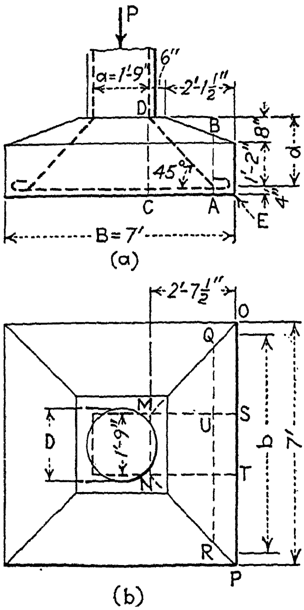


FIG. 9-6.

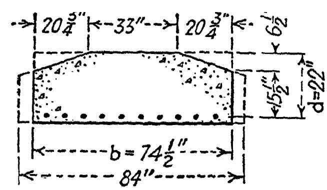


FIG. 9-7.

so that they are closely spaced under the pedestal with none outside of the width DA. These things will be illustrated by the solution of a problem.

Problem 9-2. Design a square, sloped-top footing for a circular column as shown in Fig. 9-6 for which the data are as follows: $P = 275{,}000$ lb., diameter of the column $= 24$ in., $f'_c = 2{,}500$ lb. per sq. in., $n = 12$, $f_c = 750$ lb. per sq. in., $f_s = 16{,}000$ lb. per sq. in., $v'_L = 75$ lb. per sq. in., $u = 125$ lb. per sq. in., the allowable soil pressure $= 6{,}000$ lb. per sq. ft.

In the case of a circular column, assume a square section of the same area.

$$a = \sqrt{\frac{\pi D^2}{4}} = 21.2 \text{ in.} \qquad \text{(Say 21 in.)}$$

Assume the average depth of footing $= 2$ ft., and weight $= 300$ lb. per sq. ft.

$$B^2 = \frac{P}{6{,}000 - 300} = \frac{275{,}000}{5{,}700} = 48.2 \text{ sq. ft.}$$

$$B = 7 \text{ ft. (approx.).}$$

Figures 9-6(*a*) and (*b*) picture the footing as it will be assumed.

$$CA = 1 \text{ ft. } 10 \text{ in.}, \qquad AE = 9\tfrac{1}{2} \text{ in.}, \qquad QR = 5 \text{ ft. } 5 \text{ in.},$$

$$b = \frac{RQ + PO}{2} = 6 \text{ ft. } 2\tfrac{1}{2} \text{ in.}$$

The critical section for bending in the footing may be assumed to be either at the edge of a deep pedestal like that of Fig. 9-4(*a*) or at the edge of a concrete column like that of Fig. 9-4(*b*). When a steel column rests on a billet that is directly on top of a footing like that in Fig. 9-4(*b*), it is best to assume the critical section to be about halfway between the center of the column and the edge of the billet. However, if a deep pedestal is used, the maximum bending can be assumed to be at the edge of the pedestal.

Taking moments about MN,

$$M = p \times ST \times MS \times \frac{MS}{2} + 2p \times \frac{OS}{2} \times MS \times 0.6MS$$

$$M \quad \left(1.75 \times \frac{2.62^2}{2} + 0.6 \times 2 \times \frac{2.62}{2} \times 2.62^2\right)5{,}700 = 96{,}000 \text{ ft.-lb.}$$

$$1{,}150{,}000 \text{ in.-lb.}$$

The effective section of the footing for bending at MN is shown in Fig. 9-7. It is not a rectangle. Ordinarily the compressive stress is low so that it need not be computed, and j can be assumed $= 0.88$. If this is done, and A_s is assumed to be the area of ten $\frac{3}{4}$-in. round rods (4.4 sq. in.),

$$f_s = \frac{M}{A_s jd} = \frac{1{,}150{,}000}{4.4 \times 0.88 \times 22} \quad 13{,}500 \text{ lb. per sq. in.}$$

However, in order to illustrate the design fully, a more careful calculation will be made.

Try the ten $\frac{3}{4}$-in. round rods which have been assumed previously. Taking the static moments about the neutral axis,

$$- \frac{[illegible]}{2}\Big) = n \times 4.4(22 - kd)$$

$$= 12 \times 4.4(22 - kd)$$

$$kd = 6.05 \text{ in.}, \qquad d - kd = 15.95 \text{ in.}, \qquad \text{and} \qquad k = \frac{6.05}{22} = 0.275$$

$$I_c = \frac{74.5 \times 6.05^3}{3} - \frac{2 \times 20.75 \times 6.5^3}{36} - \frac{2 \times 20.75 \times 6.5}{2}(6.05 - 2.17)^2$$
$$+ 12 \times 4.4(22 - 6.05)^2 = 16{,}600 \text{ in.}^4$$

$$S_c = \frac{I_c}{kd} = \frac{16{,}600}{6.05} = 2{,}740 \text{ in.}^3$$

$$f_c = \frac{M}{S_c} = \frac{1{,}150{,}000}{2{,}740} = 420 \text{ lb. per sq. in.}$$

$$\frac{\quad}{I_c} = \frac{\quad}{16{,}600} = 13{,}300 \text{ lb. per sq. in.}$$

This is rather conservative, but it will be accepted.

The shear at MN = the pressure upon the trapezoid $MOPN$ [Fig. 9-6(b)].

$$V = \frac{p(MN + OP)MS}{2} = 5{,}700\frac{(1.75 + 7)}{2}2.62 = 65{,}300 \text{ lb.}$$

Then, since all of the rods in the width b are assumed to be effective, the bond stress is, using $j = 0.88$,

$$u = \frac{V}{(\Sigma o)jd} = \frac{65{,}300}{10 \times 2.36 \times 0.88 \times 22} = 143 \text{ lb. per sq. in. (slightly excessive).}$$

The transverse shear at $MN = v_T = V/bkd = 65{,}300/21 \times 6.05 = 514$ lb. per sq. in. This equals $514/2{,}500 = 0.21f'_c$, which is satisfactory.

Now test the longitudinal shear at A of Fig. 9-6(a).

$$V = p\frac{(QR + OP)}{2}US = 5{,}700\frac{(5.42 + 7)}{2}0.79 = 28{,}000 \text{ lb.}$$

Assume that the section is rectangular, that $b = 65$ in., that $d = 17$ in., and that $j = 0.88$. Then the longitudinal shear is

$$v_L = \frac{V}{bjd} = \frac{28{,}000}{65 \times 0.88 \times 17} = 29 \text{ lb. per sq. in.}$$

The finished design is shown in Fig. 9-8. The dowels are hooked so as to resist moments. All of the rods in the bottom of the footing are hooked also in order to provide the necessary length for their development and to prevent bond failures near their ends. The lower layer of rods is placed about $3\frac{1}{2}$ in. above the bottom, giving about 4 in. average for the two sets. The rods which are $3\frac{1}{2}$ in. from the edges of the footing are added, but they are not counted in bending. These last rods might be omitted, but so many assumptions and approximations are used in the calculations that they will be relied upon to help reduce the bond stress. However, another way to accomplish this is to use a larger number of $\frac{5}{8}$-in. rods having relatively more surface area.

If it is desirable to use a stepped footing instead of a sloped one, the general procedure will be the same, but the computations will be more simple. The bottom edges of the pedestal are the lines about which the bending moments are usually calculated. Unless the pedestal is too wide or too shallow, it will need no

special analysis except for local compression near its top, and for shear.

9-5. Column Footings Subjected to Overturning Moments. In practical work, one frequently must design footings for columns carrying direct loads and bending, the latter being due to such things as wind pressures and lateral crane loads when the columns are fixed at their bases; sometimes footings support direct loads from columns plus horizontal shears delivered by them to the tops of the pedestals, as when the columns bend due to wind but are not fixed at their bases or when they have bracing members connecting near their bottoms; occasionally the moment may be caused by the fact that the column is located eccentrically on the footing. In such cases, the soil pressures are not uniformly distributed as in Fig. 9-3(*a*) but they may be assumed to vary uniformly across the footing.

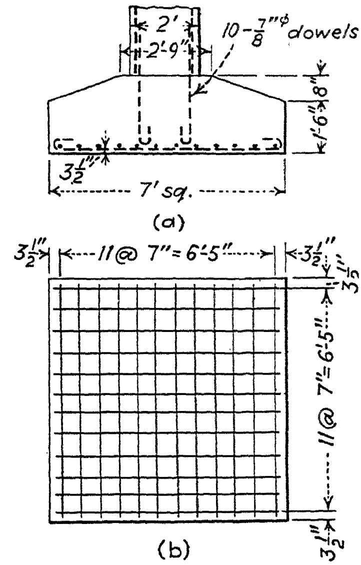


FIG. 9-8.

When these overturning moments always act parallel to one axis of the footing, it may be economical to make the footing rectangular with its long side in the direction of the overturning tendencies.

Problem 9-3. Design a spread footing for the steel columns shown in Fig. 9-9, using one of the typical columns along the north (right-hand) side of the concentrator. It supports one side of a suspended bunker about 30 ft. deep; the moment is due to wind on the outside and to crane forces from the inside of the building, and of course it is reversible. The soil is caliche, having an allowable bearing value of 5 tons per sq. ft. The allowable f_s, f_c, v'_L, and u are 18,000, 1,200, 90, and 150 lb. per sq. in., respectively; $n = 10$. Since the bin may be full or empty when the wind blows, the combination of wind and maximum vertical load produces the critical bending moment in the footing and pressure on the soil, but when the bin is empty, the wind may overturn the footing. Both cases must be investigated, the design data being the following:

$$\begin{aligned} \text{Max. } P + M &: 725{,}000 \text{ lb.} + 300{,}000 \text{ ft.-lb.} \\ \text{Min. } P + M &: 40{,}000 \text{ lb.} + 300{,}000 \text{ ft.-lb.} \\ S &: 15{,}600 \text{ lb.} \end{aligned}$$

where P = vertical load, M = moment, and S = horizontal shear, all at the bottom of the steel.

For a combination of wind and live loads, assume that the allowable bearing pressure and unit stresses may be increased 30 per cent.

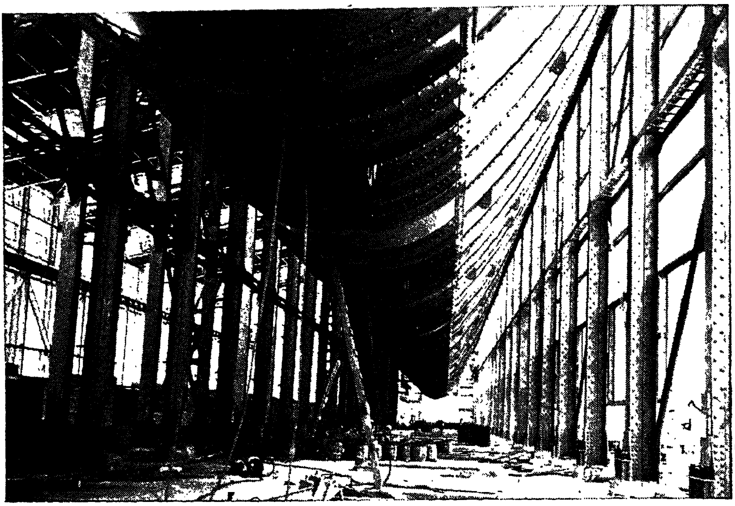


FIG. 9-9.—Suspended ore bin along north side of concentrator at the Morenci Reduction Works, Phelps Dodge Corporation, Morenci, Arizona. View looking west.

When starting such a problem, the designer should see what limiting factors affect the case—interferences with width and length, use or omission of a pedestal, size of pedestal if used, depth to bottom of footing, etc. He should choose the type of footing that he thinks is best, then see if he can use it; if something interferes, then the design should be modified as necessary.

One way to obtain trial dimensions of such a rectangular footing is to find the minimum area for direct load only, adding an estimate of the weight of the footing and soil or flooring over it. Thus, call P = 775,000 lb.

$$A = \frac{775{,}000}{10{,}000} \quad 77.5 \text{ sq. ft. (say } 8 \times 10 \text{ ft.).}$$

Next assume that, for direct loads and overturning, the pressures under the footing vary uniformly. Then the maximum pressure under the footing is

$$p = \frac{P}{A} + \frac{Mc}{I} = \frac{P}{BH} + \frac{6M}{BH^2} \tag{9-2}$$

where B and H are, respectively, the width and length of the footing, the

latter being in the direction of the rotational tendency. Then, assume H and solve for B.

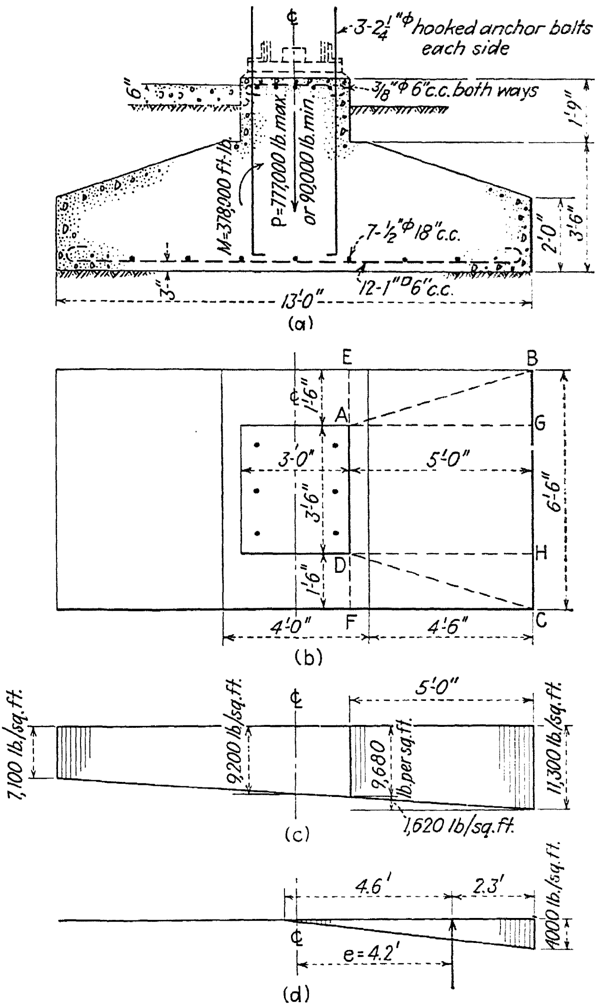


FIG. 9-10.—Footing under column supporting ore bin, Morenci Reduction Works, Phelps Dodge Corporation, Morenci, Arizona.

Assuming $H = 10$ ft., and using M at the bottom of the footing as 300,000 $+ S \times 5 = 378{,}000$ ft.-lb.,

$$10{,}000 \times 1.3 = \frac{775{,}000}{10B} + \frac{6 \times 378{,}000}{100B} \qquad B = 7.7 \text{ ft.}$$

This could be used, but test out a longer, narrower footing 6.5 ft. × 13 ft. Its area will be about the same as the broader one. Therefore, it will be adopted because it is more effective in resisting the overturning.

Before proceeding further, test the pressure for the case of overturning with the minimum vertical load P = 40,000 lb. + 50,000 lb. for the footing, etc., M = 378,000 ft.-lb.

$$\text{Eccentricity} = e = \frac{378{,}000}{90{,}000} = 4.2 \text{ ft}$$

The pressure diagram is therefore as shown in Fig. 9-10(d).

$$\left(p \times \frac{6.9}{2}\right)6.5 = 90{,}000, \qquad p = 4{,}000 \text{ lb. per sq. ft.}$$

This is satisfactory.

The trial footing is shown in Fig. 9-10. An estimate of its weight plus the floor and earth over it is 52,000 lb. The revised maximum P and M are given in Fig. 9-10(a); the pressure diagram, in Sketch (c). Since the footing is narrow, it is sloped two ways only. This makes a strong ridge clear across the top. Because of this and the many approximations made in such designs, the bending moment at the edge of the pedestal will be computed at the section EF of Sketch (b) for the pressure on the entire portion $EBCF$ with 600 lb. per sq. ft. deducted from the values given in (c) because of the weight of the footing and earth on it.

$$M_{EF} = 6.5\left(9{,}080 \times \frac{5^2}{2} + 1{,}620 \times 5 \times 0.67 \times 5\right) = 914{,}000 \text{ ft.-lb.}$$

Assume j = 0.88; d = 39 in.

$$A_s = \frac{914{,}000 \times 12}{(18{,}000 \times 1.3)0.88 \times 39} = 13.7 \text{ sq. in.}$$

For the maximum vertical load alone,

$$M_{EF} = 6.5\left(\frac{777{,}000}{84.5} - 600\right)\frac{5^2}{2} = 6.5(9{,}200 - 600)\frac{5^2}{2} = 699{,}000 \text{ ft.-lb.}$$

$$A_s = \frac{699{,}000 \times 12}{18{,}000 \times 0.88 \times 39} = 13.6 \text{ sq. in.}$$

Therefore, use fourteen 1-in. sq. hooked rods 5 in. c.c. longitudinally. Put in a few transverse spacer ties as shown in Fig. 9-10(a).

$$V_{EF} \text{ for Fig. 9-10}(c) = \left[\frac{(9{,}080 + 11{,}300)}{2} - 600\right]5 \times 6.5 = 321{,}000 \text{ lb.}$$

$$u = \frac{321{,}000}{56 \times 0.88 \times 39} = 167 \text{ lb. per sq. in.} \quad (\text{Less than } 1.3 \times 150.)$$

For vertical loads alone,

$$u = \frac{(9{,}200 - 600)5 \times 6.5}{56 \times 0.88 \times 39} = 145 \text{ lb. per sq. in.}$$

Since the punching shear for vertical loads, counting all of $EBCF$, is

$$v = \frac{(9{,}200 - 600)5 \times 6.5}{78 \times 0.88 \times 39} = 104 \text{ lb. per sq. in.},$$

inspection alone shows the longitudinal shearing stress v_L, at the bottom of a 45° pressure cone, to be satisfactory. Using Eq. (2-5a),

$$f_c = \frac{6M}{bd^2} = \frac{6 \times 914{,}000 \times 12}{78 \times 39^2} = 555 \text{ lb. per sq. in.} \quad \text{(Very safe.)}$$

A test of the loading case shown in Fig. 9-10(d), shows that the shears and moments in the footing are less than those already considered, hence the footing is satisfactory.

If the anchor bolts shown in Fig. 9-10(a) did not go to the bottom to grip the footing, reinforcement would have to be used to keep the pedestal from being ripped off. The rods under the billet are used arbitrarily to prevent cracking of the top by shear.

Whenever the designer prefers to consider the bending moment as that due to the pressure on the trapezoidal area $ABCD$ of Fig. 9-10(b), he may compute it on the basis of the pressure diagram of Sketch (c) acting on $AGHD$, a uniform pressure of 9,680 − 600 lb. per sq. ft. acting on ABG and DHC, and a triangular pressure diagram varying from 0 to 1,620 lb. per sq. ft. on these two triangular areas (remembering that the volume of a pyramid = $\frac{1}{3}$ base area × altitude, and the lever arm from AD to the center of gravity = $\frac{3}{4}$ × the altitude AG).

Suppose the footing of Fig. 9-10(a) is subjected to overturning moments both lengthwise and crosswise of the footing. It will be sufficient to compute the bearing pressure for the vertical load alone, add to it that due to the moment in one direction ($6M/BH^2$), then add also the pressure caused by the crosswise moment ($6M'/B^2H$). The bending moments in the footing may generally be computed at one side of the pedestal for $P + M$ alone or at the adjacent side for $P + M'$ only.

Problem 9-4. Check the footing shown in Fig. 9-11 for the loads and overturning moment given therein. This footing is notched out to clear a heavy machinery foundation which must be isolated from it. The maximum permissible soil pressure is 4 tons per sq. ft.; n = 10; max. f_s, f_c, v'_L, and u = 20,000, 1,200, 90, and 150 lb. per sq. in., respectively.

The weight of the footing is taken as the approximate total weight of the concrete and the earth over it, this weight being assumed to be at the center line of the column. The earth is often omitted from the calculations, especially when the footings are deep.

Since this footing is not rectangular, Eq. (9-2) must be used in its general form. A procedure yielding sufficiently satisfactory answers is the following:

1. Locate the center of gravity of the assumed bearing area.
2. Compute the moment of inertia of this area about both rectangular axes.

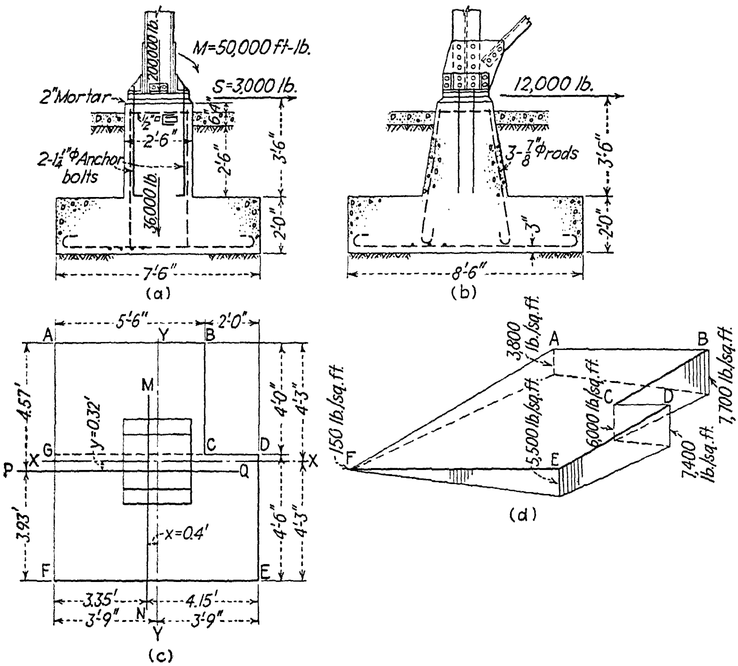


FIG. 9-11.—Unsymmetrical footing for steel column in industrial plant.

3. Combine the moments due to eccentricity of loading with those due to other causes.
4. Compute the resultant pressure by adding that due to the direct load to those caused by the moments.
5. Find the critical portion of the footing, approximate the bending moment, and test the section.

Axis *X-X*:

Part	A	l	M	l^2	I_{CG}	Al^2	ΣI
ABCG	22.	2.25	49.5	5.06	29.3	111.3	140.6
GDEF	33.8	2.	−67.6	4.	57.	135.2	192.2
	55.8		−18.1				332.8

$$y = \frac{-18.1}{55.8} = -0.32 \text{ ft.}$$

$$I_{PQ} = I_X - Ay^2 = 332.8 - 55.8 \times 0.32^2 = 327 \text{ ft.}^4$$

Axis Y-Y:

Part	A	l	M	l^2	I_{CG}	Al^2	ΣI
ABCG	22.	−1.	−22.	1.	55.5	22.	77.5
GDEF	33.8	0.	0.	0.	158.2	0.	158.2
	55.8		−22.				235.7

$$x = \frac{-22}{55.8} = -0.4 \text{ ft.}$$

$I_{MN} = I_Y - Ax^2 = 235.7 - 55.8 \times 0.4^2 = 227$ ft.^{4}
$P = 236{,}000$ lb. (Assume it at center of column.)
M_X at bottom $= 12{,}000 \times 5.5 + 236{,}000 \times 0.32 = 142{,}000$ ft.-lb.
M_Y at bottom $= 50{,}000 + 3{,}000 \times 5.5 + 236{,}000 \times 0.4 = 161{,}000$ ft. lb.

The greatest pressure is likely to be at B or D of Fig. 9-11(c). Letting the subscripts denote the point under consideration, then

$$p_B = \frac{236{,}000}{55.8} + \frac{161{,}000 \times 2.15}{227} + \frac{142{,}000 \times 4.57}{327} = 7{,}700 \text{ lb. per sq. ft}$$

$$\frac{236{,}000}{55.8} \quad \frac{161{,}000 \times 4.15}{227} + \frac{142{,}000 \times 0.57}{327} \quad 7{,}400 \text{ lb. per sq. ft.}$$

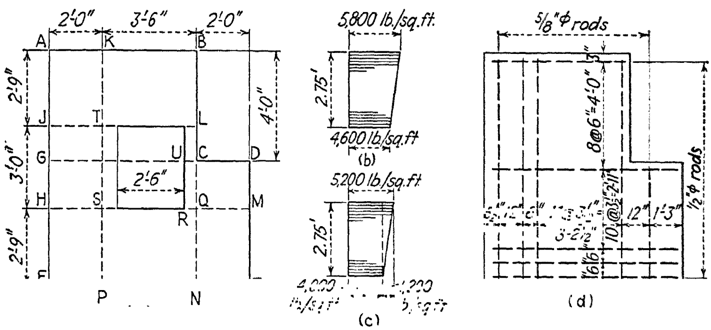


FIG. 9-12.—Reinforcement in an unsymmetrical spread footing.

The pressure diagram under the whole footing is shown in Fig. 9-11(d). The use of this in designing the footing generally involves broad approximations. A designer might resort to various detailed computations, but it is seldom worth while. His psychological reaction resulting from such tedious work may be beneficial to him, but probably the real value of the answer is not in proportion to the fussiness of the computations. Such footings are generally the exception, having little duplication. Therefore, the designer should think first whether or not his time and that of the checker of his work will cost more than the use of a few more rods in a design that is obviously safe. Each special problem requires the use of good judgment in deciding these matters.

The following computations show one way of designing this footing:

1. Looking at Figs. 9-11(d) and 9-12(a), it seems that a strong band or beam might be made running lengthwise of the footing, such as *KBNP* of

Fig. 9-12(a). Make it strong enough to carry the shears and moments from the entire portion $JABL$ about TL, or $HMEF$ about SQ. Assume it to be 3 ft. 6 in., a little wider than the pedestal.

2. Judge whether TL or SQ is the critical section. Test both if it seems to be necessary. Although the area $JABL$ is less than $HMEF$, it is subjected to larger pressures and appears to be the critical side.

3. From Fig. 9-11(d) and similar computations, get the pressures at each of the four corners, then average that at A and B, also at J and L. The resultant approximate pressure diagram is shown in Fig. 9-12(b). The average dead load of the footing and the material over it is 36,000 ÷ 55.8 = about 600 lb. per sq. ft. Then the approximate net pressure diagram is that shown in Fig. 9-12(c). The bending moment at TL is

$$M_{TL} = 5.5\left(4{,}000 \times \frac{2.75^2}{2} + 1{,}200 \times 2.75^2 \times 0.67\right) = 117{,}000 \text{ ft.-lb.}$$

$$V_{TL} = 5.5\left(4{,}000 \times 2.75 + 1{,}200 \times \frac{2.75}{2}\right) = 70{,}000 \text{ lb.}$$

4. Using Eqs. (2-6a) and (2-5a), with $d = 21$ in.,

$$A_s = \frac{117{,}000 \times 12}{20{,}000 \times 0.88 \times 21} = 3.8 \text{ sq. in}$$

Use twelve $\frac{5}{8}$-in. round rods at $3\frac{1}{2}$ in. c.c., hooked.

$$f_c = \frac{6 \times 117{,}000 \times 12}{42 \times 21^2} = 450 \text{ lb. per sq. in.}$$

$$v_L = \frac{70{,}000}{42 \times 0.88 \times 21} = 90 \text{ lb. per sq. in. without considering any } 45^\circ$$

spreading.

$$u = \frac{70{,}000}{23.5 \times 0.88 \times 21} = 161 \text{ lb. per sq. in. (near enough).}$$

5. The strip $UDMR$ of Fig. 9-12(a) seems to have the greatest bending of any crosswise to $KBNP$. Arbitrarily assume a piece 12 in. wide with an average net upward pressure of 6,400 lb. per sq. ft. For this strip, M_{UR} = 20,000 ft.-lb. and A_s required = 0.65 sq. in. Therefore, use $\frac{1}{2}$-in. round rods $3\frac{1}{2}$ in. c.c., for which u = 161 lb. per sq. in.

The final reinforcing plan is shown in Fig. 9-12(d). The top hoop and the bars in the pedestal are shown in Figs. 9-11(a) and (b).

9-6. Combined Footings. In some cases, where structures are founded upon soils having low bearing capacities, it is advisable to combine the footings of two or more columns. One common example of this occurs when the outside row of columns is close to the building line and it is impossible for the footing to spread

over on to adjacent property. The load upon the outer column is also likely to be larger than that on the inner one because of the heavy walls. The footing may be built somewhat as shown in Fig. 9-13, with an enlargement near the outer column so as to produce a uniform intensity of pressure. Sometimes an elongated pedestal may be added in order to make the structure a sort of inverted T-beam which is supported by the two columns. Of course, the fundamental objective is to secure uniform settlement and to avoid tipping of the footing.

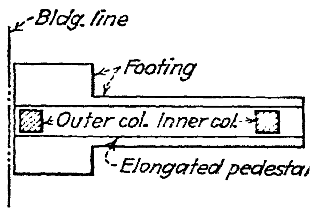


FIG. 9-13.

Many industrial buildings are equipped with heavy cranes that are carried on columns along the exterior walls. The

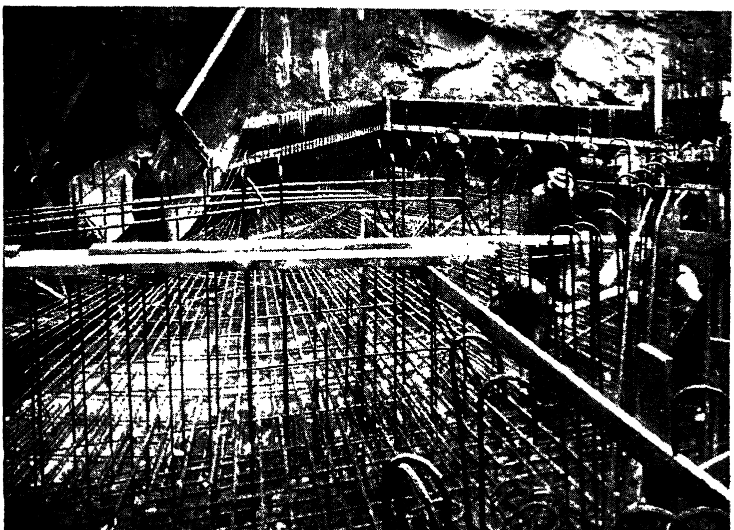


FIG. 9-14.—Reinforcement in invert of the New Jersey shaft of the Lincoln Tunnel at New York City.

foundations for these may be made with spread footings for the columns and grade beams for the walls, or they may be continuous concrete walls with pilasters and medium-sized footings at the columns. In the latter case, the wall may be designed as a

stiffening girder or a long, narrow footing to spread the crane column's load. This latter type of foundation may require more excavation and concrete, but it produces a stiff construction.

In other cases, like the invert of the New Jersey shaft of the Lincoln Tunnel (Fig. 9-14), all of the footings are combined into

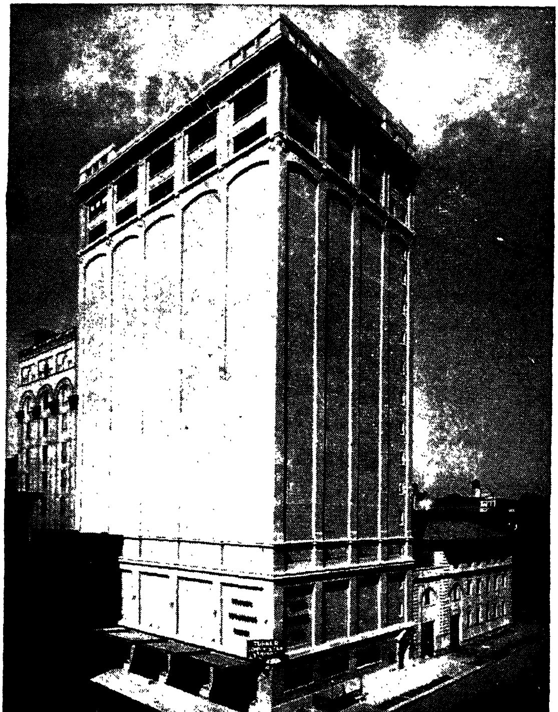


FIG. 9-15.—Merchants Refrigerating Company Building, New York City. (*Courtesy of Turner Construction Co.*)

a thick, continuous, reinforced-concrete slab or mat. This is done here in order to make a boxlike structure which will withstand hydrostatic pressures. In the case of the Merchants Refrigerating Company Building, Varick St., New York City (Fig. 9-15), a solid 5-ft. mat was used in order to spread the load to the soil.

Problem 9-5. Find the stresses in the pier which is pictured in Fig. 9-16. It is part of the viaduct of the New Jersey approach to the Lincoln Tunnel

at New York City. Assume $n = 10$ ($f'_c = 3{,}000$ lb. per sq. in.) and f_s and $f_c = 18{,}000$ and 1,000 lb. per sq. in., respectively.

This problem will illustrate the use of wooden piles as supporting members for a foundation. It is given here in order to show the nature of many practical cases.

The entire pier is designed to act as a unit, carrying four concentrated loads from the superstructure. Various conditions and combinations of loading must be investigated in the design of such a structure; but in this problem, full live load and dead load will be combined with longitudinal

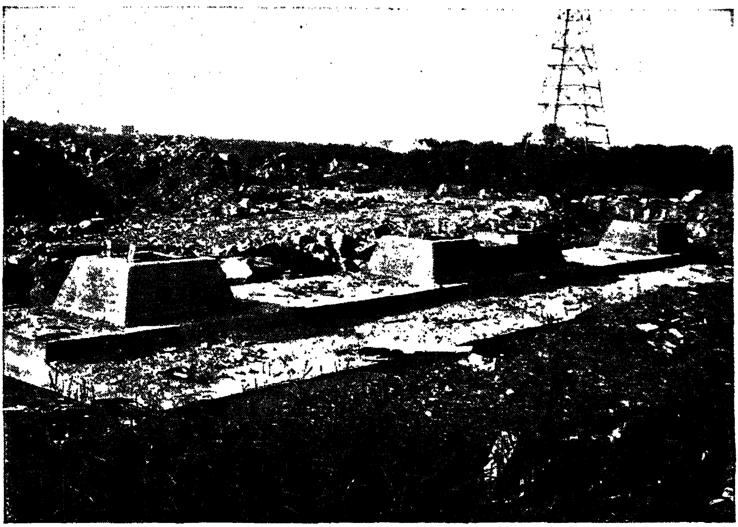


FIG. 9-16.—Foundation for six-lane viaduct, New Jersey approach to the Lincoln Tunnel, New York City.

wind and braking forces. The vertical loads are 900 kips at each pedestal; the longitudinal loads are 70 kips acting horizontally at the top of each pedestal, normal to the long axis of the pier. There are 144 piles arranged in 24 rows having 6 piles each.

Two special features must be noticed in this problem: (1) The four vertical loads are equal, meaning that the pier is not like a continuous beam which is on rigid supports; (2) the piles are supported by the frictional resistance of the surrounding material (clay) so that, for dead loads which are constant and continuously applied, the load on the piles is assumed to be distributed equally among them all on account of the plasticity or minute movement of the clay. It is therefore assumed further that for both live and dead loads, the reactions of all piles are equal because the live loads are relatively small. However, the temporary longitudinal forces are assumed to cause uniformly varying loads upon the piles across the narrow dimension of the pier, and all transverse rows of piles are assumed to act in the same manner.

The total dead load of the pier, neglecting the piles, and assuming the space that is occupied by pile heads to be solid concrete, is about 1,500 kips. This is assumed to be uniformly distributed, and it causes a load of $1{,}500/144 = 10.4$ kips per pile. The superstructure produces a load of $4 \times 900/144 = 25$ kips per pile. These give 35.4 kips vertical load on each one.

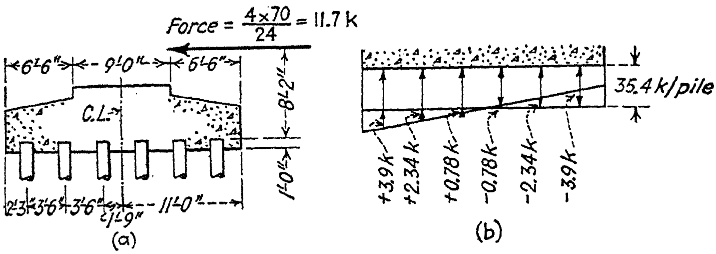


FIG. 9-17.

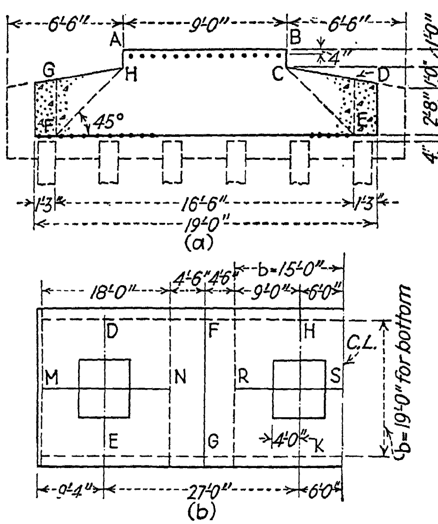


FIG. 9-18.

On the other hand, the longitudinal forces try to overturn the entire pier. If these forces are applied as shown in Fig. 9-17(*a*), the moment that acts upon one row of piles is

$$M = \frac{4 \times 70}{24} \times 8.17 = 95 \text{ ft.-kips.}$$

The piles are 3.5 ft. apart so that the vertical force on the outer piles is

$$P = \frac{Mc}{I} = \frac{95 \times 8.75}{2(1.75^2 + 5.25^2 + 8.75^2)} = \frac{832}{214} = 3.9 \text{ kips.}$$

The total vertical reactions of the piles are then as pictured in Fig. 9-17 (b). Of course, the horizontal reaction which balances the longitudinal forces comes from the bearing of the clay upon the sides of the footing and of the piles.

It now becomes necessary to make some broad and arbitrary assumptions in order to proceed efficiently with the design, because the loading conditions and the distribution of forces within the pier are too uncertain to justify a long, supposedly exact analysis. One must use good judgment and be conservative without undue wastefulness. Therefore, the following will be assumed:

1. The effective cross section of the pier which is counted upon to withstand the lengthwise bending of the pier is $ABCDEFGHA$ [Fig. 9-18(a)]. The general shape is chosen so as to have a narrow strip showing above the soil, to allow pouring a 12-in. sealing slab around the pile heads prior to making the structural base, and to produce a stiff, chunky footing.

2. The effective width of the pier for transverse bending is 18 ft. at the end pedestals and 15 ft. at the intermediate ones as shown in Sketch (b). The average point of greatest transverse bending moment is 2 ft. from the center of the pedestals.

3. The effective depth of the footing is 56 in. (see Fig. 9-21).

4. The stiffening effect of the connecting rib on top of the main footing between the two central pedestals (Fig. 9-20) is not relied upon because it is used primarily as a filler between these pedestals, and because it is poured after the main footing, causing a plane of weakness at the junction between the base and the top.

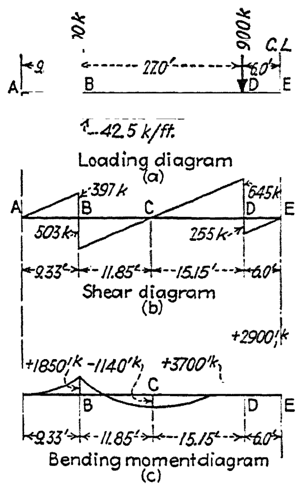


FIG. 9-19.

5. Each transverse strip supports six rows of piles.

6. The longitudinal bending moments are to be computed at the centers of the pedestals; the shears are to be calculated at the edges of the 45° shear cones under the pedestals; and the effective width for punching shear is to be the width of the base of the pedestal.

The bending moments and shears in the longitudinal strip as a whole are caused by the dead and live loads of the superstructure which are taken as

$$W \div L = 4 \times 900 \div 84.67 = 42.5$$

kips per ft. of pier. The overturning forces do not increase the total vertical loads but merely add to those on one side and relieve those on the other. Also, the weight of the pier itself does not cause bending in this strip. The shear and bending-moment diagrams are pictured in Fig. 9-19. The values are obtained by working from A to the center E.

From Eq. (2-6a), a trial value of A_s at point D is

$$A_s = \frac{3{,}700{,}000 \times 12}{18{,}000 \times 0.88 \times 56} = 50 \text{ sq. in.}$$

If 1-in. square rods are placed at $4\frac{1}{2}$ in. c.c., there will be 51 of them across the 19-ft. section shown in Fig. 9-18(*a*). Similarly, the tensile steel in the top at *C*, Fig. 9-19(*c*), should be at least

$$A_s = \frac{1{,}140{,}000 \times 12}{18{,}000 \times 0.88 \times 56} = 15.4 \text{ sq. in.}$$

Try seventeen 1-in. square rods at 6 in. c.c. Since the bending moment at *B*, Fig. 9-19(*c*), is half of that at *D*, use 1-in. square rods at 9 in. c.c.

Figure 9-20 pictures the arrangement of the longitudinal reinforcement. The assumed effective beam for bending at *D* is as shown in Fig. 9-21(*a*). Solving for the stresses that result from a positive bending moment of 3,700 ft.-kips on a rectangular section 9 ft. wide gives approximately

$$108 \times \frac{(kd)^2}{2} + 9 \times 6(kd - 4) = 10 \times 51(56 - kd)$$

$$kd = 18.5 \text{ in.}, \qquad d - kd = 37.5 \text{ in.}, \qquad k = 0.33, \qquad j = 0.89$$

$$I_c = 957{,}000 \text{ in.}^4, \qquad S_c = 51{,}800 \text{ in.}^3$$

$$f_c = \frac{3{,}700{,}000 \times 12}{51{,}800} = 860 \text{ lb. per sq. in.}$$

$$f_s = \frac{3{,}700{,}000 \times 12 \times 37.5 \times 10}{957{,}000} = 17{,}400 \text{ lb. per sq. in.}$$

Similarly, at *B* of Fig. 9-19(*c*), with $A_s = 25$ sq. in., $f_c = 625$ lb. per sq. in., and $f_s = 16{,}200$ lb. per sq. in.

The greatest shear is at the left of *D* [Fig. 9-19(*b*)]. The bottom of the 45° pressure cone is (4 + 4.67) ft. from *D*. The total shear at this point is $645 - 42.5 \times 8.67 = 277$ kips. Then the horizontal shearing stress at this point is

$$v_L = \frac{V}{bjd} = \frac{277{,}000}{108 \times 0.89 \times 56} = 51 \text{ lb. per sq. in.} \quad \text{(Narrow section.)}$$

The greatest bond stress, assuming the full shears at *B* and *D*, is at *B* because the steel area here is half of that used at *D*. Then, using the shear at the edge of the pedestal 4 ft. to the right of *B*, and assuming $j = 0.88$,

$$u = \frac{V}{(\Sigma o)jd} = \frac{503{,}000 - (4 \times 42{,}500)}{25 \times 4 \times 0.88 \times 56} = 68 \text{ lb. per sq. in.}$$

The transverse shearing stress at the edge of the pedestal 4 ft. to the left of *D* [Fig. 9-19(*b*)] is

$$v_T = \frac{V}{b(kd)} = \frac{1{,}000(645 - 42.5 \times 4)}{108 \times 18.5} = 238 \text{ lb. per sq. in.}$$

The unit stresses at point C of Fig. 9-19(c) are found by using the section of Fig. 9-21(a) as an inverted T-beam with tension in the top. Neglect the few rods that serve as temperature reinforcement in the top of the sloping

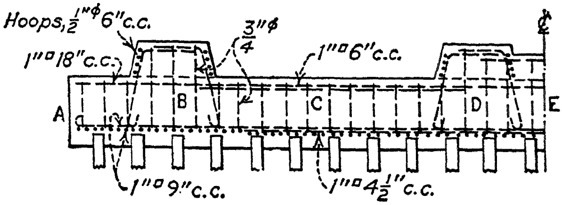


FIG. 9-20.

sides of the footing. From Fig. 9-20, A_s = 17 sq. in., and A'_s = 25 sq. in. From Fig. 9-21(a), b = 228 in.

$$228\frac{(kd)^2}{2} + 9 \times 25(kd - 4) = 10 \times 17(56 - kd)$$

$$kd = 8 \text{ in.}, \qquad d - kd = 48 \text{ in.}, \qquad I_c = 434{,}500 \text{ in.}^4$$

$$f_c = \frac{1{,}140{,}000 \times 12 \times 8}{434{,}500} = 252 \text{ lb. per sq. in.}$$

$$f_s = \frac{1{,}140{,}000 \times 12 \times 48 \times 10}{434{,}500} = 15{,}100 \text{ lb. per sq. in.}$$

The maximum bending moment in a transverse strip due to longitudinal forces is found, from Fig. 9-18(b), to occur in the section RS. Applying the pile loads in Fig. 9-17(b), the bending moment in a 1-ft. strip at a point 2 ft. from the center of the pedestal is (using six rows of piles)

$$M_L = \tfrac{6}{15}(3.9 \times 6.75 + 2.34 \times 3.25) = 13.6 \text{ ft.-kips.}$$

The bending moment in this transverse strip due to the superstructure is $M_s = 25(6.75 + 3.25) \div 3.5 = 71.4$ ft.-kips per ft. of pier, where 25 kips is

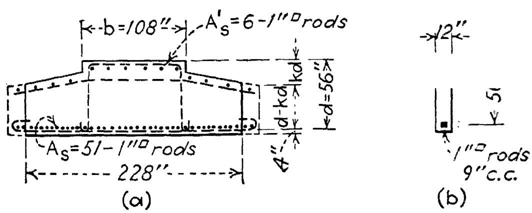


FIG. 9-21.

the load per pile (4 × 900/144). This must be combined with the effect of the longitudinal forces so that

$$M = 13.6 + 71.4 = 85 \text{ ft.-kips.}$$

The cross section of a 1-ft. strip is shown in Fig. 9-21(*b*) without considering the $\frac{3}{4}$-in. rods in the top; then

$$kd = 10.1 \text{ in.}, \qquad d - kd = 45.9 \text{ in.}, \qquad I_c = 32{,}130 \text{ in.}^4$$
$$f_c = 320 \text{ lb. per sq. in.}, \qquad f_s = 14{,}600 \text{ lb. per sq. in.}$$

Since the foregoing calculations show that the stresses at the critical points are safe, then those at other points are automatically known to be safe, and therefore need not be computed.

9-7. Abutments. The abutments of many bridges are structures that illustrate some important principles in the design of foundations. When they support an embankment as well as

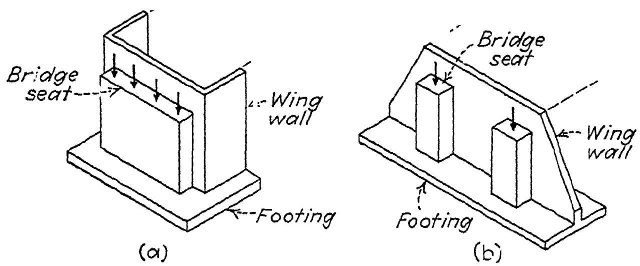


FIG. 9-22.

the superstructure of the bridge, they are a combination of a retaining wall and a pier.

Figure 9-22(*a*) pictures a U-shaped abutment. The arrows represent the assumed reactions of the superstructure. One can readily see that the abutment will act as a unit. The earth inside and behind the wing walls and bridge seat tends to burst out the former and to push the latter forward. The wing walls may serve as counterforts for the front wall. In cold climates a contraction joint with a key should be used through the middle of the abutment (or elsewhere as conditions warrant) to avoid cracking due to freezing.

Another type of abutment is shown in Fig. 9-22(*b*). This has straight wing walls with pilasters for the support of the bearings. It is an ordinary retaining wall with two large concentrated loads and two buttresses. Figure 9-23 shows a somewhat similar abutment.

These two illustrations show that the design of actual structures often becomes very complicated. It is necessary for the designer to study each case carefully, to see how he can separate

it into parts, to find the forces that act upon each portion, and to decide how he can approximate its action so that his design will be reasonable and safe.

It is not possible to give an elaborate example here, but a simple one will be illustrated.

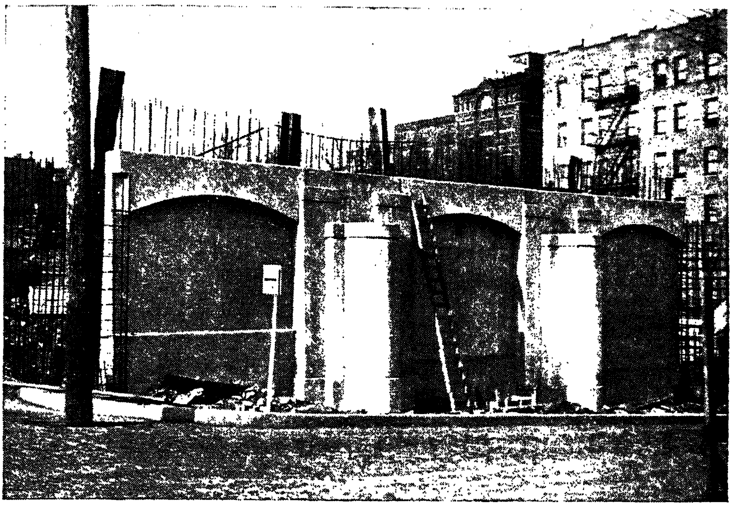


FIG. 9-23.—Abutment forming part of the New Jersey approach to the Lincoln Tunnel, New York City.

Problem 9-6. Find the maximum bending moment in the toe of the central portion of the abutment of Fig. 9-24. Assume earth = w = 100 lb. per cu. ft., $\phi = 30°$.

The bridge structure itself is assumed to be a highway overpass composed of a steel framework encased in concrete as pictured in Fig. 9-24(d). The reactions of the main girders are represented by P in Sketch (a), the bearings of the main girders being in recessed pockets. Figure 9-25 shows a bridge bearing which is being placed in a similar position. The stringers at the end of the bridge rest directly upon a shelf on the abutment, and their reactions are P'.

The portions AB and CD of the abutment are separated from the central part by expansion (or contraction) joints because the structure is so long that it is likely to develop shrinkage or temperature cracks if all of it is tied together. However, these joints are keyed so as to hold the parts in proper alignment, vertically and horizontally. These outer parts, if they were to be designed, might be treated as retaining walls by assuming a 1-ft. strip at the highest point and another near the middle. The results of the design of these strips would enable one to approximate the stresses, areas of rods, and dimensions for the rest of the section.

The central portion of the abutment will be assumed to be isolated from the rest. It must be designed for two sets of bridge loads, *viz.*, (1) dead load and applicable longitudinal forces only and (2) the combination of dead load, live load, impact, and longitudinal forces. The loads and the loading conditions are taken arbitrarily. Of course, the minimum values are for dead load alone. The "min. F" is due to longitudinal wind and temperature effects, but "max. F" has the additional loads due to traction and braking forces from live loads. Transverse wind pressure is omitted because it is not important in this particular analysis.

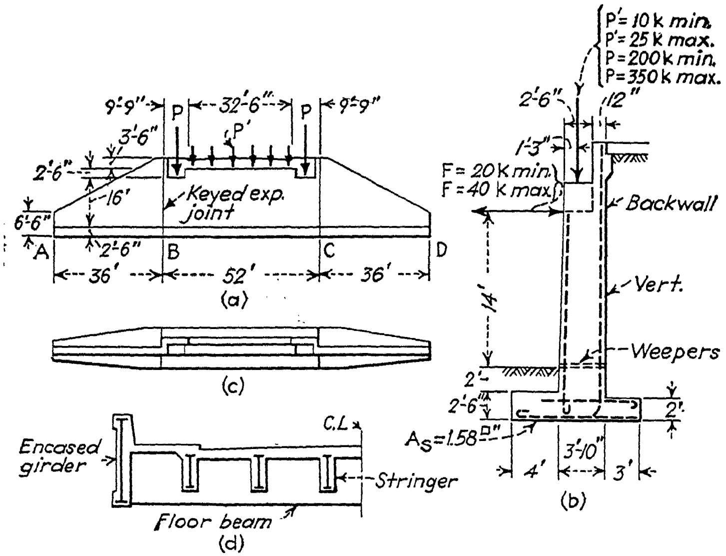


FIG. 9-24.

No surcharge will be assumed back of the abutment because the approach slab rests on the backwall and acts as a beam. However, the pressures caused by earth, temperature effects at the top, and water will be included. The magnitudes of these three forces per foot of abutment are:

1. From Eq. (8-7) of the preceding chapter, with the angle of slope of the earth δ equal to zero, $\phi = 30°$, and $w = 100$ lb. per cu. ft.,

$$P = 15h_1^2 = 15 \times 24.5^2 = 9{,}000 \text{ lb.}$$

2. The top pressure due to temperature will be the same as in Art. 8-9, or

$$P_T = 700 \text{ lb. per lin. ft.}$$

3. The top of the water table[1] will be assumed to be 8 ft. above the drain,

[1] Water pressure might be omitted in this case, but it is included for the purpose of illustration.

10 ft. above the top of the footing. Then, as illustrated in Art. 8-8,

$$P_w = 10 \times 45 \times \tfrac{10}{2} = 2{,}250 \text{ lb.}$$

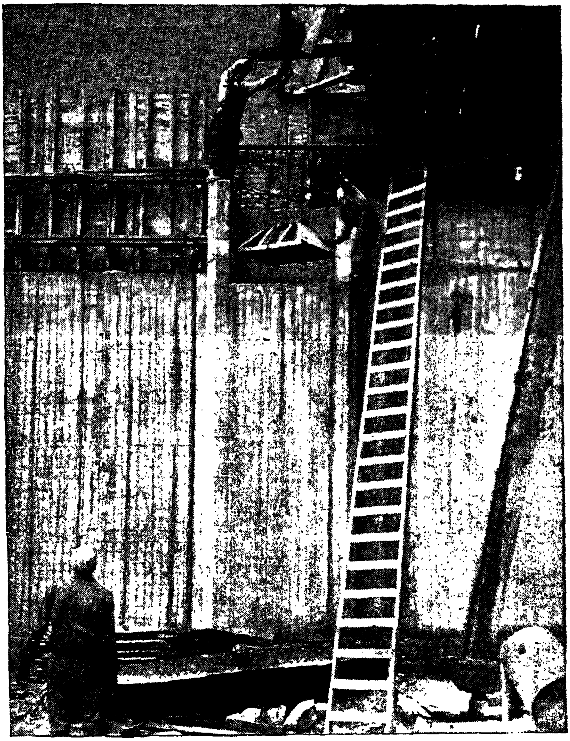


FIG. 9-25.—Setting a bearing for a steel bridge over the New York approach to the Lincoln Tunnel, New York City.

The weight of the abutment per foot must now be determined. Assume that the average cross section of the 52-ft. central portion is shown in Fig. 9-26. Take moments of all parts about A.

Portion		Weight		Lever arm		Moment
(a) 1 × 3.5 × 150	=	520	×	3.5	=	1,840
(b) 3.5 × 18.5 × 150	=	9,720	×	4.75	=	46,200
(c) 0.5 × 0.33 × 18.5 × 150	=	460	×	6.61	=	3,040
(d) 3 × 2 × 150	=	900	×	1.5	=	1,350
(e) 2.5 × 7.83 × 150	=	2,940	×	6.92	=	20,350
(f) 3 × 22.5 × 100	=	6,750	×	1.5	=	10,130
(g) 4 × 2 × 100	=	800	×	8.83	=	7,070
ΣW	=	22,090 lb.		ΣM	=	89,980 ft.-lb.

Then

$$\frac{89{,}980}{22{,}090} = 4.07 \text{ ft.}$$

One other preliminary step is the calculation of the bridge loads per foot of abutment. The entire central section of the abutment will act as a unit because it is a rigid block. The forces must be summed up and transformed into equivalent values per foot of abutment. From Fig. 9-24(b), the following can be found:

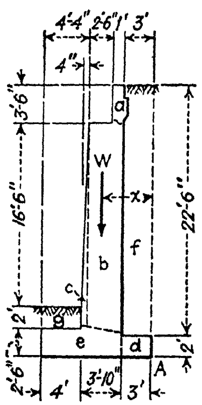


FIG. 9-26

Minimum loads:

$$\text{Vertical} = \frac{2P + 6P'}{52} = \frac{460}{52} = 8.85 \text{ kips}$$

$$\text{Horizontal} = \frac{F}{52} = \frac{20}{52} = 0.385 \text{ kip}$$

Maximum loads:

$$\text{Vertical} = \frac{2P + 6P'}{52} = \frac{850}{52} = 16.35 \text{ kips}$$

$$\text{Horizontal} = \frac{F}{52} = \frac{40}{52} = 0.77 \text{ kip.}$$

The vertical loads will be taken at a point 5.25 ft. from the tip of the heel. The horizontal forces will be assumed to be at the bottom of the girder seats, 18.5 ft. above the bottom of the footing.

Now that these fundamental loads have been found, the toe must be designed for the more severe of the two loading conditions: (1) the minimum bridge load with the corresponding lateral force; (2) the maximum bridge load with the maximum lateral force. These two combinations are pictured in Fig. 9-27 in which (a) represents the first case and (b), the second, and where W_A = the weight of the abutment, and W_B = the weight of the bridge superstructure and live load per foot of the abutment.

Consider Sketch (a) first. Take moments about A to find where the resultant R hits the base.

$$(22{,}090 + 8{,}850)x' = 22{,}090 \times 4.07 + 8{,}850 \times 5.25 + 700 \times 24.5 + 385 \times 18.5 + 9{,}000 \times 8.17 + 2{,}250 \times 3.33$$

$$x' = 7.8 \text{ ft.}$$

$$e = 7.8 - \frac{10.83}{2} = 2.38 \text{ ft.}$$

From Eq. (8-4) of the previous chapter,

$$p = \frac{2W}{3\left(\frac{B}{2} - e\right)} = \frac{2(22{,}090 + 8{,}850)}{3 \times 3.04} = 6{,}800 \text{ lb. per sq. ft.}$$

Next, consider Sketch (*b*) of Fig. 9-27. The equation of moments about A is

$$(22{,}090 + 16{,}350)x' = 22{,}090 \times 4.07 + 16{,}350 \times 5.25 + 700 \times 24.5 + 770 \times 18.5 + 9{,}000 \times 8.17 + 2{,}250 \times 3.33$$

$$x' = 7.5 \text{ ft.}$$

$$e = 7.5 - \frac{10.83}{2} = 2.08 \text{ ft.}$$

Then

$$p = \frac{2W}{3\left(\frac{B}{2} - e\right)} = \frac{2(22{,}090 + 16{,}350)}{3 \times 3.34} = 7{,}660 \text{ lb. per sq. ft.}$$

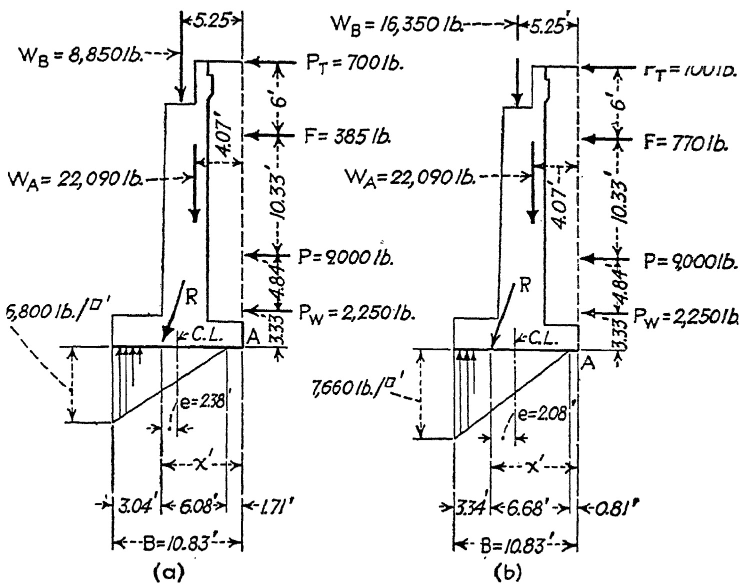


FIG. 9-27.

It is obvious that the second case produces the greater bending in the toe, but this is not always so for all abutments, since it depends upon the positions and the relative magnitudes of the forces.

The critical loading conditions upon the toe are pictured in Fig. 9-28. The moment at C is due to the pressure diagram minus the weight of the earth and the concrete. Then

$$M = 5{,}110 \times \frac{4^2}{2} + 0.67 \times 2{,}550 \times \frac{4^2}{2} - (2 \times 100 + 2.5 \times 150)\frac{4^2}{2}$$

$$M = 50{,}000 \text{ ft.-lb.}$$

If such an abutment as this is founded upon earth (Fig. 9-27), the footing must be capable of withstanding the loads that are imposed upon it before the backfill is placed. This may necessitate considerable reinforcement in the bottom of the heel.

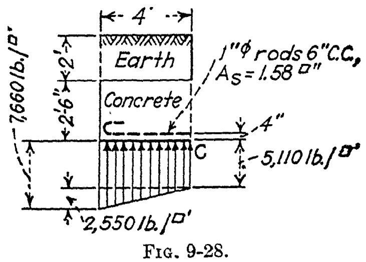


FIG. 9-28.

The varieties of construction used for abutments, and for piers also, are too extensive and too much affected by local conditions to warrant detailed discussion of them herein. However, the principles and kinds of reasoning illustrated in this chapter will generally enable one to design such structures.

Practice Problems

Problem 9-7. Assume a plain concrete pedestal like that of Fig. 9-3. It is 3 ft. square and 18 in. deep. It rests upon earth. It supports a column that is 16 in. square and that carries a load of 40,000 lb. What is the tensile stress in the concrete?

Problem 9-8. Assume a rectangular, plain concrete pedestal like that of Fig. 9-3, supported on earth. It is 3 ft. wide, 4 ft. long, and 20 in. thick. It carries a 16-in. square column which supports a load of 50,000 lb. What is the maximum tensile stress in the concrete?

Problem 9-9. Design a simple, square, reinforced-concrete footing to support an 18-in. square column which carries a load of 150,000 lb. Let f'_c = 2,500 lb. per sq. in., $n = 12$, the allowable f_c = 800 lb. per sq. in., f_s = 18,000 lb. per sq. in., the maximum shearing stress = 75 lb. per sq. in., with hooked rods, u = 125 lb. per sq. in., and the allowable soil pressure = 5,000 lb. per sq. ft.

Discussion. The footing is like that of Fig. 9-4(*a*) but without the pedestal. The solution is similar to that of Problem 9-2 without the effect of the slope.

Problem 9-10. Design a square, reinforced-concrete footing with a sloped top for the column of Problem 9-9 if all of the conditions remain unchanged except that the permissible soil pressure = 4,000 lb. per sq. ft.

Discussion. Assume that there is a flat top 24 in. square and that the slope is about 1:4 as in Fig. 9-4(*b*). Design the footing as for Problem 9-2.

Problem 9-11. Design a square, stepped footing like that of Fig. 9-4(*a*) to support a column load of 300,000 lb. The column is 24 in. in diameter, c = 3.5 ft., $n = 12$, the allowable f_c = 800 lb. per sq. in., f_s = 18,000 lb. per sq. in., the max. v_L = 75 lb. per sq. in., with hooked rods, u = 125 lb. per sq. in., and the allowable pressure on the soil = 5,000 lb. per sq. ft. Detail the footing.

Discussion. Follow the same procedure as for Problem 9-2 with c instead of a as the dimension that determines the starting points of the 45° sloped shear or distribution lines.

CHAPTER 10

MISCELLANEOUS STRUCTURES

10-1. Introduction. The purpose of this chapter is to illustrate the design of several important types of reinforced-concrete construction which may be handled by one who does not have a thorough knowledge of the analysis of indeterminate structures. Where necessary, the Code or other specifications will be resorted to in the determination of moments, shears, etc.

10-2. Positive Bending Moments in One-way Slabs. Slabs that have their main reinforcement extending in one direction (one-way slabs) and that are supported on two opposite sides

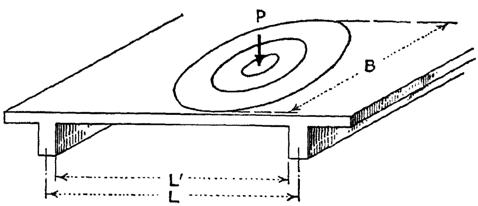


FIG. 10-1.

by steel beams, or slabs which are an integral part of a series of concrete T-beams, are often used in bridge and building construction. When the live loads are uniformly distributed, such slabs are designed as if they were a series of adjacent beams, each 1 ft. wide. However, this should not be done when the live loads are large concentrations.

When a concentrated load is placed upon a slab between the supports as pictured in Fig. 10-1, the slab will deflect so that its surface is minutely concave on top. Contours of this surface might look somewhat like those in the sketch. The limit of these deflections in one direction will be the clear span L'; in the other direction it will be some distance such as B. This lateral spreading is due to a combination of the resistance of the slab to shear and bending.

This lateral spreading of the deflections indicates that a considerable width of the slab is subjected to bending moments,

both M_x at right angles to the supports and M_y parallel to the ... magnitudes of these moments at any given point have some ... the relative positions of the point in ques-

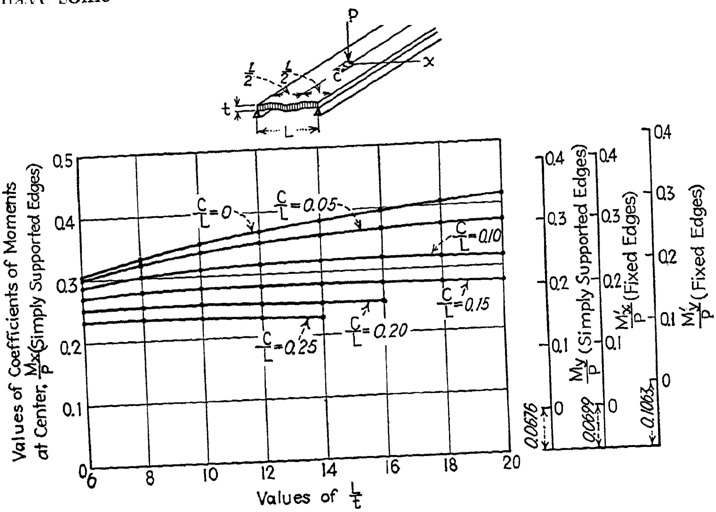


FIG. 10-2.

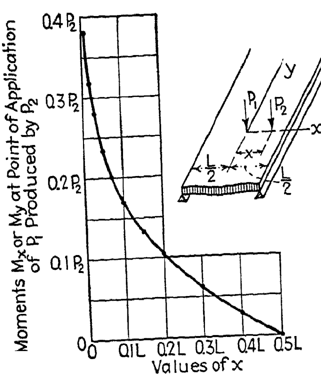


FIG. 10-3.

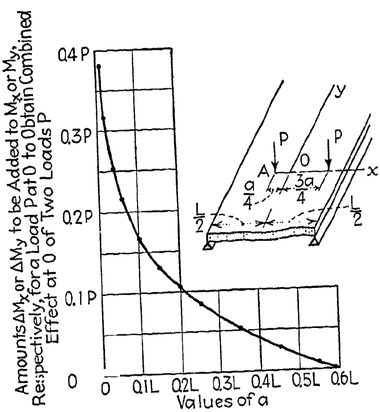


FIG. 10-4.

tion and the load. Furthermore, the total bending moment at any one point will be the summation of the individual moments that are caused by each concentrated load which affects that point.

The data that have been prepared by Dr. H. M. Westergaard[1] will be used for the calculation of positive bending moments in large one-way slabs. These data are general, and they facilitate the solution of problems that involve loads of any magnitude in any position. They will be used without going through the mathematical work by means of which they were derived.

Figure 10-2 (except for the changes in symbols which are necessary to agree with those used herein) gives Dr. Westergaard's curves showing the bending moment M_x for a unit load per foot of width at the center of a simply supported slab when a load is placed at its center. The three scales at the right of the

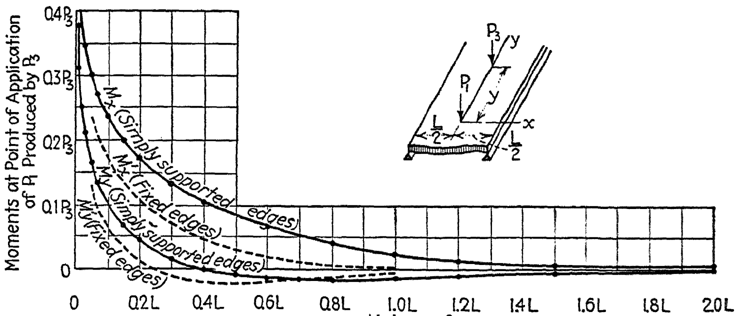


FIG. 10-5.

diagram are to be used for the three other cases to which they apply. When the ends of the slab are restrained but not fixed, the coefficients may be found by interpolation between the values that are given for zero restraint (simply supported) and for 100 per cent restraint (fixed).

Figures 10-3, 10-4, 10-5, and 10-6 give the additional bending moments (at the center of the slab) which must be added to those obtained from Fig. 10-2 in order to include the effects of adjacent concentrations. Each condition of loading is illustrated clearly in the figures. However, it should be noticed that these four diagrams do not consider any effect from the thickness of the

[1] See Computations of Stresses in Bridge Slabs Due to Wheel Loads, *Public Roads*, March, 1930. This investigation is a mathematical analysis of the bending moments in an isotropic slab with two opposite edges simply supported; the slab is very long in the direction parallel to the supports; the concentrated loads are distributed over a circular contact surface; and the supports are unyielding.

slab or from the diameter of the loaded area, but this is a satisfactory procedure unless the loads are too close together.

The curves are entirely general, but the designer must consider the relation of bridge slabs to the direction of traffic in

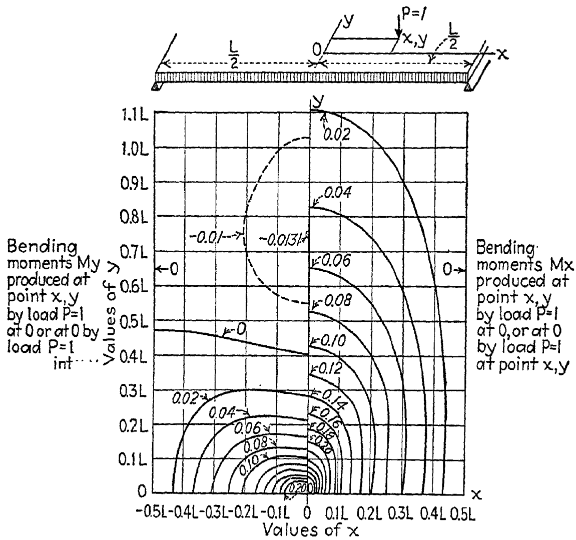


FIG. 10-6.

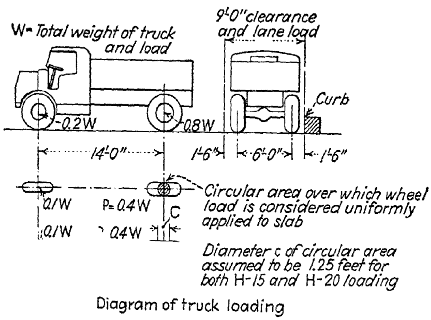


FIG. 10-7.

order to place the loads properly. The loading diagram for an H-20 truck is shown in Fig. 10-7.

It is well known that the restraint of the ends of a slab will decrease the magnitude of the positive moment. However,

owing to the deflection of the supports and the rotation of the ends of the slabs because of loads on one span but not on the adjacent one, the amount of this restraint is uncertain. In the case of a continuous slab which rests upon steel beams as pictured in Fig. 10-8(*a*), it is advisable to design the slabs as simply supported ones when the stringers are over 16 or 18 ft. long and when the beams are not more than about 5 or 6 ft. c.c. because the deflections of the beams are appreciable. In such a case, it is also advisable to use the same reinforcement for both positive and negative moments. If the steel stringers are very stiff or if the spans of the slabs are large, it is safe to assume 50 per cent restraint at the supports. However, in the case of monolithic reinforced-concrete construction like that of Fig. 10-8(*b*), the

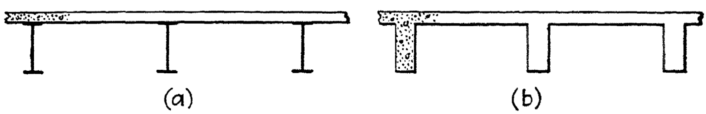


FIG. 10-8.

slabs may be considered to have 75 per cent restraint for live loads.[1]

The formulas for the bending moments that are caused by uniformly distributed dead loads or live loads may be assumed as follows:

1. Positive M at the center of a simply supported slab $= wL_1^2/8$, where L_1 is the clear span plus the depth of the slab, and w = the intensity of load per square foot.

2. Positive M at the center of the end spans of two or more spans $= wL'^2/10$, where L' is the clear span.

3. Positive M at the center of interior spans when there are three or more spans $= wL'^2/12$.

4. Negative M at interior support when there are two spans $= wL'^2/10$ if the span does not exceed 10 ft., or $wL'^2/8$ if the span is greater than 10 ft.

5. Negative M at first interior support when there are more than two spans $= wL'^2/12$ if the span does not exceed 10 ft., or $wL'^2/10$ if the span is greater than 10 ft.

[1] Modifications of the computed bending moments can be made to allow for stringer deflections by using the principles of moment distribution as given in Art. 13-8 (Chap. 13). The deflection of the stringer is the distance d through which the ends of the slab are offset. Also see Nathan M. Newmark, A Distribution Procedure for the Analysis of Slabs Continuous over Flexible Supports, University of Illinois, *Bull.* 304.

6. Negative M at other interior supports for any span $= wL'^2/12$.
7. Shear at outside edge of first interior support $= 1.2wL'/2$.
8. Shear at other supports $= wL'/2$.
9. The lateral moment M_y for uniformly distributed loads is zero.

These values listed above apply when the intensity of the uniformly distributed live load does not exceed three times that of the dead load and when the longer of two adjacent spans does not exceed the smaller by more than 20 per cent. L' is the clear span for positive moments and the average of the two adjacent clear spans for negative moments. Furthermore, the foregoing values apply to reinforced-concrete structures in which the slabs and beams are poured monolithically. If the slab is supported by steel I-beams, the span in all cases should be the distance center to center of beams.

TABLE 10-1.—LATERAL REINFORCEMENT FOR ONE-WAY SLABS
(Percentage of main reinforcement)

Direction of main reinforcement with respect to direction of traffic	Span length, feet	In middle half of span, per cent	In outer quarter of span, per cent
	0 to 10	45	30
Parallel....	10 to 20	35	25
	Over 20	25	15
Transverse.	0 to 10	65	45

The recommended percentage of lateral reinforcement[1] (for M_y) is given in Table 10-1. One must remember that the effective depth is decreased because the lateral rods are above the main ones.

Problem 10-1. Figure 10-9 shows part of the Bronx approach of the Bronx-Whitestone Bridge, at New York City, the sketch being a partial cross section of the deck. Design the slab AB to withstand the positive bending moment caused by two H-20 trucks in any position traveling lengthwise of the structure (parallel to the stringers). Assume that the point A is 75 per cent fixed[2] for live loads; c, the diameter of the loaded con-

[1] See Erps, Googins, and Parker, Distribution of Wheel Loads and Design of Reinforced Concrete Bridge Floor Slabs, *Public Roads*, October, 1937.

[2] When the student is familiar with moment distribution (Chap. 13), such "assumptions" can be obtained readily by trial analyses.

tact surface, $= 1.25$ ft.; $f_c = 1{,}000$ lb. per sq. in.; $f_s = 20{,}000$ lb. per sq. in.; and $n = 8$ (a rich concrete).

1. *Dead-load Moments.* Assume that the slab is 12 in. thick. Then the dead load, including the $2\frac{1}{2}$-in. concrete pavement, is 182 lb. per sq. ft. Therefore, using formula No. 2 previously stated,

$$\text{Positive } M = \qquad = 3{,}780 \text{ ft.-lb.}$$

2. *Live-load + Impact Moment at Center of Span.* Although the point of maximum bending may be slightly to the right of the center of the span, it is satisfactory to compute the live-load moment at the center of AB and then

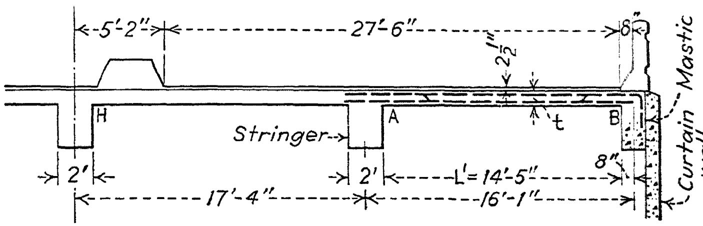


FIG. 10-9.

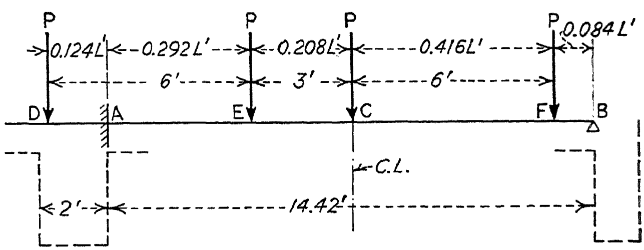


FIG. 10-10.

to add it directly to the dead-load moment. Also, since A is assumed to be 75 per cent fixed but B is hinged, the effect at the center line is equivalent to 38 per cent restraint at both ends.

The positions of the loads in Fig. 10-10 are chosen so as to place one load at the center. The load at the left of A is over the stringer, and it can be disregarded. The wheel at F is slightly closer to the curb than is necessary, but it is conservative.

The bending moments at C due to these concentrated loads are found as follows:

a. Rear Wheel at C, Using Fig. 10-2.

$$\frac{c}{L'} = \frac{1.25}{14.42} = 0.087; \qquad \frac{L'}{t} = \frac{14.42}{1} = 14.42.$$

If simply supported, $M_x = 0.33P$; if 100 per cent fixed, M_x is decreased by practically $0.07P$; then, for 38 per cent restraint, $M_x = 0.33P - 0.38 \times 0.07P = 0.303P$. Similarly, $M_y = 0.26P - (0.26 - 0.22)0.38P = 0.245P$.

b. Rear Wheels at E and F, Using Fig. 10-3. For E, $x = 0.208L'$; for F, $x = 0.416L'$. Then, using the same ratios as for the load at C to approximate the effect of restraint,

$$M_x \quad 0.1P \times \frac{0.303}{0.33} = 0.092P \text{ for } E.$$

$$M_x = M_y \quad 0.025P \times \frac{0.303}{0.33} = 0.023P \text{ for } F.$$

c. Front Wheel in Line with C, Using Fig. 10-5. The front wheels of the trucks are $14/14.42 = 0.97L'$ from the rear ones. Therefore, $y = 0.97L'$; interpolation for 38 per cent restraint yields $M_x = 0.02P$; and $M_y = -0.015P$.

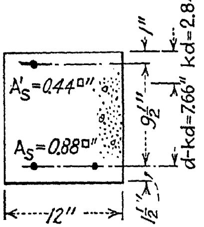


FIG. 10-11.

d. Front Wheels in Line with E and F. Figure 10-6 indicates that these loads cause only a slight effect at C for simply supported slabs, but Fig. 10-5 also indicates that, for fixed slabs, wheel loads even near the center line produce negligible moments. It is therefore reasonable to assume that other loads nearer to the supports may be disregarded.

e. Total Moments at C:

The impact factor is

$$I = \frac{50}{125 + 14} = 0.36.$$

Therefore,

$$16{,}000 \times 1.36 = 21{,}800 \text{ lb.}$$

and

$$M_x = 9{,}550 \text{ ft.-lb.,} \qquad \text{and} \qquad M_y = 7{,}520 \text{ ft.-lb.}$$

3. *Design of Slab.* The total bending moment from $DL + LL + I$ is

$$M_x = 3{,}780 + 9{,}550 = 13{,}330 \text{ ft.-lb.}$$
$$M_y = 7{,}520 \text{ ft.-lb.}$$

The section in Fig. 10-11 is assumed and tested, yielding f_c 885 lb. per sq. in. and $f_s = 19{,}100$ lb. per sq. in., which is satisfactory.

The amount of reinforcement in the bottom of the slab to withstand M_y will be taken as 0.65 of that for M_x, or $A_s = 0.65 \times 0.88 = 0.57$ sq. in. (Table 10-1). Therefore, use ¾-in. round rods 9 in. c.c. in the middle half

and 12 in. c.c. in the outer quarters of the slab. The top ties are arbitrarily made $\frac{3}{4}$-in. round rods 18 or 24 in. c.c.

10-3. Negative Bending Moments in Continuous One-way Slabs. The calculation of the magnitudes of the negative bending moments at the supports of large, continuous, one-way slabs, due to concentrated loads, is an important subject about which there does not seem to be sufficient data to permit easy and exact computations.

The procedure that is set forth herein[1] is based upon that which is used in the design of continuous beams, with an arbi-

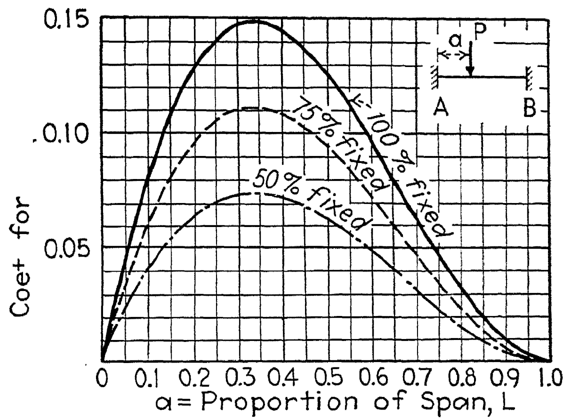


FIG. 10-12.—Influence lines for negative moment at left end for various restraints of ends.

trary correction of the bending moments to allow for the fact that a large area of the slab participates in supporting the loads. The three fundamental assumptions upon which the method is founded are as follows:

1. The total negative moments in the slab are the same as those which would occur in a prismatic beam having the same span and loads.

2. The width of the portion of the slab, at a support, that resists the negative bending moment due to any load will vary somewhat in proportion to the distance from the support to that load. This width is called the "effective width" b_e. This is equivalent to assuming that the load on a 1-ft. strip is $P \div b_e$.

[1] This method is empirical, without experimental proof, but it seems to give results that are reasonable.

3. The total negative bending moment per foot of width of slab is assumed to be the sum of the various moments per foot which are caused by the individual loads, the moment due to each one being distributed over its particular effective width.

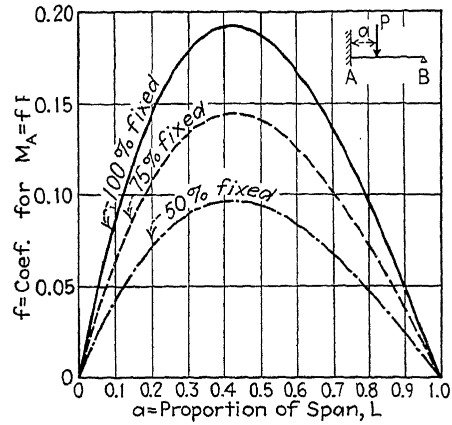


FIG. 10-13.—Influence lines for negative moment at left end for various restraints of left end.

Figures 10-12 to 10-15, inclusive, are influence lines which will facilitate the calculation of the total moments—also the total reactions.

The effective width b_e is the width of an imaginary rectangular strip of the slab across which the concentrated load may be

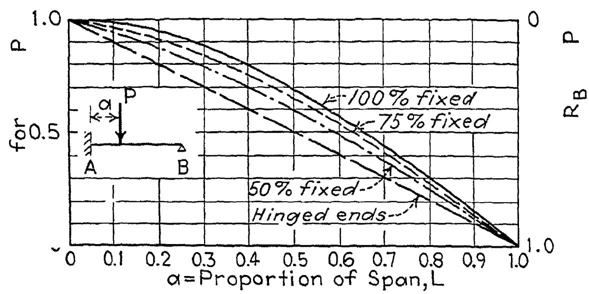


FIG. 10-14.—Influence lines for reactions for various restraints of left end.

considered to be uniformly distributed along a line through the load and parallel to the supports and for which the calculated unit stresses are taken as the greatest unit stresses that actually will exist in the slab under the action of that particular load.

The magnitude of b_e is not really known, but it is assumed conservatively to be

$$b_e = 2c + 1.4d \qquad \text{or} \qquad b_e = 2t + 1.4d \qquad (10\text{-}1)$$

in which c = the diameter of the contact surface under the load. d = the distance from the center of the load to the edge[1] of the

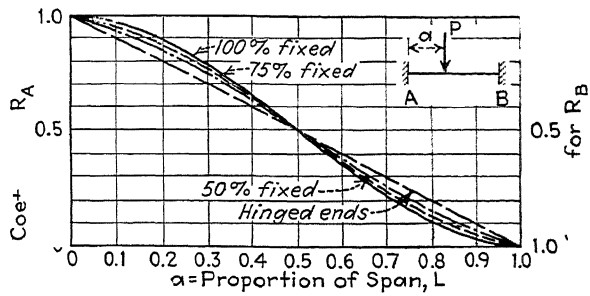


FIG. 10-15.—Influence lines for reactions for various restraints of ends.

support that is being considered, and t = the thickness of the slab, all being expressed in feet. The part of Eq. (10-1) that gives the greater effective width is to be used. The meaning of the formula is shown in Fig. 10-16. The minimum value of b_e will be assumed to be $5\sqrt{t}$, where t is the slab thickness in feet.

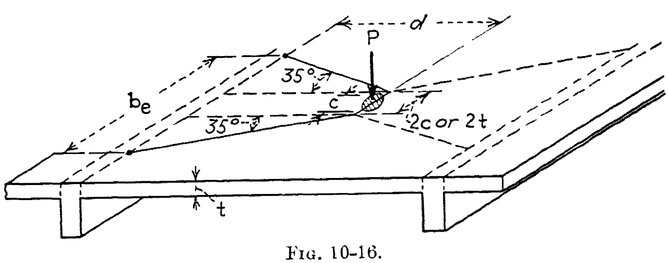


FIG. 10-16.

If the distance from the edge of the support to a load is less than the depth of the slab, it is probable that the load will have little effect upon the bending moment in the slab because it will tend to go rather directly into the support. If more adequate data to determine b_e are developed in the future, the

[1] In the case of steel beams, d should be measured from the center line of the beams. See Fig. 10-7 for the magnitude of c.

designer may use them without necessarily changing the fundamental method of attacking the problem.

Figure 10-17 indicates the general principle for the summation of the effects of multiple loads. In Sketch (*a*), the loads are on the same strip of slab; in (*b*), they are on separate strips, but, obviously, part of the slab is affected by both loads, the moments being additive if the overlap is considerable (about 20 per cent).

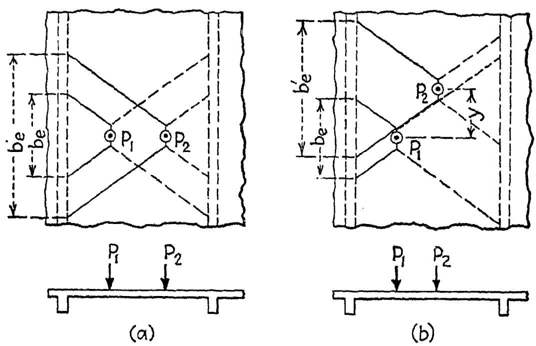


FIG. 10-17.

When the overlap is slight, one may add 50 per cent of the effect of the lesser of the two moments. When a series of concentrations such as two H-20 trucks 9 ft. c.c. can be placed parallel to the supports, Eq. (10-1) should be used with varying d and different combinations of wheels to plot a zigzag curve for $\Sigma(P/b_e)$; then an average curve of $\Sigma(P/b_e)$ can be drawn in terms of d for design purposes, giving about $P/4$ for the equivalent load upon a 1-ft. strip of a very large slab (18 to 20 ft.).

Of course, M_y, the bending moment that is parallel to the support, is assumed to be zero at the support.

Problem 10-2. Design the slab AB of Fig. 10-9 for the negative bending moment at A, using the same data as for Problem 10-1.

It must be noticed at once that the degree to which point A is restrained will depend somewhat upon the loading. It is possible to use a truck in each of the three lanes, in which case A should be considered 100 per cent fixed, or two trucks can be used so as to load the span AB but so as to leave HA almost unloaded. In the latter case, the slab is assumed to be 75 per cent fixed at A for live loads. From the principles of continuity, it is clear that the three trucks loading both spans will give the larger bending moment at A; but for the purpose of illustrating the method of analysis, both conditions will be tested.

1. *Dead-load Moment at A:*

$$M_a \quad \frac{182 \times 14.42^2}{10} = 3{,}780 \text{ ft.-lb.}$$

2. *Live-load + Impact Moment at A for Two Trucks.* An examination of Fig. 10-13 shows that the wheel loads will cause the maximum moment at A in a rectangular beam if they are between $0.1L$ and $0.8L$ from A. On the other hand, Fig. 10-16 indicates that the bending moments per foot of width

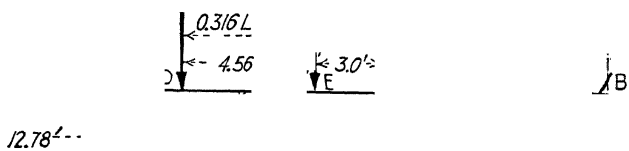


FIG. 10-18.

decrease as d increases. Therefore, determining the critical positions of the loads requires some experimentation. The positions given in Fig. 10-18 will be tested, the wheel at D being neglected.

a. Effective widths, using Eq. (10-1), are

$$b_e \text{ for load at } E = 2 \times 1.25 + 1.4 \times 1.44 = 4.5 \text{ ft.},$$

or, assuming $t = 1$ ft.,

$$b_e = 5\sqrt{t} = 5 \text{ ft.}$$
$$b_e \text{ for load at } F = 2 \times 1.25 + 1.4 \times 4.44 = 8.7 \text{ ft.}$$
$$b_e \text{ for load at } G = 2 \times 1.25 + 1.4 \times 10.44 = 17.1 \text{ ft.}$$

b. Rear Wheels at E, F, and G, Using Fig. 10-13 *for* 75 *Per Cent Fixed End at A:*

For E, $\quad M_a =$

For F, $\quad M_a = 0.135 \quad 14.42 \quad = 0.224P$

For G, $\quad \dfrac{0.095 \times P \times 14.42}{17.1} = 0.080P$

Total $= 0.491P$.

c. Front Wheels. An examination of Fig. 10-17 shows that the effect of the front wheels can be neglected because they are 14 ft. from the rear ones. Even in the case of the one that is in line with G, the assumed spreading will not overlap the effective width of the rear-wheel load at E.

3. *Live-load + Impact Moment at A for Three Trucks Placed as in Fig.* 10-19:

a. Effective Widths:

$$b_e \text{ for load at } E = 2 \times 1.25 + 1.4 \times 3.33 = 7.2 \text{ ft.}$$
$$b_e \text{ for load at } F = 2 \times 1.25 + 1.4 \times 6.33 = 11.4 \text{ ft.}$$
$$b_e \text{ for load at } G = 2 \times 1.25 + 1.4 \times 12.33 = 19.8 \text{ ft.}$$

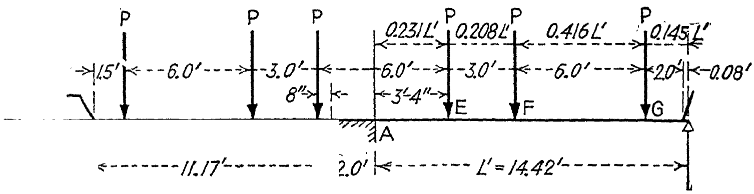


FIG. 10-19.

b. Rear Wheels at E, F, and G, Using Fig. 10-13 *for* 100 *Per Cent Fixed End at A:*

For E, $\quad M_a \quad 0.158P \times 14.42 \quad 0.316P$

For F, $$M_a = \frac{0.192P \times 14.42}{11.4} = 0.243P$$

For G, $$M_a \quad \frac{0.07P \times 14.42}{19.8} = 0.051P$$

$$\text{Total} = 0.610P.$$

4. *Design of Slab.* The moment at A for three trucks is greater than for two trucks. Therefore, use this moment for the design of the slab.

$$\begin{aligned} M_a \text{ for } DL & & &= 3{,}780 \text{ ft.-lb.} \\ M_a \text{ for } LL + I &= 0.610P &= 0.610 \times 21{,}800 &= \underline{13{,}300} \text{ ft.-lb.} \\ & & \Sigma M_a &= 17{,}080 \text{ ft.-lb.} \end{aligned}$$

If the slab is the same as that of Fig. 10-11 with three $\frac{3}{4}$-in. round rods in the top and one in the bottom, then $k = 0.316, j = 0.9, f_c = 915$ lb. per sq. in., and $f_s = 15{,}800$ lb. per sq. in. The top reinforcement should be arranged so as to stop one rod and bend down the second one out of the set of three in each foot of width, running the third rod clear across the top. The point of cutoff and bending should be determined by the use of the moment diagram as explained in Figs. 3-5 and 4-12. In the bottom of the slab, the rods will therefore be spaced 4 and 8 in. c.c., alternately, near the center, one rod extending for the full length.

Although the end B of the slab is assumed to be hinged, it is advisable to arrange the reinforcement at this point so as to provide for a negative moment which is about two-thirds of the positive moment at the center. The real magnitude of this restraint at B depends upon the torsional resistance of the marginal beam. This can be approximated by the method given

in Art. 16-10 (Chap. 16). The rods should be bent down near the outside of the edge support. Figure 10-9 indicates suggested details.

Continuous deck or floor construction like that in Fig. 10-9 presents difficulties in arranging construction and expansion joints. The slabs and beams should be poured monolithically if it is possible. When construction joints are necessary, try to place them parallel to the main reinforcement and at the columns that support the stringers. The expansion joints should be provided by using twin columns and edge supports so as completely to separate adjacent portions of the structure into units about 60 to 100 ft. long.

10-4. Bending Moments in Edge Supports. When a concentrated load is placed near the free edge of a slab, the distribution of the load cannot occur in the way that has been assumed for continuous slabs. It is necessary to support the edge, also to find a means of computing the bending moment in this edge support.

Figure 10-21 pictures part of a highway bridge with concentrated loads at the center of the span. The curves marked P_1 to P_6, inclusive, are plotted by using Westergaard's data in Fig. 10-5. They show the moment along the center line for each load as it would be if the slab were of infinite width. The lower curve which is marked ΣM_x is plotted by summing up the ordinates of the various curves that overlap, thus making one diagram that gives the total bending moment in each strip of the slab.

Naturally, the structure must withstand the entire bending moment that the loads produce. It is therefore reasonable to assume that the edge support must make up for all of the slab that is missing outside the curb line; in other words, it must carry all of the bending moment which is represented by the area between the base line and the ΣM_x curve beyond the curb. This can be calculated for any condition by plotting a diagram similar to Fig. 10-21. Erps, Googins, and Parker[1] have found that, for two H-20 trucks 9 ft. c.c. with the outer wheel 1 ft. 6 in. from the curb, the following formulas will give reasonably close values for the moment M_E in the edge support of a highway bridge slab:

[1] Distribution of Wheel Loads and Design of Reinforced Concrete Bridge Floor Slabs, *Public Roads*, October, 1937.

Simply supported slabs.................. $M_E = 0.01PL^2$.
50 per cent end restraint.................. $M_E = 0.008PL^2$.
75 per cent end restraint.................. $M_E = 0.007PL^2$.
100 per cent end restraint................. $M_E = 0.005PL^2$.

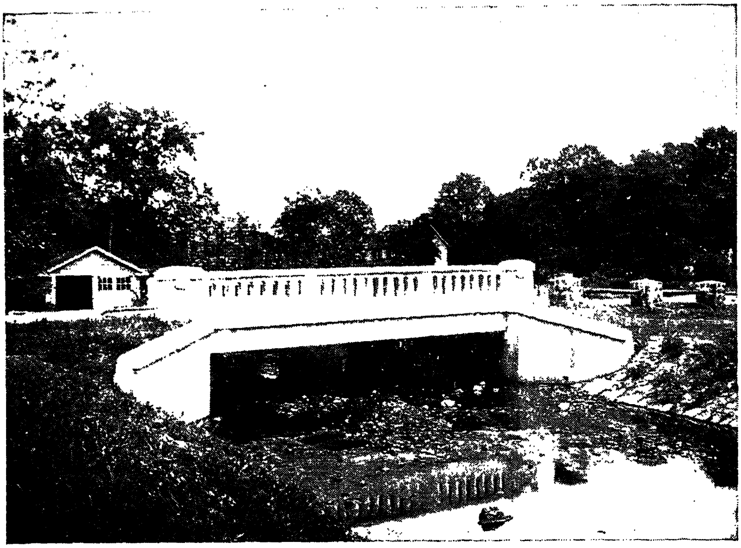


FIG. 10-20.—Highway bridge at South Orange, N.J. (*Courtesy of H. J. Finley.*)

In these formulas, L = the span in feet and P = the load in pounds. The data apply for spans of 5 to 25 ft., inclusive, when the loads are at the center of the span.

Problem 10-3. The highway bridge in Fig. 10-20[1] is a flat-slab structure which is shown in section in Fig. 10-21. Assume that the specified loading calls for one H-20 truck in one lane and H-15 trucks in the other two lanes (Fig. 10-7). Find the live-load plus impact bending moment in the part of the slab that supports the outer lane; also, in the edge support. Assume L = 30 ft. The ends of the slab are simply supported.

This structure has a rather long span for a simply supported slab. Rigid frames should be considered for longer spans. Actually, the bridge is skewed slightly, but this fact will be neglected.

It is possible to place three trucks side by side on the roadway, but not more than one truck can be in any one lane. Therefore, the locations of the wheels for maximum bending near the left edge of the slab will be as shown in Fig. 10-21.

[1] Built by the Engineering Department of Essex County at South Orange, N. J.

1. *Dead-load Moment.* The average dead load for the entire slab including the 2-in. wearing surface is approximately 1.87 × 150 = 280 lb. per sq. ft. Therefore, using Fig. 10-22, the following are found:

For slab, $M = \dfrac{280 \times 30^2}{8} = 31{,}500$ ft.-lb.

For edge support,

$$M = (1.33 \times 2.21 + 0.5 \times 2.25)\frac{150 \times 30^2}{8} = 68{,}600 \text{ ft.-lb.}$$

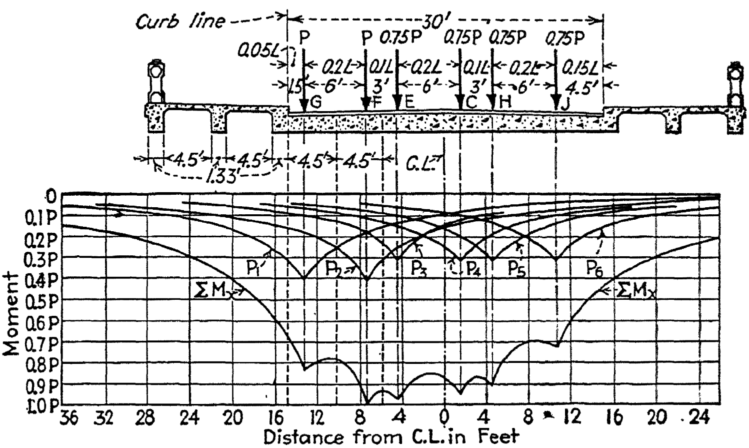


FIG. 10-21.

2. *Live-load + Impact Moment.* The usual criterion for maximum bending would place the rear wheels 1.4 ft. off the center line of the span. However, since there will be little difference in the results, and since Westergaard's curves give data for the center of the span, the rear wheels will be located at the center. The front wheels will be within 1 ft. of the end of the span, and their effects can be neglected.

The ΣM_x curve in Fig. 10-21 is developed by using Fig. 10-5 for each wheel, remembering that the H-20 truck is placed at the left side and that the H-15 truck wheel loads are $0.75P$. The effects of the areas of contact and of the slab thickness are disregarded. From the figure, the area within the ΣM_x curve is found to be approximately as follows:

Inner 4.5 ft., $M = 0.14PL = 4.20P$
Outer 4.5 ft., $M = 0.117PL = 3.51P$
Edge support, $M_E = 0.241PL = 7.23P$.
Since the impact factor for a 30-ft. span is 0.323,

$P = 1.323 \times 16{,}000 = 21{,}200$ lb., giving the following moments:

Inner strip, $M = \dfrac{4.20 \times 21{,}200}{4.5} = 19{,}800$ ft.-lb. per ft.

Outer strip, $M \quad \dfrac{3.51 \times 21{,}200}{4.5} \quad 16{,}500$ ft.-lb. per ft.

Edge support, $M_E \quad 7.23 \times 21{,}200 = 153{,}000$ ft.-lb., total.

Problem 10-4. Find the stresses in the curb section of the bridge of Fig. 10-21 if the details are as given in Fig. 10-22. Assume $n = 12$.

This problem is selected in order to illustrate a very troublesome point which often arises in practical design, viz., localized overstressing of projecting parts. It should not be overlooked or neglected.

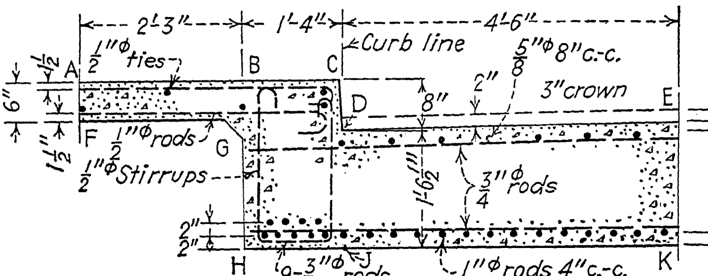


FIG. 10-22.

The curb piece is used to support the edge of the slab; therefore the two parts must be rigidly connected. It is not correct to assume that they can act independently. When the loads are applied, the entire structure in the vicinity of the curb tends to deflect as a unit. The neutral axis in the curb section AC cannot be at a relatively high position above D, whereas that of the slab DE is considerably below D. The effect is therefore a combination of bending and torsion which causes a concentration of compression along the line AC, with the center of compression C_c somewhere near G. The center of the tensile forces C_T will be near the steel just above HK and somewhere near the middle of HK.

One must be rather arbitrary but conservative in his assumptions regarding the design of the members in such a situation. In this case, assume that half of the first panel of the sidewalk, all of the curb piece, and the outer half of the outside lane will act as a unit.

The total moment in this member, including 100 lb. per sq. ft. live load on the sidewalk, will be as follows:

$$M_{DL} = 150(2.25 \times 0.5 + 1.33 \times 2.21 + 1.71 \times 4.5)\frac{30^2}{8} = 198{,}000 \text{ ft.-lb.}$$

$$M_{LL} \text{ on sidewalk} = 3.58 \times 100 \times \frac{30^2}{8} = 40{,}300 \text{ ft.-lb.}$$

$$M_{L+I} \text{ on outer lane (Problem 10-3)} = 16{,}500 \times 4.5 = 74{,}200 \text{ ft.-lb.}$$

$$\text{edge support (Problem 10-3)} = 153{,}000 \text{ ft.-lb.}$$

$$= 465{,}500 \text{ ft.-lb.}$$

In order to illustrate the procedure clearly, the analysis of the section will be made with more refinement than is usually the case in practice. The various steps are as follows, using the data for the member as shown in Fig. 10-23, in which the rods near DE are neglected:

1. *Location of Neutral Axis:*

$$27 \times 6(kd - 3) + \tfrac{16}{2}(\ d)^2 + \tfrac{54}{2}(kd - 8)^2 + 11 \times 1(kd - 2) = 12 \times 12.47(24.5 - kd) + 12 \times 1.76(22.5 - kd)$$

$$kd \quad 10.46 \text{ in.}, \qquad d - kd = 14.04 \text{ in.}$$

Probably the real neutral axis is not straight, but it may be somewhat as shown by the dotted line in Fig. 10-23. However, an attempt at an exact analysis does not seem to be justified.[1]

2. *Calculation of I_c:*

$$I_c = \frac{27 \times 6^3}{12} + 27 \times 6 \times 7.46^2 + \frac{16 \times 10.46^3}{3} + \frac{54 \times 2.46^3}{3} + 11 \times 8.46^2 + 12 \times 12.47 \times 14.04^2 + 12 \times 1.76 \times 12.04^2.$$

Therefore, $I_c = 49{,}220$ in.^{4}

3. *Section Moduli for Transformed Section:*

$$S_c = \frac{}{kd} = \frac{}{10.46} = 4{,}700 \text{ in.}^3$$

$$S_s = \frac{I_c}{n(d - kd)} = \frac{49{,}220}{12 \times 14.04} = 292 \text{ in.}^3$$

4. *Unit Stresses:*

$$f_c = \frac{M}{S_c} = \frac{465{,}500 \times 12}{4{,}700} = 1{,}190 \text{ lb. per sq. i}$$

$$f_s = \frac{M}{S_s} = \frac{465{,}500 \times 12}{292} = 19{,}100 \text{ lb. per sq.}$$

The results of the foregoing analysis will be accepted, and torsion will be neglected. However, the results show the danger of excessive stresses in the concrete when a portion of it projects above the general level; also in the steel when some of it is far below the plane of the main part of the reinforcement. The designer should try to make the difference between the main slab and the projecting part as moderate as possible.

The transfer of shear also needs attention. If C_C and C_T of Fig. 10-23 represent the resultant compressive and tensile forces, respectively, there must be longitudinal shearing stresses on vertical planes between these two forces. On the other hand, if the concrete is cracked below the neutral axis because of the strain in the steel, the ability of the concrete to transfer these shears is reduced somewhat. It is therefore desirable to use sub-

[1] For a better bending theory, see Hardy Cross, The Column Analogy, Engineering Experiment Station, University of Illinois, *Bull.* 215.

stantial ties along the planes of the rods near DE and HK, stirrups in $BCJH$, and hooked or bent ties joining AB to the curb, doing this so as to knit the structure together thoroughly.

When the slab and sidewalk of Fig. 10-23 cannot be poured monolithically, it is necessary to have a vertical construction joint along DJ or a horizontal one along D, the latter being preferred because the stirrups will tie the bottom to the upper part of $BCJH$. However, the joint must be roughened to transfer the horizontal shear.

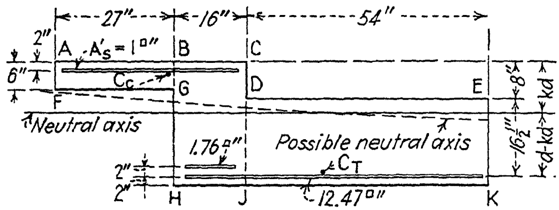


FIG. 10-23.

10-5. Shear in One-way Slabs. Experiments that have been made by Spangler[1] upon one-way, simply supported, concrete slabs with thicknesses of 2.5 to 6.5 in., with spans of 3.5 to 10 ft., and with widths of 5 to 7.5 ft. indicate that the actual shear per foot of width of such slabs may not differ greatly from Westergaard's theoretical results for thin flat plates[2] provided the loads are more than $0.25L$ from the support at which the shear is desired. However, when the loads are closer to the support, the experimental values are much less than the theoretical ones—probably because of the spreading that results from the thickness of the slab. Spangler recommends that the effective width of the slab for shear should be

$$b_e = 5\sqrt{t} \tag{10-2}$$

where t is the thickness of the slab in feet, and b_e is in feet. This will be adopted as a minimum. However, since shearing stresses must accompany changes of bending moments, and since Westergaard's data show that the moments are spread through wide areas in long-span slabs, the effective width in shear will be

[1] The Distribution of Shearing Stresses in Concrete Floor Slabs under Concentrated Loads, Iowa Engineering Experiment Station, Iowa State College, *Bull.* 126.

[2] See Computations of Stresses in Bridge Slabs Due to Wheel Loads, *Public Roads*, March, 1930.

assumed to be the same as that which has been used for negative bending moments at the supports, i.e.,

$$b_e = 2c + 1.4d \qquad \text{or} \qquad 2t + 1.4d. \tag{10-1}$$

A comparison of the values of b_e when computed by an approximation from the data of Westergaard, Spangler, and the second term of Eq. (10-1) is shown in Fig. 10-24. The introduction

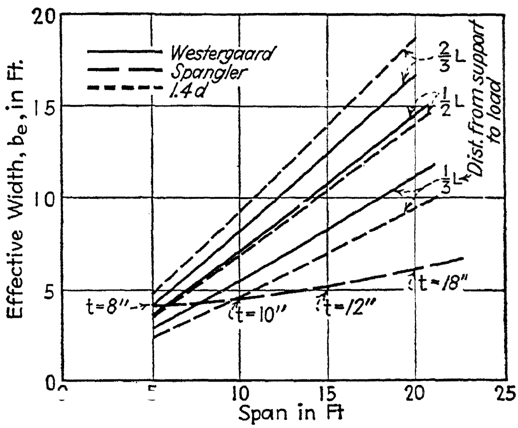


FIG. 10-24.—Effective widths of slabs for the computation of shearing stresses.

of the term $2t$ or $2c$ in Eq. (10-1) will increase b_e. It is apparent that Spangler's formula will control for short spans and for cases in which the loads are near the supports.

When the effective widths for various loads overlap, the shear per foot of slab is assumed to be the sum of the shears that are due to each individual load that affects that particular strip, in the same manner as that shown in Fig. 10-17 and described for negative moments in Art. 10-3.

Westergaard's data show that the intensity of the shear is increased when the support is fixed instead of hinged, but it is not necessary to make any change in the effective width for such cases. However, one should be conservative in the selection of the permissible unit shearing stress because of the cracking of the concrete which may occur in the top of the slab at such points of high negative moment.

Spangler's experiment indicated that the maximum shearing stresses when loads were placed near the free edge of a slab were

two or three times as great as when these loads were more centralized. In general, for large slabs, it is advisable to design the edge support to withstand all of the shear that is represented by the part of the effective width (or effective widths for multiple loads) that lies ouside the edge of the slab.

Problem 10-5. Find the shearing stress at A in the slab AB of Fig. 10-18 for the loading condition that is shown in the picture. Assume $f'_c = 3{,}750$ lb. per sq. in.

The dead-load reaction per foot will be (Art. 10-2)

$$\frac{1.2 \times wL'}{2} = 0.6 \times 182 \times 14.42 = 1{,}570 \text{ lb.}$$

The live-load + impact reaction per foot may be found by taking the positions of the loads as in Fig. 10-18, scaling the corresponding ordinates for R_a from Fig. 10-14 for 75 per cent fixed end, and dividing the results by the proper effective widths which were computed in Part 2a of Problem 10-2, as follows:

For E, $$R_a = \frac{0.97 \times 21{,}800}{5} = 4{,}230 \text{ lb.}$$

For F, $$R_a = \frac{0.83 \times 21{,}800}{8.7} = 2{,}080 \text{ lb.}$$

For G, $$R_a = \frac{0.37 \times 21{,}800}{17.1} = 470 \text{ lb.}$$

The total reaction per foot is $\Sigma R_a = 8{,}350$ lb.

The analysis of the slab in Problem 10-2 gave $k = 0.316$, $j = 0.9$, and $d = 11$ in. Therefore,

$$v_L = \frac{V}{bjd} = \frac{8{,}350}{12 \times 0.9 \times 11} = 70 \text{ lb. per sq. in.}$$

$$v_T = \frac{V}{bkd} = \frac{8{,}350}{12 \times 0.316 \times 11} = 200 \text{ lb. per sq. in.}$$

Therefore, no stirrups are required, because the Code permits $0.03f'_c = 0.03 \times 3{,}750 = 113$ lb. per sq. in.

10-6. Concentrated Loads on Two-way Slabs. A two-way slab is defined as one that has supports along all four sides and that has reinforcement extending in the directions of L_1 and L_2 (Fig. 10-25).

When a load is placed at the center of such a slab, the point C will deflect a definite amount. If one imagines that AB and ED

are unit strips of the slab centered on C, the deflections of AB and ED will be equal at C. On this basis, if these strips were acting alone, the proportion of this concentrated load that would be carried by each strip would vary inversely as the cubes of the spans. However, the distribution of the load to other portions of the slab complicates the problem.

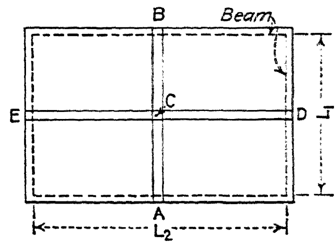


FIG. 10-25.

Westergaard's studies show that the bending moments in a simply supported slab of this character can be determined by using the data of Fig. 10-5 or 10-6. For instance, let Fig. 10-26(a) be a plan of a slab of infinite length with a span L_1. Apply a downward load P, at E, with coordinates x, y; apply a second load P acting upward, at F, a distance $2a$ from E. These

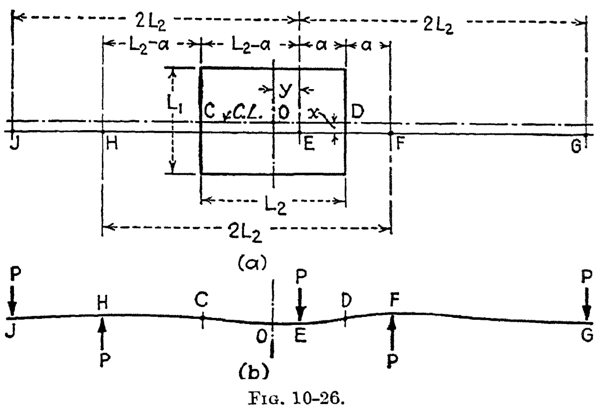


FIG. 10-26.

two forces counteract each other so as to cause zero deflection and zero moment at D. This is equivalent to the presence of a simply supported edge along D, as far as the action of the slab between O and D is concerned.

If a second upward force P is added at H of Fig. 10-26, with equal downward forces at J and G, the practical effect is to cause zero deflection and zero moment at C as well as at D. This is equivalent to having hinged supports along four sides

of a slab with spans L_1 and L_2. Figures 10-2 to 10-6 can then be used to find M_x and M_y, considering both the upward and the downward forces.

If the long side of a two-way slab exceeds twice the short side, it is best to design the short span as a one-way slab and to use a little more lateral reinforcement than would be placed in the one-way slab.

When the edges of a two-way slab are fixed, or when those on two opposite sides are fixed while the others are hinged, the negative bending moments in the slabs at the supports are very uncertain.[1] For a conservative design, it is recommended that, when L_2 of Fig. 10-26 exceeds $1.5L_1$, the slab should be considered as a one-way slab in computing the negative moment in the long sides; when loads are close to any support, the negative moments should be computed as for one-way slabs; and when the loads are near the center of a panel in which L_2 is less than $1.5L_1$, the negative moments may be computed in both directions as for a one-way slab with the portions of the load that are carried by each strip assumed to be inversely proportional to the cubes of the spans.

Generally, the shears in two-way slabs will be critical when the loads are near the supports. It will be conservative to calculate these shears as for one-way slabs.

Problem 10-6. Assume a two-way slab as pictured in Fig. 10-26 for which $L_1 = 15$ ft., $L_2 = 20$ ft., $t = 1.33$ ft., and $c = 1.25$ ft. Find the bending moments M_x and M_y for the load P at O.

Since y is zero, $a = 0.67L_1$, and $y + 2a = 1.34L_1$. Then, for the upward forces at H and F, $y = \pm 1.34L_1$, and $x = 0$; for the downward forces, $y = \pm 2.68L_1$, and $x = 0$. Therefore, using Fig. 10-2, with $c/L_1 = 0.083$ and $L_1/t = 11.3$, for the load at O, $M_x = 0.325P$, and $M_y = 0.257P$.
Then Fig. 10-5 gives the following:
For loads at H and F,

$$M_x = -2 \times 0.012P, \qquad \text{and} \qquad M_y = +2 \times 0.008P.$$

For loads at J and G,

$$M_x \text{ and } M_y = \text{practically zero.}$$
$$M_x = 0.301P, \qquad \text{and} \qquad M_y = 0.273P \text{ ft.-lb. per ft.}$$

Problem 10-7. Assume the same slab as in Problem 10-6. Find M_x and M_y at O for a load P at $x = 1$ ft. and $y = 3$ ft.

[1] See Nathan M. Newmark, A Distribution Procedure for the Analysis of Slabs Continuous over Flexible Supports, University of Illinois, *Bull.* 304.

For all loads, $x = L_1/15 = 0.067L_1$. For E, $y = 0.2L_1$; for F, $y = \frac{17}{15}L_1 = 1.13L_1$; for H, $y = -\frac{23}{15}L_1 = -1.53L_1$; for G, $y = \frac{43}{15}L_1 = 2.86L_1$; and for J, $y = -\frac{37}{15}L_1 = -2.47L_1$. Using Fig. 10-6 for E and F, and Fig. 10-5 for H, gives

For load at E, $M_x = +0.157P$, and $M_y = +0.05P$
For load at F, $M_x = -0.019P$, and $M_y = +0.009P$
For load at H, $M_x = -0.010P$, and $M_y = +0.005P$
Therefore, $\Sigma M_x = 0.128P$, $\Sigma M_y = 0.064P$.

The loads at G and J are neglected. The moments are in foot-pounds per foot of width of slab.

10-7. Uniformly Distributed Loads on Two-way Slabs. Two-way slabs that are part of large, continuous, monolithic floor systems are statically indeterminate. The Joint Committee has given their design considerable study. The recommendations in the Code[1] are based partly upon theory and partly upon tests. They are believed to be reliable and to yield safe results; hence they will be adopted herein.

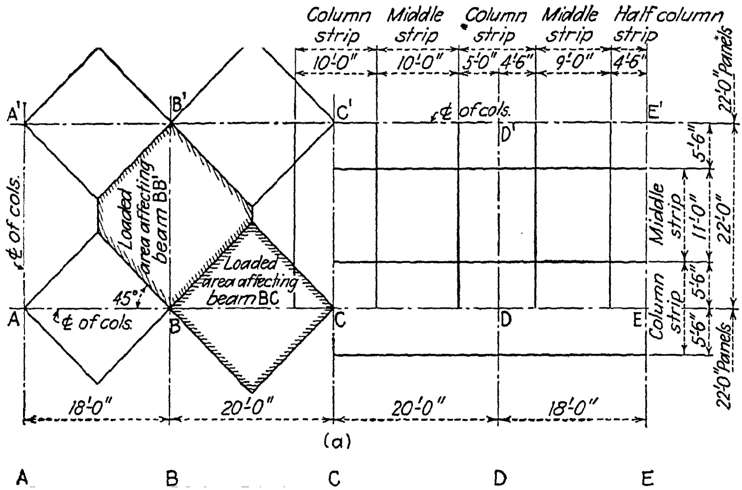
The following outline is an attempt to state briefly the rules given in the Code:

1. Slabs must be monolithic with supports, solid or ribbed both ways, and supported on all four sides with edges continuous or discontinuous.
2. Span = center to center of supports, or clear span + twice the slab thickness, whichever is the smaller.
3. S denotes short span.
4. m denotes ratio of short span to long span. Min. m (for computations) = 0.5.
5. Middle strip is a strip across panel, one-half panel in width, symmetrical about center line of panel. If m is less than 0.5, middle strip width = long span minus short span.
6. Column strip = remainder outside of adjacent middle strips and centered along support, as pictured in Fig. 10-27(a).
7. w = uniformly distributed load (panel fully loaded).
8. Principal design sections:
   A. For negative moments: along edges of panel at faces of supports.
   B. For positive moments: along center lines of panel.
9. Bending moment coefficients—as shown in Table 10-1A. Interpolate for intermediate values of m. For column strip moment coefficients, use two-thirds of value used for middle strip as average, varying from maximum at edge of middle strip, where value equals that of middle strip, to one-third that value at edge of panel.
10. Reinforce top and bottom of exterior corners of panels for same moment as positive moment in middle strip. Critical section in top of

[1] Data based largely upon H. M. Westergaard, Formulas for the Design of Rectangular Floor Slabs and Supporting Girders, A.C.I. *Proc.* p. 26, 1926.

slab is perpendicular to diagonal; in bottom, parallel to diagonal. Effective area of diagonal rod = $A \times$ sine of angle between rod and critical section.

11. When varying spans or loads on opposite sides of supports cause unequal negative moments, distribute two-thirds of unbalanced moment to the two spans in proportion to their I/L (stiffness factors). When the



(b)

FIG. 10-27.—Assumed distribution of loads to beams supporting two-way slabs with uniform loads.

difference between the unadjusted negative moments is large (perhaps 20 per cent of the larger), investigate mid-span positive moments as for continuous beams, or assume such moment = 1.5 $\times$ the original end moment in that panel minus the average of the end moments in the adjacent panels.

12. Distribution of load for shear in slabs and for design of supporting beams assumed to be uniform load on area within 45° lines from corners and median line of panel parallel to long side (Fig. 10-27), giving the following total loads for design of beams, considering panel on one side of beam:

(*a*) Shears:

(1) For short spans: $W_S = \dfrac{wS^2}{4}$. (10-3)

(2) For long spans: $\dfrac{wS^2}{4}\left(\dfrac{2-m}{m}\right)$ (10-4)

NOTE: Adjust reactions to take care of effect of different end moments.

TABLE 10-1*A*.—BENDING MOMENT COEFFICIENTS C FOR TWO-WAY SLABS
For moments in middle strips in ft.-lb. per ft. width of slab

Bending moments	Short span S Values of m						Long span, all values of m
	1.0	0.9	0.8	0.7	0.6	0.5 and less	
Case 1. Interior panels							
Negative moment at continuous edge........	0.033	0.040	0.048	0.055	0.063	0.083	0.033
Positive moment at midspan...............	0.025	0.030	0.036	0.041	0.047	0.062	0.025
Case 2. One edge discontinuous							
Negative moment at continuous edge.........	0.041	0.048	0.055	0.062	0.069	0.085	0.041
at discontinuous edge...	0.021	0.024	0.027	0.031	0.035	0.042	0.021
Positive moment at midspan................	0.031	0.036	0.041	0.047	0.052	0.064	0.031
Case 3. Two edges discontinuous							
Negative moment at continuous edge.........	0.049	0.057	0.064	0.071	0.078	0.090	0.049
at discontinuous edge...		.028	0.032	0.036	0.039	0.045	0.025
Positive moment at midspan................	0.037	0.043	0.048	0.054	0.059	0.068	0.037
Case 4. Three edges discontinuous							
Negative moment at continuous edge	0.058	0.066	0.074	0.082	0.090	0.098	0.058
at discontinuous edge...	0.029	0.033	0.037	0.041	0.045	0.049	0.029
Positive moment at midspan.	0.044	0.050	0.056	0.062	0.068	0.074	0.044
Case 5. Four edges discontinuous							
Negative moment at discontinuous edge	0.033	0.038	0.043	0.047	0.053	0.055	0.033
Positive moment at midspan................	0.050	0.057	0.064	0.072	0.080	0.083	0.050

Moment = CwS^2 for both long and short spans. w = load in lb. per sq. ft. Slabs interposed between heavy masonry walls are considered as discontinuous.

(b) Approximate uniform load per foot of beam for computing bending moments:

(1) For short span: $w_S = \frac{wS}{3}$. (10-5)

(2) For long span: $w_L = \frac{wS}{3}$ (10-6)

13. Minimum slab thickness t 4 in. but not less than

$$\left(S + \frac{S}{m} - \frac{N}{10}\right)\frac{1}{72}\sqrt[3]{\frac{2,500}{f'_c}} \tag{10-7}$$

where short span S is in inches and N is total length in inches of slab periphery that is continuous with adjacent slabs.

Problem 10-8. Assume a building floor like that in Fig. 10-27 with several bays 22 ft. long. The live load = 150 lb. per sq. ft., and f'_c = 3,000 lb. per sq. in. Find the maximum bending moments and shear in the interior bay shown, using the procedure given in the preceding outline.

Assume S = 18 ft., $m = \frac{18}{22} = 0.82$.

$$\text{Min. } t = \left(216 + \frac{216}{0.82} - \frac{2 \times 216 + 264}{10}\right)\frac{1}{72}\sqrt[3]{\frac{2,500}{3,000}} \quad 5.4 \text{ in.}$$

For the interior panel, t = 5.3 in.

TABLE 10-2.—BENDING MOMENTS IN MIDDLE STRIPS OF SLABS
For Problem 10-8

Panel	Edge	Center of span	m	Coef. C	S, ft.	S^2	$M = CwS^2$, ft.-lb. per ft.	Min A_s, sq. in.
	AA'	.....	0.82	0.026	18	324	−1,900	0.29
	BB'	.....	0.82	0.054	18	324	−3,940	0.57*
$AA'B'B$	AB	.....	0.82	0.041	18	324	−2,980	0.45
	$A'B'$	.....	0.82	0.041	18	324	−2,980	0.45
		Short	0.82	0.040	18	324	+2,910	0.44
		Long	0.82	0.031	18	324	+2,260	0.34
	BB'	.....	0.91	0.039	20	400	−3,520	0.57*
	CC'	.....	0.91	0.039	20	400	−3,520	0.53
$BB'C'C$	BC	.....	0.91	0.033	20	400	−2,970	0.45
	$B'C'$	.....	0.91	0.033	20	400	−2,970	0.45
		Short	0.91	0.030	20	400	+2,700	0.41
		Long	0.91	0.025	20	400	+2,250	0.34

* Adjusted by distribution of unbalanced M with j = 0.88; d = 4.5 in.

Assume t = 6 in. throughout, and w = 150 + 75 = 225 lb. per sq. ft. Then interpolate in Table 10-1A to find the moment coefficients and to compute the data given in Table 10-2.

The negative moments along BB' for the middle strip appear to differ by 3,940 − 3,520 = 420 ft.-lb. This should be adjusted as follows:

$$\text{Outer panel: } \frac{I}{L} \quad \left(\frac{12 \times 6^3}{12}\right) \div 216 \quad \frac{216}{216} = 1.0$$

$$\text{Interior panel: } \frac{I}{L} = \left(\frac{12 \times 6^3}{12}\right) \div 240 = \frac{216}{240} = 0.9$$

$$M \text{ for outer panel: } 3{,}940 - \frac{1}{1.9} \times 420 \times 0.67 = 3{,}790 \text{ ft.-lb.}$$

$$M \text{ for interior panel: } 3{,}520 + \frac{0.9}{1.9} \times 420 \times 0.67 = 3{,}650 \text{ ft.-lb.}$$

Having the bending moments, the reinforcement can be determined. Furnishing the required steel areas with a practical bar arrangement is not as simple as it may appear. Generally, part of the bottom steel should extend throughout the slab in both directions; some may be bent up to resist negative moments, although the slab is too thin for this to be very advantageous.

The greatest shear will be in the middle strip of the short span in panel $AA'B'B$. It is 225 × 9 = 2,025 lb. for a 12-in. strip.

In some cases the stiffness of such large thin slabs may need to be investigated.

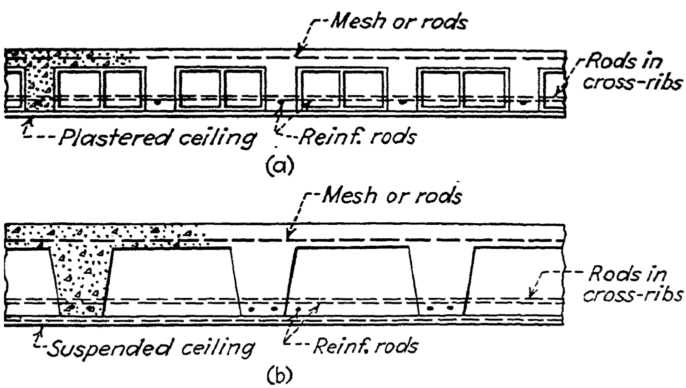


FIG. 10-28.

10-8. Tile-filled and Ribbed Floors. In building construction for which the live loads are not large, it is often economical to use large slabs with hollow-tile filler blocks as pictured in Fig. 10-28(a) or with hollow spaces which are produced by the use of thin steel forms, or "pans," as shown in Sketch (b). Both of these types may be built with the one-way or two-way system of reinforcement, but the latter is generally more rigid. The two-way system is made by using square or rectangular tiles or

pans which are separated on all four sides so as to form a series of small T-beams at right angles to each other.

The design of these special floors is based upon the same principles as those which have been explained for solid slabs.

10-9. Beams Supporting One-way and Two-way Slabs. The beams that support one-way slabs—like the stringers at A and B (Fig. 10-9)—are to be designed to support the reactions that are delivered to them by the slabs. When dealing with concentrated moving loads, these should be placed so as to give the maximum stresses in each individual beam. If these loads are within 2 or 3 ft. of the beam, the reactions from them should be treated as concentrated loads; otherwise they may be assumed to be distributed uniformly over a length that is to be determined by Eq. (10-1).

For beams that support two-way slabs, the Code specifications have been given in Art. 10-7 for uniformly distributed loads. The designing of the beams will be similar to the calculations shown in Figs. 4-21 to 4-23.

The effects of concentrated loads on beams that support two-way slabs are to be determined in the same general manner as for beams that carry one-way slabs.

10-10. Flat Slabs. The term "flat slabs" denotes large, rectangular slabs of approximately uniform thickness which are supported on columns but which have no beams or girders to carry these slabs—except possibly at the outside of the structure or at openings. Decreased height of each story, excellent lighting and ventilation, better fire resistance because of the absence of projecting corners, easier form work, and better economy for heavy uniform loading—these are some of the advantages of flat-slab construction. Figure 10-29 shows its use for both an elevated railway and a warehouse.

The essential parts of this type of construction are pictured in Fig. 10-30. This shows that, for design purposes, the slab is divided each way into column strips, which serve the purpose of beams between the columns; and middle strips, which may be regarded as suspended spans that are carried by the column strips for two-way systems—reinforcement parallel to the column rows in both directions. In a four-way system, as indicated in Fig. 10-30, the middle strips are crossed by two sets of diagonal reinforcement which carry the loads more directly to the columns. However, the four-way system is more complicated, and the

"packing up" of the four layers of rods is objectionable if the slab is relatively light. Figure 10-31 should be studied very carefully because it shows a two-way system that is under construction.

When a flat slab is loaded, it deflects on all sides of the columns which tend to punch through the floor. In fact, an exaggerated example can be seen when one props up a large canvas with a system of poles. Negative moments exist around the edges and

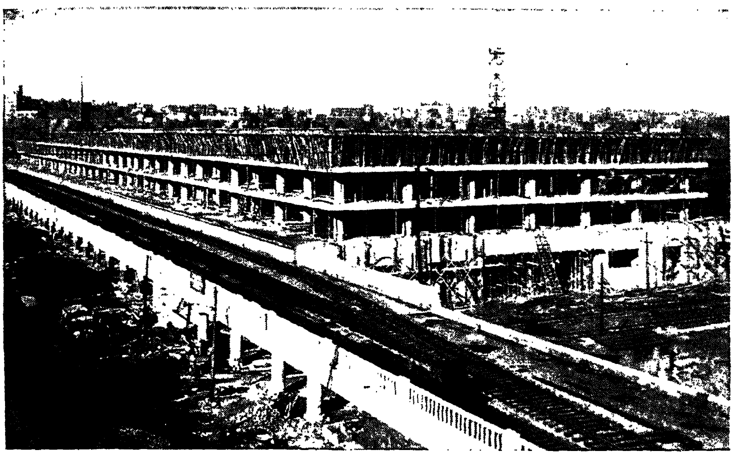


FIG. 10-29.—Construction of Lackawanna Terminal Warehouse, Jersey City, N. J. (*Courtesy of Turner Construction Co.*)

across the tops of the column capitals which are flared out in order to reduce these moments and the shears—the same being the function of the drop panels. The determination of the magnitudes and distributions of the stresses is exceedingly complex if one attempts to compute them theoretically. However, the empirical rules which have been determined from experiments and which are set forth in the Code will be sufficient for practical designs.

The following is, in general, an outline of recommendations given in the Code, with some additional explanations:

A. Limitations for applicability of specifications:
1. Slabs must be rectangular and monolithic with columns. They may have intermittent drop panels or continuous ones forming paneled ceilings, or they may have none at all.
2. There must be three or more rows of panels in each direction.
3. Maximum ratio of length to width of panel = 1.33.

4. Widths of drop panels must approximate one-third of panel length.
5. Maximum thickness through drop panels = 1.5 times thickness of slab outside of drops.

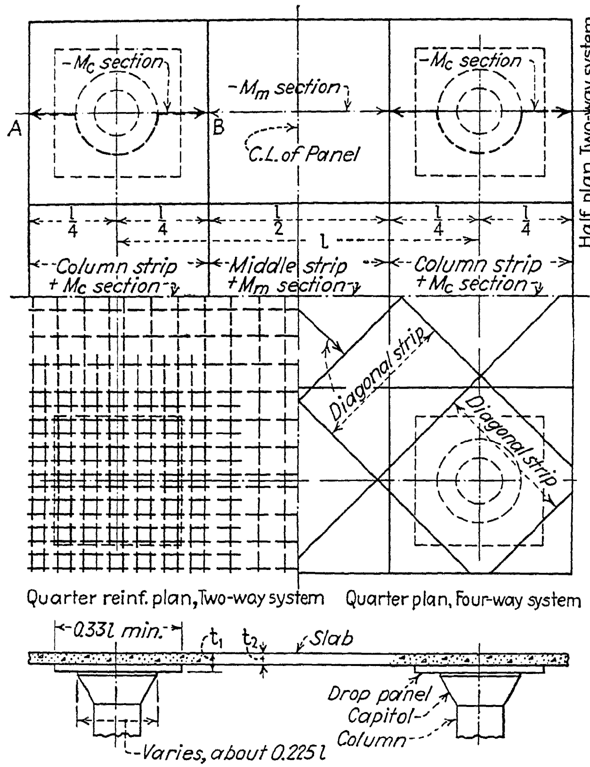


FIG. 10-30.

6. Dimensions of adjacent panels must not vary by more than 15 per cent.

B. Principal design sections (see Fig. 10-30):
1. Negative moments—along edges of panel. Notice that $-M_c$ sections for column strips follow perimeters of capitals.
2. Positive moments—along center lines of panel.
3. Width of column head section for calculating compression and for resisting negative moments = width of drop panel or $0.5l_1$ where no drop panel is used. l_1 = length of panel perpendicular to span of strip being considered.
4. In two-way system, reinforcement to resist computed bending in a strip must lie within that strip itself.

5. In four-way or other systems, reinforcement to resist computed bending in a strip = sum of normal areas of all rods in the strip times the cosine of the angle between each bar and the longitudinal axis of the span.

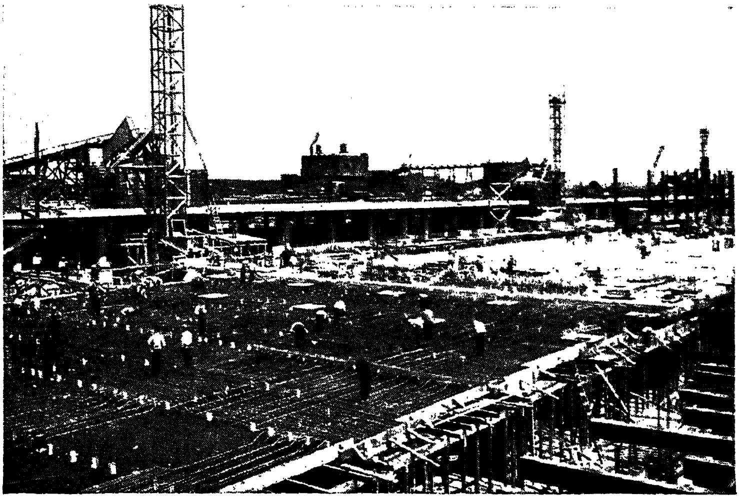


FIG. 10-31.—Construction of Lackawanna Terminal Warehouse, Jersey City, N. J. (*Courtesy of Turner Construction Co.*)

C. Bending moments:

1. For interior panels, the total bending moment M_o in the direction of either side of the panel for a uniformly distributed load of *w* lb. per sq. ft. is

$$M_o = 0.09(l - \tfrac{2}{3}c)^2 w l_1. \tag{10-8}$$

The symbol c = effective diameter in feet of a round capital at its junction with slab or drop panel except that one must disregard any part of the capital outside the largest 90° cone that can be included within the outline of the capital. If the capital is rectangular, c is the diameter of a circle having an area equal to the base of a similarly inscribed 90° pyramid.

2. Distribution of total moment M_o at principal design sections shall be as shown in Table 10-3. These moments should not be changed by more than 6 per cent, and the numerical sum of positive and negative moments at the principal design sections must not be reduced.

3. The minimum reinforcement at any section of any strip or band is $A_s = 0.0025bd$.

4. Discontinuous panels (one or two adjacent edges):

a. Beams at edges of discontinuous panels are assumed to restrain the slab. Columns having a stiffness factor (I/h) at least 1.5

TABLE 10-3.—MOMENTS TO BE USED IN DESIGN OF FLAT SLABS
For interior panels

Strip	Flat slabs without drops		Flat slabs with drops	
	Negative	Positive	Negative	Positive
Slabs with two-way reinforcement				
Column strip.	$0.46M_o$	$0.22M_o$		$0.20M_o$
Middle strip..	$0.16M_o$			
Slabs with four-way reinforcement				
Column strip.			$0.54M_o$	
Middle strip..	$0.10M_o$	$0.20M_o$	$0.08M_o$	

TABLE 10-4.—FACTORS BY WHICH BENDING MOMENTS IN TABLE 10-3 ARE TO BE MULTIPLIED TO COMPUTE BENDING MOMENTS IN DISCONTINUOUS PANELS

	Strip	Angle with edge	Discon-tinuous edge	Center of span	First interior column row	At exterior column
Restrained edge	Column.....	90°	0.80	1.15	1.15	
	Column.....	0°	....	1.15†	1.15†	
	Middle.....	90°	0.80	1.30	1.30	
	Middle.....	0°				
	Half column, along edge	0°	....	0.25	....	0.25
Unrestrained edge	Column.....	90°	0*	1.50	1.15	
	Column.....	0°	....	1.15†	1.15†	
	Middle.....	90°	0*	1.50	1.30	
	Middle.....	0°				
	Half column, along edge	0°	....	0.25	....	0.25

NOTE: When designing an outside corner panel, make column and middle strips each way the same as given for those 90° to the discontinuous edge.

* Minimum reinforcement at any section = $0.0025bd$.

† For corner panels with two discontinuous edges; also seems to be desirable for others.

times that of the slab (I/l) are equivalent to such beams where I of the slab is based on a width equal to center to center of columns and the thickness is $t_2 + \frac{1}{3}t_1$.

b. Bending moments given in Table 10-3 shall be corrected as shown in Table 10-4.

c. Extend positive and negative reinforcement to within 2 in. of discontinuous edge, and anchor the latter.

5. If there is an interior beam or bearing wall along center line of columns, multiply the moments of Table 10-3 by the following:

a. 1.30 for the negative moment of middle strip.

b. 0.25 for the moments in column strip above and parallel to the beam or wall.

D. Thickness of slab and drop panel:

1. To avoid excessive stress at middle of column strip due to nonuniform distribution of negative bending moments, change thickness computed on basis of moments and allowable unit stresses for balanced design to the following:

a. Interior panels, with or without drops:

$$t_1 = 1.2d + 1\tfrac{1}{2} \text{ in.,} \tag{10-9}$$

computing width of column strip as full width of strip without drops or width of drop alone if used. Also, minimum thickness outside drops is

$$t_2 = 0.67t_1. \tag{10-10}$$

b. Exterior panels:

Vary Eq. (10-9) to agree with modifications of moments as set forth previously. If exterior panel widths equal interior ones, t_1 for exterior may be about $1.08t_1$ for interior. Compressive reinforcement may be used in the slab to maintain t_1 throughout, or exterior panels may be correspondingly reduced in width.

Minimum thickness at center of panel:

a. Floor slabs with drop panels, $4\frac{1}{2}$ in.
Floor slabs without drop panels, 5 in.

b. Roof slabs with drop panels, $3\frac{1}{2}$ in.
Roof slabs without drop panels, 4 in.

c. Floor slabs: end panels, $0.030l$.

Floor slabs: interior panels, $0.026l\sqrt[3]{\dfrac{2{,}500}{f'_c}}$.

d. Roof slabs: end panels, $0.025l\sqrt[3]{\dfrac{2{,}500}{f'_c}}$.

Roof slabs: interior panels, $0.023l\sqrt[3]{\dfrac{2{,}500}{f'_c}}$

E. Beams in flat slab construction:

1. Beams (or equivalent) are to be used at all discontinuous edges; they are to be monolithic with slab and to frame into columns.
2. Edge beams are to carry uniformly distributed load of one-fourth that on adjacent panel—plus all direct loads.
3. Interior beams are to carry uniformly distributed load of one-fourth that on both adjacent panels—plus all direct loads—as a T-beam.
4. If interior and intersecting beams interfere with flat-slab action, frame entire panel as beam-and-slab construction.

F. Column capitals, brackets, and drops:

1. Minimum effective diameter of capital $= 0.20l$, where $l =$ average span length of rectangular panels.
2. Brackets may replace capitals on exterior columns if slope is not flatter than 45° and brackets are as wide as column. Value of $c = 2 \times$ distance from center of column to where bracket is $1\frac{1}{2}$ in. deep.
3. Minimum length of drop $= \frac{1}{3}l$. Increase if shear requires it.
4. Use drops on exterior columns if used on interior ones.

G. Points of inflection (assumed):

1. For interior symmetrical panels, distance from center of span $= 0.3$ times distance between two sections of critical negative moments without drops or 0.25 times this distance with drops.
2. For end panels, locate as for restrained beams.

H. Arrangement of reinforcement:

1. Arrange rods for intermediate as well as critical sections.
2. Extend some positive and negative reinforcement at least $\frac{1}{15}l$ beyond assumed point of inflection.
3. Locate lapped splices in regions of small stress.
4. Arrange at least 4/10 of bars to pass continuously through two sections of critical negative moments and intermediate one of positive moment.
5. At least one-fourth of positive reinforcement in column strips is to extend 20 diameters or more into drops or to within $\frac{1}{12}l$ of column centers if no drops are used.
6. Negative reinforcement of column strips must be inside the strips. (With drop panels, it may be desirable to place a large part of it over the drop panel.)

I. Shear and diagonal tension:

1. Maximum allowable v_L at distance d (effective depth) beyond capital or bracket and concentric with or parallel to it:
   a. $0.03f'_c$ if at least 50 per cent of total negative reinforcement of column strip passes through the section.
   b. $0.025f'_c$ if 25 per cent of total negative reinforcement of column strip passes through the section.
   c. Interpolate for values between the preceding ones.
2. Maximum allowable v_L at distance d beyond the drop panel and parallel thereto $= 0.03f'_c$, but most of the negative reinforcement of the column strip should be within the strip and above the drop.

J. Openings in flat slabs.

Openings may be used if moments given in Table 10-3 can be safely resisted.

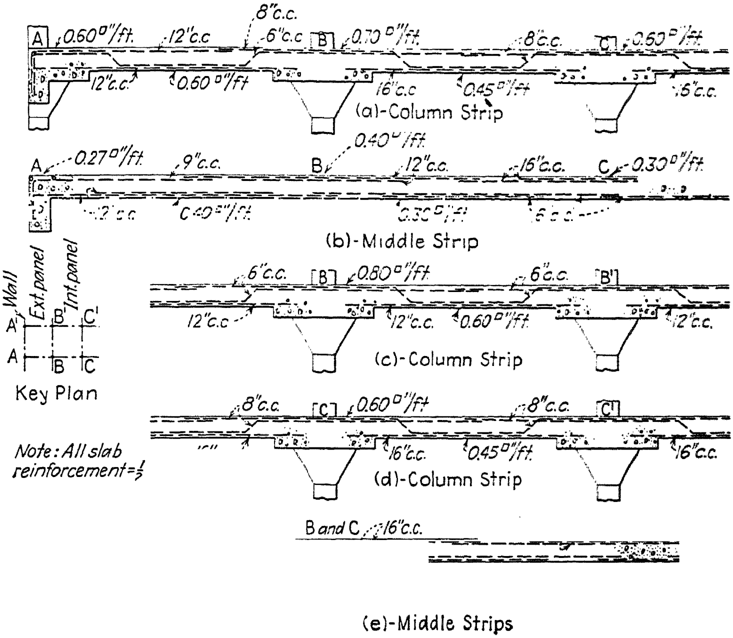


FIG. 10-31*A*.—Schematic diagrams of flat-slab reinforcement.

Problem 10-9. Assume that a building is 6 panels wide and 15 panels long. It is two-way flat-slab construction with drop panels. All spans are 20 ft. both ways; the live load = 250 lb. per sq. ft.; the drop panels are 7 ft. square; and c for the capitals is 4.5 ft. Design the slabs of the exterior and the first interior panels of a typical intermediate bay as shown in the key plan in Fig. 10-31*A* if f'_c = 3,000 lb. per sq. in., n = 10, and the allowable f_s and f_c = 18,000 and 1,050 lb. per sq. in., respectively. Use the Code's procedure. Assume a strong edge beam at the exterior wall so that the slab is restrained.

Discussion. The numbers and letters given along the right-hand margin of the calculations refer to the applicable item in the preceding outline. Assume the average load $w = 250 + 110 = 360$ lb. per sq. ft.

1. Interior panel:

$M_o = 0.09(20 - \frac{2}{3} \times 4.5)^2 360 \times 20 = 187{,}000$ ft.-lb. *C*1

Column strip: $-M_c = -0.50 \times 187{,}000 = -93{,}500$ ft.-lb. *C*2

$+M_c = +0.20 \times 187{,}000 = +37{,}400$ ft.-lb. *C*2

Middle strip: $-M_m = -0.15 \times 187{,}000 = -28{,}000$ ft.-lb. *C*2

$+M_m = +0.15 \times 187{,}000 = +28{,}000$ ft.-lb. *C*2

From Table 5, Appendix, $K = 169$ for balanced design. D1

Width of column strip for compression = 7 ft. B3

$$\sqrt{\frac{M}{Kb}} = \sqrt{\frac{93{,}500}{169 \times 7}}$$

$t_1 = 1.2 \times 9 + 1.5 = 12.3$ in. (say 13 in. to allow for extra thickness of exterior panel slabs) D1a D1b

$t_2 = 0.67 \times 13 = 8.7$ in. (say 9 in.) D1a

Shear near edge of capital:

$$V = \left[10 \times 20 - \frac{\pi}{2}\left(2.25 + \frac{11.5}{12}\right)^2\right]360 = 66{,}200 \text{ lb.} \qquad I1a$$

$$v_L = \frac{66{,}200}{(\pi \times 38.5)0.88 \times} = 54 \text{ lb. per sq. in. [E}$$

Shear near edge of drop panel:

$$V = \left[10 \times 20 - \left(3.5 + \frac{7.5}{12}\right)\left(7 + \frac{2 \times 7.5}{12}\right)\right]360 = 60{,}000 \text{ lb.}$$

I2

$$\frac{60{,}000}{[2(42 + 7.5) + (84 + 2 \times 7.5)]0.88 \times 7.5} = 46 \text{ lb. per sq. in.}$$

2. Exterior panel:

Column strip:

$-M_c$ at outer edge $= -93{,}500 \times 0.80 = -74{,}800$ ft.-lb. C4b

$+M_c$ at center $= +37{,}400 \times 1.15 = +43{,}000$ ft.-lb.

$-M_c$ at interior column $= -93{,}500 \times 1.15 = -107{,}500$ ft.-lb.

Middle strip:

$-M_m$ at outer edge $= -28{,}000 \times 0.80 = -22{,}400$ ft.-lb. C4b

$+M_m$ at center $= +28{,}000 \times 1.30 = +36{,}400$ ft.-lb.

$-M_m$ at interior col. line $= -28{,}000 \times 1.30 = -36{,}400$ ft.-lb.

Half column strip parallel to edge:

$-M_c$ at column $= -93{,}500 \times 0.25 = -23{,}400$ ft.-lb. C4b

$+M_c$ at center $= +28{,}000 \times 0.25 = +7{,}000$ ft.-lb.

Inspection of v_L for the interior panel shows that, even though the bending moments at the first interior columns are greater than for the second interior ones, the change in the shear will be so small that the exterior slabs are automatically known to be able to resist the shearing stresses safely.

3. Reinforcement.

Figure 10-31*A* shows the arrangement of reinforcing adopted. Equation (2-6*a*) is used for computing A_s, assuming that all strips are 10 ft. wide except the half column strip along AA'.

10-11. Cellular Construction. Cellular construction of reinforced concrete is advantageous for relatively long spans because it permits the use of deep members without undue dead load. The resistance of such sections to bending can be utilized to good

advantage structurally and economically. Figure 10-32 gives one example of this type of construction.

The design of cellular structures involves the same general principles that are used in other cases. However, there are many detail points which should be considered carefully.

FIG. 10-32.—Schmitz Park Bridge, Seattle, Wash. (*Courtesy of Portland Cement Association.*)

Figure 10-33 shows half of the cross section of the Summit Avenue Bridge[1] over the New Jersey approach to the Lincoln Tunnel in Union City, N.J. It is a two-span continuous structure, and it is shown in more detail in Fig. 10-34, which will be used as an example in order to clarify the following discussion:

1. The calculation of the resistance of such a section to bending moments is similar to that for T-beams. Each cell is made to carry its particular portion of the loads. The transformed-section method may be used for computing I_c. The engineer can use his judgment in simplifying the details of the section in order to facilitate the calculations, as will be shown in Problem 10-10.

In this structure, the 12-in. slabs between the cells, and the parapets, were poured after the forms for the cells were removed, so these parts should be disregarded in the computation of I_c,

[1] This structure is unusually deep, but it is made to carry a 36-in. diameter sewer through the central cell.

this being a conservative assumption. However, the slab will participate somewhat in supporting the live loads and the dead load of the deck. The parapet was separated into units by deflection (or expansion) joints in order to prevent its participation in the bending action.

When one is designing a rigid frame or a continuous structure, it is necessary to design it for both positive and negative bend-

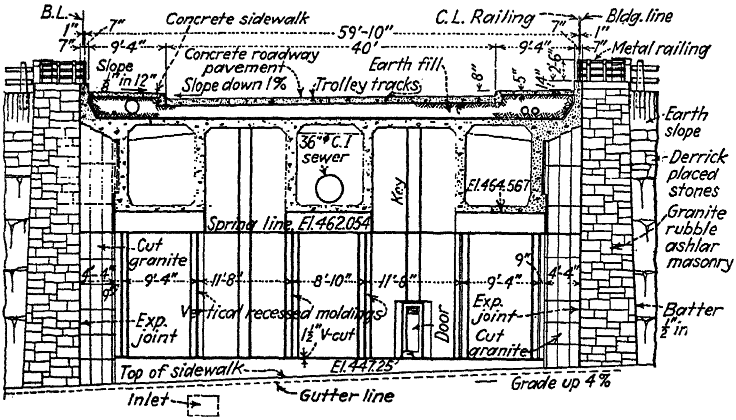


FIG. 10-33.—Section of Summit Avenue Bridge, New Jersey approach to the Lincoln Tunnel in Union City, N.J.

ing moments. It is obvious that, in this case, there is considerable concrete to resist the compressions due to the former; but in the case of negative moments, the resistance to compression is confined to the U-shaped bottom part of the cell. Therefore, near the points of large negative bending moments, the side walls and the bottom slab should be thickened considerably on the inside.

2. The placing of the longitudinal reinforcement must be studied carefully. When the intrados of the structure is curved, the rods should be concentrated largely near the bottom of the side walls; otherwise the slab may buckle, or the rods may spall the concrete because of insufficient ties to carry the radial components which result from the curvature of the rods. The detail of the central cell in Fig. 10-34 shows the arrangement of the steel for positive moments; that of the outer cell, for negative moments.

The rods must be arranged so that they can be spliced by lapping without undue hindrance in the placing of the concrete;

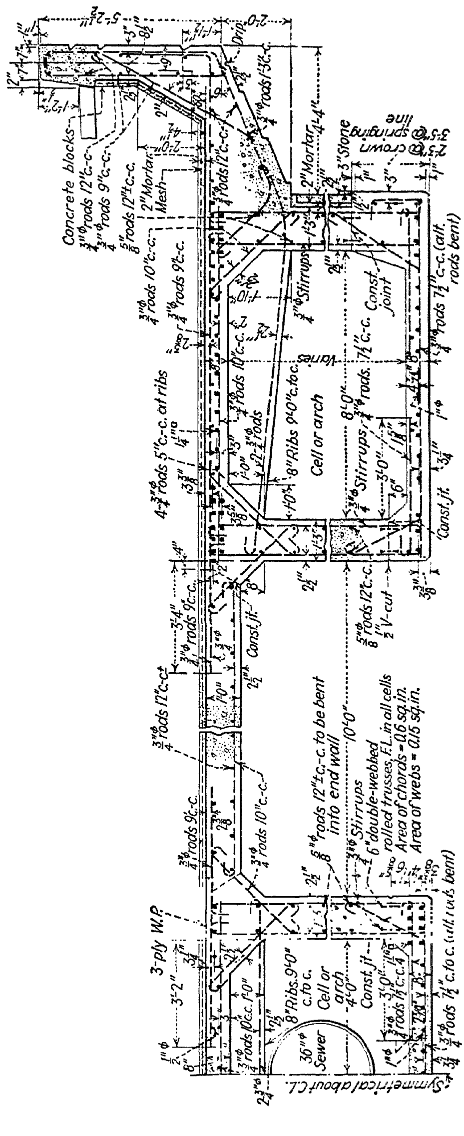

Concrete blocks
2" Mortar
Mesh
Drip
2" Mortar
3" Stone
2.5' @ crown
3.5' @ springing line
4-3/4"φ rods 5" c.-c. at ribs
Cell or arch
8" Ribs 9'-0" c. to c.
3/4"φ Stirrups
Varies
Const. joint
Const. jt.
1/2" V-cut
3-ply W.P.
36"φ Sewer
Cell or arch
Symmetrical about C.L.
5/8"φ rods 12"± c.-c. to be bent into end wall
3/4"φ Stirrups
6" double-webbed rolled trusses, F.L. in all cells
Area of chords = 0.6 sq. in.
Area of webs = 0.5 sq. in.

FIG. 10-34.

they must be adequately encased; and they must be tied in to resist the radial component of the tension.

3. The provision for longitudinal shearing stresses is one of the most difficult problems in such designs, especially if the structure is continuous. The intensities of these stresses may be computed from the formula $v_L = V/b'jd$, where b' is the total width of the vertical walls of the cell—two walls in this case—and j is the distance from the centroid of compression (approximately) to the lowest row of rods. These shears will be very large in most cases; hence construction joints are points of possible weakness. Also, thin walls are to be avoided on general principles.

It is necessary to pour the bottom slab first. Then, as shown in the picture, it is advisable to provide large fillets at its junction with the side walls in order to facilitate the transfer of shearing stresses into the slab and in order to make room for stirrups, corner reinforcement, and hooked rods from the slab.

Technically, this construction joint should be "saw-toothed" in order to provide the maximum shearing strength. However, this is difficult to do, and it would cause a ragged appearance. Therefore, the joint should be carefully and deeply scored crosswise except at the exposed edge. In this particular structure, reinforcing trusses were used across the joint.

4. The vertical shear must be provided for in the side walls. In such heavy, deep structures, the complications that result from bending up the longitudinal rods usually exceed the advantages of their use in resisting the shear. It is generally advisable to use vertical stirrups alone for this purpose, to use them throughout the structure with spacings not over 18 or 24 in. c.c. In this bridge, no allowance was made for the shearing strength of the concrete alone, but the stirrups were designed to withstand the full shear at a unit stress of 22,000 to 25,000 lb. per sq. in.

5. The reactions at the supports must be investigated. The compressive stresses under the side walls are likely to be excessive. One remedy is the provision of heavy end or cross walls at the bearing points, thickening of the side walls, and adequate shear reinforcement to make the cross walls effective.

6. A cellular structure should also have adequate diaphragms or cross ribs so as to avoid distortion and consequent cracking near the corners of the cells. These diaphragms may or may not

extend between the cells. In Fig. 10-34, the cross ribs are used only in the cells.

Problem 10-10. Find the resisting moment of the outer cell of the bridge of Fig. 10-33 if the details are as given in Fig. 10-35. Assume that the allowable f_c and f_s equal 1,000 and 18,000 lb. per sq. in., respectively, and that $n = 10$.

An examination of Fig. 10-35 shows the approximations that have been made before attempting to compute the I_c of this section. The fillets are

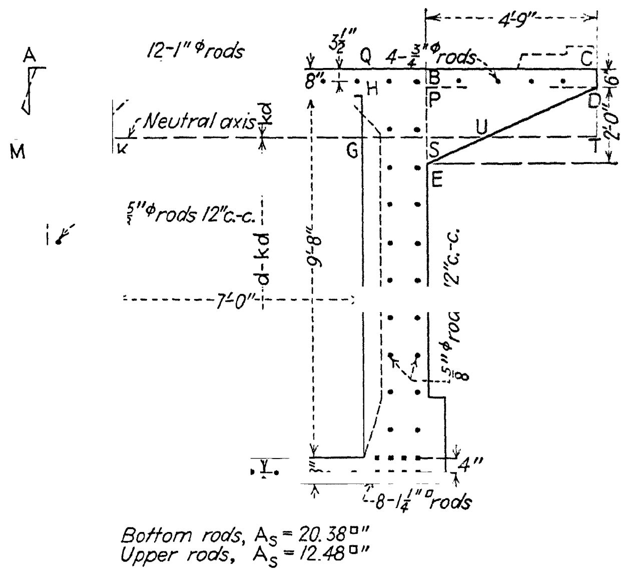


FIG. 10-35.

replaced by the use of rectangular portions at AM, JK, and HG. The ⅝-in. round ties in the webs are equivalent to a sheet of steel 0.1 in. thick (4 × 0.3 ÷ 12) extending from about 12 in. above the bottom of the section to the neutral axis, the top ties being neglected.

The detailed calculations are as follows:

$$\times\ 8(kd - 4) + [ARKM + QBSG]\frac{(27 + 21)}{2}(kd)^2 + [BCTS]$$

$$\frac{57}{2}(kd)^2 - [TDU]\frac{57}{24 \times 6}(kd - 6)^3 + 9 \times 11.24(kd - 3.5) = 10 \times$$

$$20.38(128.5 - kd) + 10 \times 12.48(124.5 - kd) + \frac{10 \times 0.1}{}(120 - kd)^2.$$

The bracketed letters indicate the portion that is being considered.

$kd = 22.5$ in., $\quad d - kd = 106$ in., $\quad k = 0.175, \quad I_c = 4{,}273{,}000$ in.^{4},
$S_c = 190{,}000$ in.^{3}, $\quad S_s = 4{,}030$ in.^{3}

1,000 15,800 ft.-kips.

$$M_s = \frac{18{,}000 \times 4{,}030}{12 \times 1{,}000} = 6{,}040 \text{ ft.-kips (controlling } M\text{)}.$$

In all probability, tension in the concrete of such a deep, massive structure will be partially effective so that the neutral axis will not be so high as it appears to be in these computations.

Practice Problems

Problem 10-11. Assume a large, simply supported slab similar to Fig. 10-1. Let $L' = 10$ ft., $B =$ infinity (for the purpose of this problem), and the thickness of the slab $= 10$ in. Apply a concentrated load $P = 15{,}000$ lb. at the center of the span. The contact surface has a diameter $=$

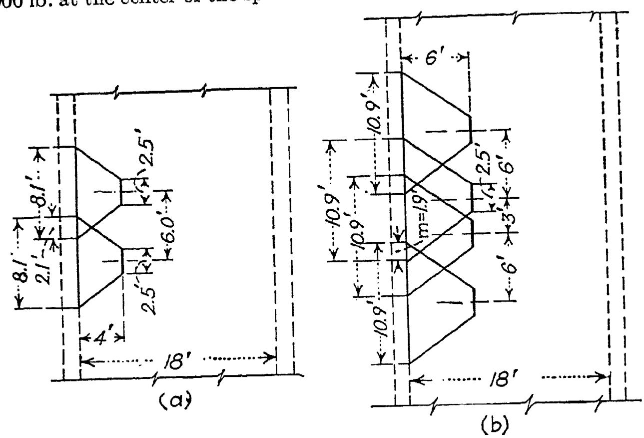


FIG. 10-36.

12 in. Find M_x and M_y per ft. of width of slab at the point of application of P, using Fig. 10-2.

Problem 10-12. Assume that a 20-ton truck like that of Fig. 10-7 runs on the slab of Problem 10-11. Place both rear wheels on a line along the middle of the span. If $c = 1.25$ ft., find M_x and M_y at one of the loads, using Figs. 10-2 and 10-5.

Problem 10-13. Assume a large, simply supported slab similar to that of Fig. 10-1. $L' = 18$ ft., B is infinite, and the thickness of the slab $= 12$ in. Find M_x and M_y for the following cases:

1. Dead load alone.
2. Both rear wheels of a 20-ton truck (Fig. 10-7) placed along the center of the span parallel to the supports, using Figs. 10-2 and 10-5.

3. Both rear wheels of a 20-ton truck placed on a line at right angles to the support. One wheel is 1.5 ft. to the left of the center line: the other is 4.5 ft. to the right. Use Figs. 10-2 and 10-4.

4. Recompute 3 if one wheel is on the center line and the other is 3 ft. from the support, using Figs. 10-2 and 10-3.

5. One rear wheel at the center of the span and the axle at an angle of 30° with respect to the line down the center of the span, using Figs. 10-2 and 10-6.

Problem 10-14. Assume that the slab of Problem 10-13 is fixed at the supports for dead loads and 50 per cent fixed for live loads. Find M_x and M_y if both rear wheels of a 20-ton truck are placed along the center of span, using Figs. 10-2 and 10-5.

Problem 10-15. Using the methods of Art. 10-3, find the approximate maximum negative bending moment at the support of a slab like that of Fig. 10-8(*b*) for the loading conditions given below. Assume a series of spans 18 ft. long (clear span) with infinite length parallel to the beams. Call the thickness 13 in. Assume that the slab is 75 per cent fixed at both supports. Use the data for 20-ton trucks.

1. One truck with its rear axle parallel to the beams and 4 ft. from the support. Notice that $b_e = 2.5 + 1.4 \times 4 = 8.1$ ft. but that the effects of the two wheels overlap $8.1 - 6.0 = 2.1$ ft., which is about 20 per cent, as shown in Fig. 10-36(*a*), so that the load for a 1-ft. strip of slab may be called $= 2P/8.1$.

2. Two trucks with their rear axles on a line parallel to the beams and 6 ft. from the support. The trucks are 9 ft. c.c (3 ft. between adjacent wheels). For this case, $b_e = 2.5 + 1.4 \times 6 = 10.9$ ft. The wheels overlap as shown in Fig. 10-36(*b*). The strip m in the picture is affected by the arbitrarily assumed distribution of three wheels, but the overlap of those that are 9 ft. apart is less than 20 per cent of b_e so that the effective load will be called $\dfrac{2P + 0.5P}{10.9}$.

3. One truck with its rear axle perpendicular to the beams. The near wheel is 5 ft. from the support; the other wheel is 11 ft. from the same beam. In this case, b_e for the near wheel is 9.5 ft.; for the far one it is 17.9 ft. In terms of a 1-ft. strip of slab, the near load will be $P/9.5$; the other, $P/17.9$.

Problem 10-16. Using the methods of Art. 10-5, find the shearing stresses in the slab of Problem 10-15 for the following conditions of loading, assuming $j = 0.88$:

1. Part 1 of Problem 10-15.

2. Part 2 of Problem 10-15.

3. One truck with its rear axle perpendicular to the beams. The near wheel is 1 ft. from the beam, and the other is 7 ft. away.

Problem 10-17. Find M_x and M_y at the center of a slab that is simply supported along all four sides as in Fig. 10-26(*a*). Let $L_1 = 15$ ft., $L_2 = 20$ ft., and the slab thickness $= 12$ in. The rear axle of a 20-ton truck is at the center line of span L_1; one wheel is at O; the other is 6 ft. to the right of O.

CHAPTER 11

ANALYSIS OF RIGID FRAMES BY METHOD OF WORK

11-1. Introduction. The term "rigid frame" is generally used to denote a structure, or part of a structure, composed of members that are "rigidly" connected at their joints, the continuity of the members producing a structure that is statically indeterminate; i.e., the effect of a load that is placed upon one member

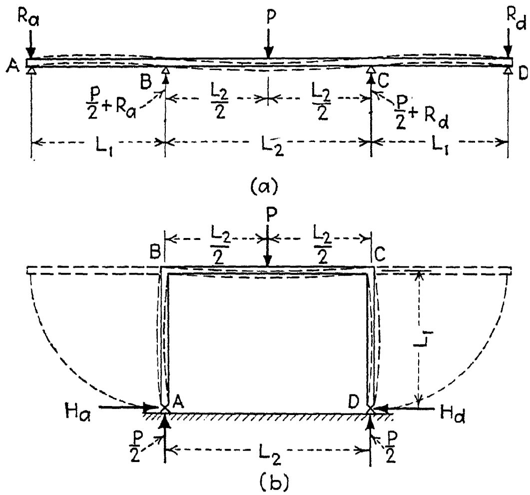


FIG. 11-1.

carries over to other members, and the stresses in the individual parts cannot be found by the use of the three equations of statics $\Sigma V = 0$, $\Sigma H = 0$, and $\Sigma M = 0$ alone. Furthermore, a rigid frame generally depends upon the flexural strength of its members and the stiffness of its joints to resist the external forces that it carries.

In reality, the three-span continuous beam of Fig. 11-1(a) is a rigid frame. A load P on span L_2 will cause the beam to curve, somewhat as shown by the dotted lines. This action sets up

internal resisting moments at B and C. The stiffness and the strength of the beam itself will cause it to "kick upward" at the ends. To hold these ends in place, there must be downward reactions at A and D.

On the other hand, if the ends of this beam are turned down so as to make an inverted U with hinges at A and D, as shown in Fig. 11-1(b), the load P will cause, in general, the same flexural distortions as before, and therefore it will set up the same resisting moments at B and C. Again, the continuity of the structure

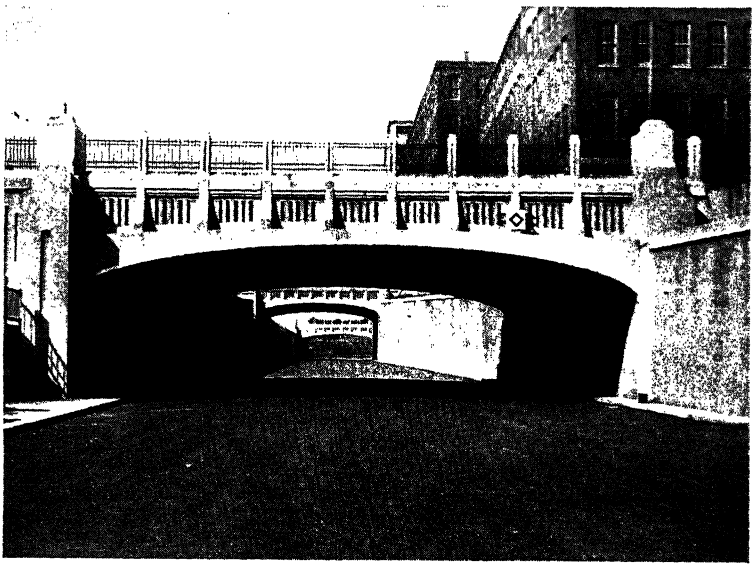


FIG. 11-1A.—Rigid-frame bridge over Raymond Boulevard, Newark, N. J.

will carry shearing forces to A and D; this time these forces will be horizontal thrusts which must be resisted by equal but opposite horizontal reactions at A and D. At the same time, there will be vertical reactions of $P/2$ at A and D along with the horizontal ones. The structure of Fig. 11-1(b) may be called a "rigid frame with hinged supports." The angular shape at B and C has not destroyed the continuity of the frame.

It is important for an engineer to understand the fundamental principles of the analysis and the design of restrained or continuous frames, because it is very difficult to design practical, reinforced-concrete structures that have perfectly hinged joints.

If he fails to consider the actual or probable restraints at the connections, he must not be disappointed when he finds unexpected cracks appearing in his structures after they are built.

Rigid-frame bridges will be dealt with in the illustrative problems in this text because they are rightfully coming into a position of importance and because they will serve to bring out most of the necessary principles that need consideration. Figure 11-1*A* is a photograph of such a bridge at Newark, N. J. The economy of such construction comes about through the restraining moments at the ends of the horizontal member—decreasing the bending moments at the center—also through the substitution of vertical walls with top and bottom supports instead of abutments of the retaining-wall type.

Inasmuch as the sections of such bridges are generally tapered or curved, the moments of inertia of the members are not constant. This makes it advisable to use herein a very general and widely adaptable method of analysis, viz., the Maxwell-Mohr method of work. This method of analysis will be briefly explained. After the individual has learned this fundamental method thoroughly, he can then more easily use such excellent methods as the "distribution of fixed-end moments" which has been developed by Prof. Hardy Cross,[1] a method that is illustrated in Chap. 13.

11-2. The Method of Work in General. Briefly, the analysis of an indeterminate structure upon the basis of work is founded upon the equality of the work done by the external forces and the energy that is stored up by the internal forces. For instance, if the force P is applied to the fixed beam of Fig. 11-2(a), it will cause the beam to deflect a small distance δ_b; this force acting through a distance produces work. At the same time, restraining moments M_C and M_D are set up as shown in Fig. 11-2(b). There are internal resisting moments in the beam with their accompanying stresses and deformations of the fibers. These stresses

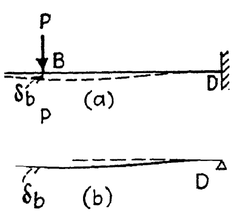


FIG. 11-2.

[1] Analysis of Continuous Frames by Distributing Fixed-end Moments, A.S.C.E. *Trans.*, Vol. 96, pp. 1–156.

acting through small distances also produce work. Therefore, to have equilibrium and to have the energy used equal to the energy stored, the work that is done by the external forces must be equal to that which is done in overcoming the internal forces. Furthermore, if the work produced or the deformation caused by a force of 1 lb. can be ascertained, then any force P must produce P times as much work or deformation as that which the 1-lb. force produced.

In the articles that are to follow, the deformations due to shearing forces and direct loads will be omitted because they are unimportant in most cases.

The following symbols and their meanings must be noted carefully:

P = any applied external load.
Q or q = a dummy or unit load of 1 lb.
m = a dummy or unit moment of 1 in.-lb. or 1 ft.-lb.
M_p = the moment at any point of the member due to P.
m_q = the moment at any point of the member due to Q.
m_m = the moment at any point of the member due to m.
δ_a, δ_b, etc. = the linear deflections of the member at the corresponding points (A, B, etc.) due to the external load P, for the actual structure.
δ'_a, δ'_b, etc. = the linear deflections of the member at the corresponding points (A, B, etc.) due to the external load P for the case of the statically determinate system.
δ_{a-1}, δ_{a-2}, etc. = linear deflections of the determinate structure, the first subscript denoting the point at which the deflection is sought, the second denoting the particular unit load that is being considered.
θ_a, θ_b, etc. = the angular rotations at a given point due to P for the actual structure.
θ'_a, θ'_b etc. = the angular rotations of a given point due to P for the determinate structure.
θ_{a-1}, θ_{a-2}, etc. = the angular rotations of the determinate structure where the first subscript denotes the point at which the rotation is desired and the second indicates the particular unit couple that is being considered.
f_p = the intensity of fiber stress at any point due to P.
f_q = the intensity of fiber stress at any point due to Q.
f_m = the intensity of fiber stress at any point due to m.

Of course, the analysis of an indeterminate structure involves the strength of its members. In other words, tentative sizes must be chosen for these members before the structure can be analyzed.

11-3. Linear Deflection of a Beam. The formulas by which the deflection of a beam can be found will be developed by direct application to a general problem. For instance, assume that the deflection of the beam of Fig. 11-3(*a*) is to be found at *B* when the load P is applied.

Whenever the deflection of a beam is to be found at a designated point, the first step is to apply a unit (or dummy) load at that point and acting in the direction of this desired deflection. Therefore, place the unit load *Q* at *B*, causing deflections at *B* and *C* as shown in Fig. 11-3(*b*). The magnitudes of these deflections need not be considered. Then gradually apply the real load *P* which will produce additional deflections of δ_b at *B* and

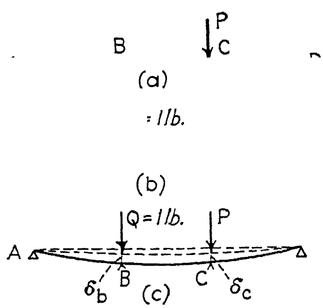


FIG. 11-3.

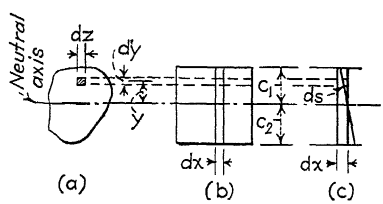


FIG. 11-4.

δ_c at *C*. Since the force *Q* was already on the beam before *P* was applied, the full magnitude of *Q* acted through the distance δ_b. Therefore, the work done by *Q* (1 lb.) alone during the application of *P* is

$$\text{Work} = U = Q\ \delta_b = \delta_b. \tag{11-1}$$

The work that has been done by *P* will not be considered at present.

Now investigate the internal work in the beam. When *Q* was applied in Fig. 11-3(*b*), it caused a resisting moment m_q in the beam. If Fig. 11-4 represents this member, then the intensity of stress at any point, due to *Q*, is

$$f_q = \frac{m_q\, y}{I}$$

and the total stress on any particle is

$$f_q\, dz\, dy = f_q\, dA = m_q\ {}'\, dA$$

This is the condition before P is applied, but P itself causes more stresses and more deformations of the fibers. The strains may be represented as in Fig. 11-4(c), where ds is the change of length of the fiber due to P in the distance dx, but this deformation is proportional to M_p. Therefore, the rate of strain is

$$\frac{f_p}{E} = \frac{\quad}{EI}$$

and the total strain is

Since Q is acting fully throughout the application of P, the internal work done or the energy stored in the fiber in question by Q alone must be equal to the stress in the fiber caused by Q times the distance through which it acts. Therefore, it is

$$dU = \qquad \frac{y\,dA)}{I}\,\frac{(M_p\,y\,dx)}{EI} = \frac{M_p m_q\,dx(y^2\,dA)}{EI^2}.$$

For the strip of beam, dx long, the summation of $y^2\,dA$ over the entire cross section $= \int_{-c_2}^{+c_1} y^2\,dA = I$, so that the work done upon this strip is

$$dU = \frac{M_p m_q\,dx}{EI}.$$

If this work is summed up for the entire length of the beam of Fig. 11-3,

$$U = \int_D^A \frac{M_p m_q\,dx}{EI}. \tag{11-2}$$

Because the external work shown in Eq. (11-1) equals the internal work done upon or the energy stored in the fibers of the beam as given in Eq. (11-2),

$$= \int_D^A \frac{M_p m_q\,dx}{EI}.$$

For any system of loads this may be restated as follows:

The deflection at any point of a beam for any system of external loads equals the integral for the entire length of the member of the bending moment caused by the external loads

times the bending moment due to a 1-lb. load applied at the point and in the direction of the desired deflection times dx/EI. If the computed deflection is positive, the assumed direction of the unit load is that of the real deflection; if the sign is negative, the assumed direction of the unit load is opposite to that of the real deflection.

11-4. Volume Method of Integration for Members with Constant *I*. The general formula for linear deflection as given in the preceding equation is

$$\frac{M_p m_q \, dx}{EI}. \tag{11-3}$$

This means that part of the expression to be integrated is the product of two moments M_p and m_q. M_p = the bending moment due to the external loads; m_q = the bending moment due to an imaginary unit load at the point where the deflection is desired and in the direction of that displacement. It is essential to find an easy way to perform the integration of these products with dx.

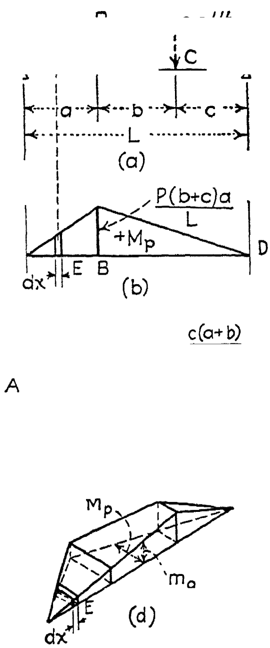


FIG. 11-5.

A very convenient method of doing this will be explained by using the beam of Fig. 11-5(a) as an illustration and finding the deflection at C due to the load P.

Draw the bending-moment diagram for the load P as shown in Fig. 11-5(b); then draw the corresponding diagram for q = 1 lb., as pictured in Fig. 11-5(c). At any point E, the ordinates M_p and m_q can be determined. If dx is represented in Figs. 11-5(b) and (c) as a differential length along the axis of the beam, it will set off a differential strip of each bending-moment diagram. Next, draw an imaginary solid as pictured in the isometric view in Fig. 11-5(d), having the M_p diagram of (b) as its base and having its altitudes at any point equal to the corresponding ordinates of (c). A differential slice

of this solid is represented in the figure. It has the dimensions M_p, m_q, and dx. Therefore,

$$dV = M_p m_q \, dx$$

where V = the volume. It is now obvious that

$$V = \int dV = \int M_p m_q \, dx. \tag{11-4}$$

In other words, assuming E and I to be constant, δ of Eq. (11-3) may be expressed as $1/EI$ times the volume of the solid having a base equal to one of the bending-moment diagrams and having altitudes equal to the corresponding ordinates of the other bending-moment diagram.

The numerical value for the volume of the solid may be found by using the prismoidal formula

$$V = \frac{L}{6} \qquad + \qquad + 4m_1m_2) \tag{11-5}$$

where the terms are as shown in Fig. 11-6. Of course, the total volume must be broken up into parts in accordance with the

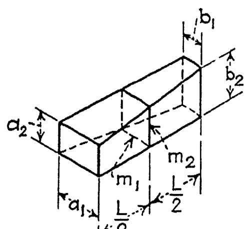


FIG. 11-6.

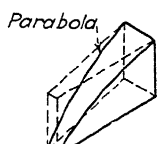


FIG. 11-7.

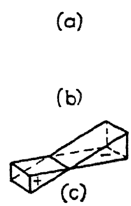


FIG. 11-8.

requirement that each part must be bounded by straight lines. In the case of a parabolic curve bounding one of the diagrams, the solid will be shaped as shown in Fig. 11-7. V for some of these solids is given in Table 4 of the Appendix. Other curves may be approximated by breaking them into short chords. Naturally, the total volume is the sum of all of its parts.

If the bending-moment diagrams are such that parts of one or both have opposite signs, the parts of the volume must have the same signs as the algebraic product of the proper ordinates. This is pictured in Fig. 11-8 where the base of (c) is the rectangle (b) and the altitudes are those of the triangles in (a). The sign

convention to be applied is immaterial as long as it is consistent and correctly used. Generally, a plus sign will be used herein to denote bending moments that cause compressive stresses in the outside or top fibers of the members, and minus will denote those moments which cause tensile stresses in these same fibers.

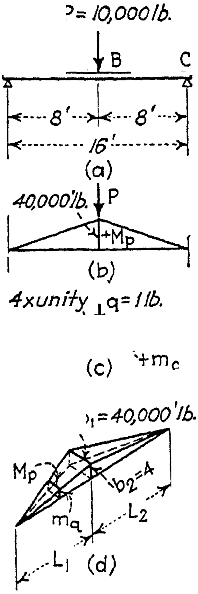


FIG. 11-9.

The application of this method will become familiar as it is used in subsequent problems. Its chief advantages are the ease of visualizing the integration and the ease of avoiding errors in signs. Table 4 in the Appendix shows the formulas for the volumes of solids which are frequently encountered when one is using this method of integration.

Problem 11-1. Find the deflection at the center of the simply supported beam of Fig. 11-9(a) due to the loading shown. Neglect the dead load of the beam. Assume $I = 4{,}000$ in.^{4}, and $E = 3{,}000{,}000$ lb. per sq. in.

Draw the bending-moment diagram for P in Fig. 11-9(b); apply $q = 1$ lb. at B, and draw the diagram of Sketch (c). The solid resulting from combining (b) and (c) is pictured in (d).

From Eqs. (11-3) and (11-4),

$$\frac{M_p m_q \, dx}{EI}, \qquad \text{or} \qquad EI\delta$$

$$EI\delta = \text{volume of left part} + \text{volume of right part}$$

$$EI\delta \quad 2L_1, \qquad + b_1 b_2 + \cdot$$

since both parts are equal in volume.

$$3{,}000{,}000 \times 4{,}000\delta = \frac{2 \times 8}{6}[0 + 40{,}000 \times 4 + 4 \times 20{,}000 \times 2]12^3.$$

$$\delta = 0.12 \text{ in.}$$

It is very important to be sure that the equation for δ is in the right units As a check on this, it may be written as follows:

$$EI\delta = \frac{L}{6} \qquad + b_1b_2 + 4m_1m_2)$$

$$\frac{\text{lb.}}{\text{in.}^2}\,\text{in.}^4\delta = \text{ft.(ft.-lb.)(ft.} \times \text{unity)}\,12^3$$

$$\delta = \frac{\text{in.}^3\text{ lb. in.}^2}{\text{lb. in.}^4} = \text{in.}$$

Note that m_q is expressed as foot-pounds per 1-lb. load.

11-5. Angular Deflection of a Beam. In order to develop the formulas by which the angular deflection of a beam can be found, assume that the angular deflection at B in the beam of Fig. 11-10(a) due to the load P is desired.

The angular deflection of a beam at a given point due to an external force means the rotation of the tangent to the neutral surface of the beam from its original position to its final location after the external force has been applied. The first step is therefore to apply a unit (or dummy) couple at the point where the angular deflection is desired. This will cause deformation of the beam as pictured in Fig. 11-10(b), but the magnitude of this deformation is of no importance. At the same time, this unit couple causes a downward reaction at A, an upward reaction at D, and an internal resisting moment m_m in the beam which produces fiber stresses f_m in the material. When P is applied, the additional strains are proportional to M_p. Then, by reasoning similar to that used to determine linear deflection, the following are found:

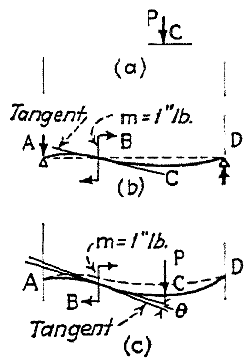


FIG. 11-10.

$$f_m = \frac{m_m y}{I}.$$

The stress in a fiber due to $m_m = f_m\,dA = \dfrac{m_m\,y\,dA}{I}.$

The total strain in a fiber due to $M_p = ds = y \times d\theta = \dfrac{M_p y\,dx}{EI}$

$$\frac{{}_p\,dx}{EI}$$

The work done upon a fiber by $m_m = (f_m\, dA)y\, d\theta =$

$$\frac{(m_m\, y\, dA)}{I}\,\frac{(M_p\, y\, dx)}{} =$$

The work done by m_m upon a strip dx(dx being constant)

$$dU = m_m \times d\theta = \frac{M_p m_m\, dx}{EI}$$

Therefore,

$$U = \int m_m \times d\theta.$$

The total work for the unit couple acting at full magnitude through the angle θ must equal the total internal work or energy stored in the whole beam because of the action of this couple. Therefore,

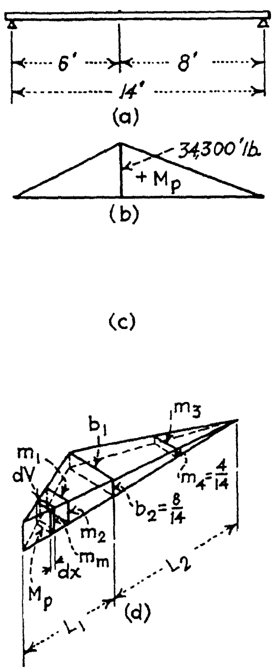


FIG. 11-11.

$$\frac{M_p m_m\, dx}{EI}. \qquad (11\text{-}6)$$

For any system of loads, this may be expressed as follows:

The angular deflection at any point of a beam for any system of external loads equals the integral for the entire length of the member of the bending moment caused by the external loads times the bending moment due to a unit couple of 1 in.-lb. applied at the point where the angular deflection is desired, all multiplied by dx/EI. If the sign of θ is positive, the rotation is in the same direction as that of the assumed unit moment; if negative, the rotation is the opposite.

Equations (11-3) and (11-6) appear to be similar, but the former gives a value that is measured in inches; the latter gives answers in radians. The terms m_q and m_m are different quantities and must not be interchanged. Sometimes these equations are written

$$= \int \frac{Mm\, dx}{} \qquad \text{and} \qquad = \int \frac{Mm\, dx}{}$$

where M = the moment caused by the external forces, and m = the moment due to the unit load or to the unit couple. However, it is easier to avoid errors if the subscripts p, q, and m are used to clarify the different terms.

Problem 11-2. Find the angular deflection at A of the beam of Fig. 11-11(a), neglecting the dead load of the beam. Assume I = 3,000 in.^{4}, and E = 3,000,000 lb. per sq. in.

Draw the bending-moment diagram for P, as shown in Fig. 11-11(b); apply a unit moment at A, acting clockwise, and draw the resulting bending-moment diagram as pictured in (c); and construct the solid of (d) by using (b) as a base and (c) as the curve for altitudes.

Using Eq. (11-6),

$$\int \frac{M_p m_m \, dx}{EI} \tag{11-6}$$

$$dV = M_p m_m \, dx$$

$$EI\theta = \int dV = \frac{L_1}{6}(0 + \text{i} \qquad + 4m_1m_2) + \frac{L_2}{6}(b_1b_2 + 0 + 4m_3m_4)$$

$$3{,}000{,}000 \times 3{,}000 \times \theta = \frac{6}{6}\left(0 + 34{,}300 \times \frac{8}{14} + 4 \times 17{,}150 \times \frac{11}{14}\right)12^2 + \frac{8}{6}\left(34{,}300 \times \frac{8}{14} + 0 + 4 \times 17{,}150 \times \frac{4}{14}\right)12^2.$$

Therefore,

$$\theta = 0.002 \text{ radian}$$
$$\theta = 0.002 \times 57.3 = 0.115°.$$

11-6. Redundants in a Hinged-end Frame. Assume a symmetrical rigid frame on hinged supports as pictured in

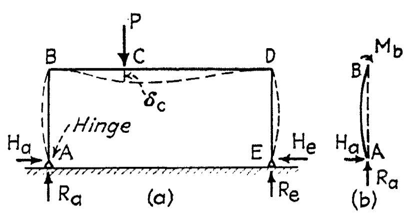


FIG. 11-12.

Fig. 11-12(a). When the load P is applied at C, the frame distorts somewhat as shown by the dotted lines, causing horizontal reactions H_a and H_e at the supports, along with the vertical reactions R_a and R_e. There are no resisting moments at A and E. I is constant.

It is important for one to visualize the distortions of rigid frames in order to enable him to understand just what to expect from their actions when they are loaded. A few basic facts should be noted as follows:

1. If the action is such as to close the joint as shown in Fig. 11-13(a), there will be tension in the outside fibers (+) and compression in the inside ones (−).

2. If the action opens the joint as indicated in Fig. 11-13(b), the tensile stresses are in the inside fibers, and the compressive ones are in the outside fibers.

3. When a member like AB of Fig. 11-13(c) with restrained ends is loaded as shown, there will be two points of contraflexure in the member, with stresses as indicated by the signs in the figure.

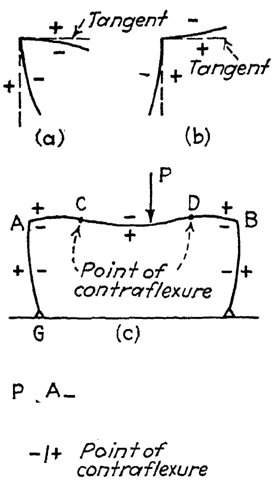


FIG. 11-13.

4. When a member like AB of Fig. 11-13(d), with restrained ends, has its ends rotated as pictured, there will be one point of contraflexure, with stresses as shown by the signs in the figure.

From the three equations for the equilibrium of statically determinate structures, when applied to Fig. 11-12(a),

$$\Sigma V = P + R_a + R_e = 0,$$
$$\Sigma H = H_a + H_e = 0,$$
$$\Sigma M = 0; \text{ or,}$$

taking moments at B as shown in Fig. 11-12(b) for a specific case,

$$M_b + R_a \times 0 - H_a(AB) = 0.$$

There are four reactions, but only three equations. Therefore, the frame is indeterminate in the "first degree." The extra force (or reaction, in this case) is called a "redundant," or "redundant force."

When a rigid frame is to be analyzed, the first step is to determine what redundants are acting upon it. The frame may then be cut (in imagination) so as to make it statically determinate, with the redundants applied as external forces.

Take the frame of Fig. 11-12(a) as an example, with H_a as the redundant: $H_e = -H_a$. Cut the frame at A, and assume it to be on a frictionless roller as pictured in Fig. 11-14(a). When the

load P is applied, BD will deflect, the legs (vertical members) will tilt, and the roller will move to A', a deflection of δ_a'. However, the structure is now determinate, and $H_e = 0$. Next, apply the redundant H_a, which must be sufficient to force the point A' back to A again, causing a deflection of $-\delta_a'$ and an equalizing reaction of H_e as pictured in Fig. 11-14(b).

The formula for the redundant H_a will now be developed. Apply the customary unit load at A, acting horizontally. In this case, let it act toward the right. From Eq. (11-3), remember-

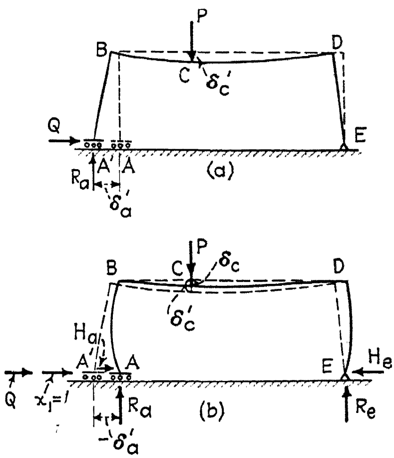


FIG. 11-14.

ing that δ_a' is the deflection at A for the determinate structure with a load P at C,

$$= \int \frac{M_p m_q \, dx}{EI}.$$

Instead of H_a, apply a unit load of 1 lb. at A'. Call it x_1 as shown in Fig. 11-14(b), and treat it as an external load. Apply a second unit (dummy) load Q at A'. The force x_1 will cause a deflection of

$$_{1-1} = \int \quad dx$$

in the determinate system, δ_{1-1} meaning the deflection produced by x_1 in the direction of x_1.

Obviously, if 1 lb. at A causes a deflection of δ_{1-1}, the force H_a will cause a deflection of $H_a(\delta_{1-1})$. However, H_a must be sufficient to force the point A' back to A again, causing a deflection of $-\delta'_a$. Therefore,

$$\frac{\delta'_a}{} = -\frac{\displaystyle\int \frac{M_p m_q\,dx}{EI}}{\displaystyle\int \frac{m_1 m_q\,dx}{EI}} = -\frac{\displaystyle\int \frac{M_p m_1\,dx}{EI}}{\displaystyle\int \frac{m_1^2\,dx}{EI}}$$

Since Q and x_1 have the same magnitude and direction in the formulas for δ'_a and δ_{1-1}, it is possible to substitute m_1 in place of m_q, thereby securing the form of Eq. (11-7) as given in the last expression for it.

When the sign of the right-hand side of Eq. (11-7), after integration, is positive, it means that H_a acts in the direction of the assumed unit load x_1; if the sign is minus, it shows that H_a is opposite to the assumed direction of the unit load x_1.

Problem 11-3. Find the horizontal reaction at A of the hinged rigid frame of Fig. 11-15(a). Consider E and I to be constant. Neglect the weight of the frame. Then find the bending moments at B and C.

Cut the frame at A, adding an imaginary frictionless roller. The distortion of the frame will be as shown in Fig. 11-15(b); the bending-moment diagram for M_p will be as pictured in (c), the sign being positive because the tension is in the bottom fibers. Next, apply the unit load $x_1 = 1$ lb. at A, with its equal but opposite reaction at E. The frame will curve as sketched in (d); the bending-moment diagram for x_1(which is called m_1) is drawn in (e), the diagram being minus because the tensions are in the outside and top fibers.

From Eq. (11-7),

$$H_a = -\frac{\delta'_a}{\delta_{1-1}} = -\frac{\int M_p m_1\,dx}{\int m_1^2\,dx}$$

since EI will cancel out (being constant).

The integration called for in Eq. (11-8) will be performed by the volume method as follows, constructing the figures for the solids mentally only:

(1) $$\int M_p m_1\,dx.$$

The volume will be that constructed by combining Figs. 11-15(c) and (e):

For AB,
 Vol. $= 0$ because the ordinates of (c) $= 0$.
For DE,
 Vol. $= 0$ because the ordinates of (c) $= 0$.
For BD,

$$\text{Vol.} = -\frac{2 \times 16}{6}(0 + 200{,}000 \times 15 + 4 \times 100{,}000 \times 15) = -48{,}000{,}000.$$

Therefore,
 $\delta_a' = -48{,}000{,}000$ (neglecting EI).
The sign of δ_a' is negative because M_p is plus and m_1 is minus.

(2) $$n_1^2\,dx.$$

The volume will be that constructed by combining Fig. 11-15(e) with itself:

For AB,

$$V = \frac{15}{6}(0 + 15 \times 15 + 4 \times 7.5 \times 7.5) \qquad = +1{,}125$$

For DE,

$$V = \text{the same as for } AB \qquad = +1{,}125$$

For BD,

$$V = \frac{32}{6}(15 \times 15 + 15 \times 15 + 4 \times 15 \times 15) = \frac{32}{6}(6 \times 15^2) \qquad = +7{,}200$$

$$\Sigma = +9{,}450.$$

Therefore,

$$\delta_{1-1} = +9{,}450.$$

The sign of δ_{1-1} is plus because "minus times minus equals plus." Therefore,

$$H_a = \frac{-\delta_a'}{\delta_{1-1}} = \frac{+48{,}000{,}000}{9{,}450} = +5{,}080 \text{ lb.}$$

The bending moments at B and C are the resultants of the bending moments that are caused by the external forces and by the redundants. Combining the ordinates of the diagram of Fig. 11-15(c) with H_a times the corresponding ordinates of (e) at B and C gives

$$M_{BA} = +M_p - H_a m_1 = 0 - H_a \times 15 = -5{,}080 \times 15 = -76{,}200 \text{ ft.-lb.}$$
$$M_{CB} = +M_p - H_a m_1 = +200{,}000 - 5{,}080 \times 15 = +123{,}800 \text{ ft.-lb.}$$

The use of the two subscripts in the expression M_{BA} is a means of denoting the moment at B in the member BA. The bending-moment diagram for

the frame is shown in the left half of Fig. 11-15(f); the shears and direct loads in the members are given in the right half.

Problem 11-4. Solve Problem 11-3 again, changing the moment of inertia of the deck BD to $2I$ (double that of the legs).

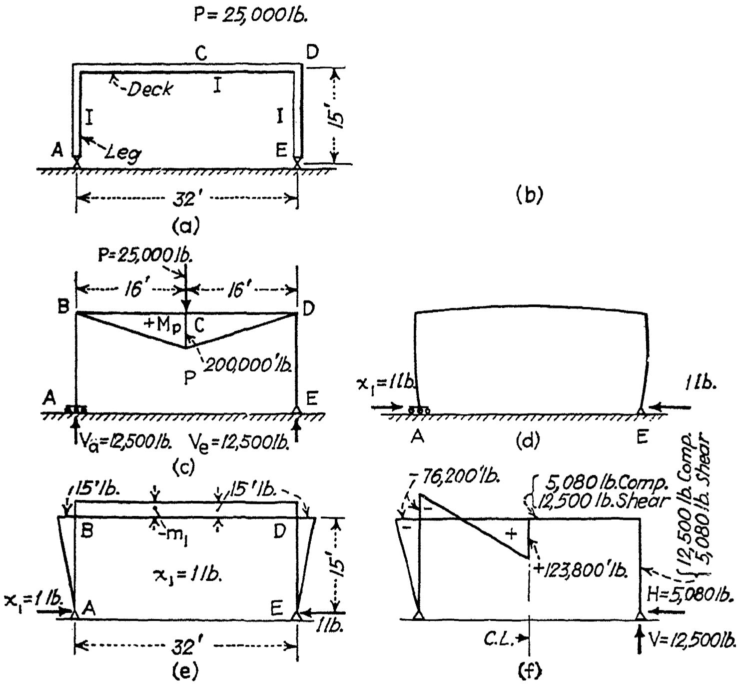


FIG. 11-15.

Construct the diagrams for M_p and m_1. They will be identical with Figs. 11-15(c) and (e), respectively. However, when it comes to the integration, E can be canceled, but I cannot be eliminated, so that Eq. (11-8) becomes

$$H_a = -\int \frac{M_p m_1 \,(}{I}$$

The moment of inertia of AB and DE is to be taken as I, whereas that of BD is to be $2I$. These values are to be used in the denominators of the expressions for the integration.

The integration is as follows, combining Figs. 11-15(c) and (e):

$$\int \frac{M_p m_1\, dx}{I}.$$

$AB = 0$
$DE = 0$

$$BD = \frac{-2 \times 16}{6 \times 2I}(0 + 200,000 \times 15 + 4 \times 100,000 \times 15) = -\frac{24,000,000}{I}.$$

Therefore,

$$\delta' = \quad 24,000,000$$

(2)

$$\int \frac{m_1^2\, dx}{I}.$$

$$AB = \frac{15}{6 \times I}(0 + 15 \times 15 + 4 \times 7.5 \times 7.5) \qquad 1,125$$

$$DE = AB \qquad 1,125$$

$$BD = \frac{32}{6 \times 2I}(15 \times 15 \times 6) \qquad +3,600$$

$$+5,850$$

Therefore,

$$5,850$$

$$H_a = \frac{\delta'}{\delta_{1-1}} = \frac{24,000,000 \times I}{5,850 \times I} =$$

$$M_{BA} = 0 - H_a \times 15 = -4,100 \times 15 = -61,500 \text{ ft.-lb.}$$
$$= +200,000 - H_a \times 15 = +138,500 \text{ ft.-lb.}$$

Compare the results of the preceding problem with those of this one. The beam or deck of the latter is stiffer than the former because I is larger. Therefore, in the latter case, BD carries a larger share of the total bending moment due to P; it (BD) deflects less; it rotates joint B less, in a clockwise direction; BA is curved less severely, and therefore it has a smaller bending moment in it. The values of M_{BA} and M_{CB} for both cases confirm these statements. It is advisable to study the actions of these two frames carefully in order to understand the behavior of structures.

11-7. Redundants in a Fixed-end Frame. Assume a symmetrical rigid frame which has the legs fixed as pictured in Fig. 11-16(a). When the load P is applied, the frame will distort somewhat as shown by the dotted lines. This will cause resisting moments at A and E along with the horizontal and

vertical reactions, as indicated in the picture. There are six unknowns, or three more than can be found by the three equations of statics. Therefore, the structure is indeterminate in the third degree.

It is necessary to find the three redundants. To do so, it is possible to make the structure determinate by cutting it in various ways, choosing any three of the six unknowns as the redundants. In order to illustrate the principles of this matter, make the structure determinate by cutting the leg entirely at A as shown in Fig. 11-16 (b). M_a = a redundant moment; H_a = a redundant horizontal reaction ($H_e = -H_a$); and V_a = a redundant vertical reaction. The fixed end at E then makes the frame a cantilever beam which is supported at E by the forces V_e, H_e, and the moment M_e, being therefore determinate. On the other hand, assume that it is desirable to cut the structure some other way. For instance, cut the leg at A, and add a roller as shown in Fig. 11-17(a). H_a is then a redundant reaction, and M_a is a redundant moment. Next, cut the leg at E so as to replace the fixed end with a hinge. M_e is then a redundant moment. The frame is again determinate because it is supported by the three forces V_a, V_e, and H_e: ($H_a = -H_e$).

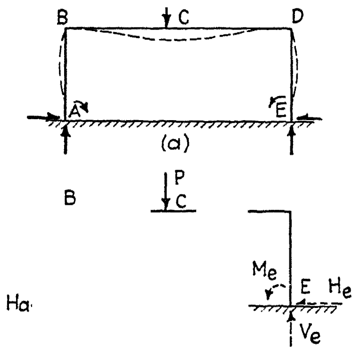


FIG. 11-16.

The formulas for the redundants might be found for the system of Fig. 11-16(b) or Fig. 11-17(a). The latter will be used for the demonstration because it is entirely general, and the distortions may be visualized more easily.

Apply a unit horizontal force x_1 at A and unit couples x_2 and x_3 at A and E, as pictured in Fig. 11-17(b). The unit force x_1 will cause a horizontal deflection at A of δ_{1-1}, an angular deflection of θ_{1-2} at A, and an angular deflection of θ_{1-3} at E, as may be seen by examining Fig. 11-17(c) which shows the unloaded determinate frame in dotted lines. Notice again that the first subscript means the number of the force or moment that produces

the deflection, and the second subscript indicates the direction of that deflection (the unit force to which it is parallel or the unit couple whose rotation it follows). The unit couple x_2 at A will cause an angular deflection of θ_{2-2}, where θ_{2-2} means the angular rotation at A due to the unit couple x_2 in the direction of x_2; it will cause another angular rotation of θ_{2-3} at E; it will also produce a linear deflection or movement at A equal to δ_{2-1} as

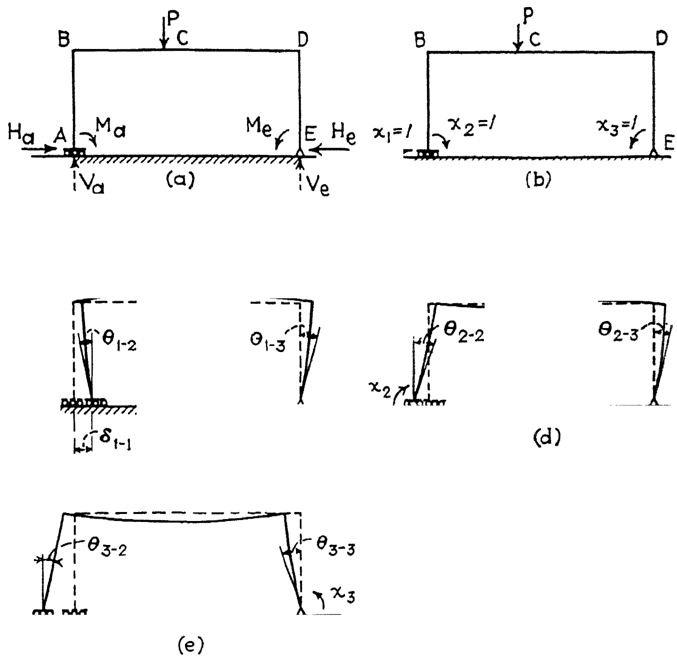


FIG. 11-17.

shown in Sketch (*d*). In the same manner, the unit couple x_3 will cause a linear deflection of δ_{3-1} at A and angular deflections of θ_{3-2} at A and θ_{3-3} at E [Sketch (*e*)].

However, the actual linear deflection at $A = 0$. The actual angular deflections at A and $E = 0$ because these points are fixed. Therefore, the algebraic sum of all of the deflections caused by the redundants at these points, and the deflections caused by the external loads acting upon the determinate structure, must equal 0.

$$
\begin{aligned}
H_a \times \delta_{1-1} + M_a \times \delta_{2-1} + M_e \times \delta_{3-1} + \delta_a' &= 0. \\
H_a \times \theta_{1-2} + M_a \times \theta_{2-2} + M_e \times \theta_{3-2} + \theta_a' &= 0. \qquad (11\text{-}9) \\
H_a \times \theta_{1-3} + M_a \times \theta_{2-3} + M_e \times \theta_{3-3} + \theta_e' &= 0.
\end{aligned}
$$

However these formulas are not readily usable as they appear here.

The terms δ_{1-1}, δ_{2-1}, θ_{1-2}, etc., in Eq. (11-9) are linear or angular deflections due to the redundants pictured in Fig. 11-17(*b*). These unit forces and couples x_1, x_2, and x_3 are to be treated as external loads which cause M_p in Eqs. (11-3) and (11-6). Then change M_p to m_1, m_2, or m_3, as the case may be. Also, remember that m_q and m_m in Eqs. (11-3) and (11-6) are the moments caused by other unit forces or other unit couples acting in the direction of the desired deflection. On this account, change m_q to m_1, and m_m to m_2 or m_3, as the case may be. It is therefore possible to rewrite Eq. (11-9) as follows:

$$H_a \int \frac{m_1 m_1\, dx}{EI} + M_a \int \frac{m_2 m_1\, dx}{EI} + M_e \int$$

$$H_a \int \frac{m_1 m_2\, dx}{EI} + M_a \int \frac{m_2 m_2\, dx}{EI} + M_e \int \frac{m_3 m_2\, dx}{EI} + \int \frac{M_p m_2\, dx}{EI} = 0.$$

$$dx + M_e \int \frac{m_3 m_3\, dx}{EI} + \int \frac{M_p m_3\, dx}{EI} = 0.$$

$$(11\text{-}10)$$

Obviously,

$$
\begin{aligned}
\delta_{1-2} &= \int \frac{m_1 m_2\, dx}{EI} = \int \\
\delta_{1-3} &= \int \frac{m_1 m_3\, dx}{EI} = \int \\
\delta_{2-3} &= \int \frac{m_2 m_3\, dx}{EI} = \int m_3 m_2\, dx
\end{aligned}
$$

When all of these integrals are determined, the three equations can be solved for H_a, M_a, and M_e. It must be remembered that dx represents a differential length along the axis of the frame, not always a horizontal distance.

In order to set up general equations for this or any other case, let the redundants $= x_1$, x_2, etc. Let $\delta =$ linear or angular deflection in general. Let the subscript p represent the external loads. Then

$$
\begin{aligned}
x_1\,\delta_{1-1} + x_2\,\delta_{2-1} + \cdots x_n\,\delta_{n-1} + \delta_{p-1} &= 0. \\
x_1\,\delta_{1-2} + x_2\,\delta_{2-2} + \cdots x_n\,\delta_{n-2} + \delta_{p-2} &= 0. \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots &\cdots \\
x_1\,\delta_{1-n} + x_2\,\delta_{2-n} + \cdots x_n\,\delta_{n-n} + \delta_{p-n} &= 0.
\end{aligned}
\tag{11-11}
$$

These are the Maxwell-Mohr work equations.

An examination of Eq. (11-10) will show that E may be canceled from all of the terms if it is constant for the material of the structure. This is usually the case.

As a check upon the arrangement of the coefficients in Eq. (11-11), note that the first subscript of all of the x_1 terms is 1; for x_2, 2; etc. The second subscript for x_1, x_2, etc., and also for the δ_p terms should be 1 in the first equation, 2 in the second equation, 3 in the third, etc. Furthermore, the "symmetry of the elastic equations" is also a guide in making sure that the coefficients are correct. For instance, using Eqs. (11-10), and writing only the subscripts, the form is

$$
\begin{aligned}
(1-1) + (2-1) + (3-1) &= -(p-1) \\
(1-2) + (2-2) + (3-2) &= -(p-2) \\
(1-3) + (2-3) + (3-3) &= -(p-3).
\end{aligned}
$$

Problem 11-5. Find the bending moments at A, B, and C for the fixed-end frame shown in Fig. 11-18(a). Neglect the dead load of the structure, and assume E to be constant.

This frame will be analyzed in two ways in order to illustrate the fact that the bending moments are not dependent upon the method of solving the problem—they are dependent upon the loads and the characteristics of the structure. The frame should be studied so as to find a way to solve the problem with the minimum labor—make as many as possible of the coefficients δ equal to zero or to each other.

1. *First Solution.* Cut the frame as shown in Fig. 11-18(b), adding a roller at A and a hinge at E. The moment diagrams for P, x_1, x_2, and x_3 are shown in Figs. 11-18(c) to (f), inclusive. The work equations are, from Eqs. (11-10) and (11-11),

$$
\begin{aligned}
x_1\,\delta_{1-1} \quad \cdot\, x_2\,\delta_{2-1} + \quad \delta_{3-1} + \delta_{p-1} &= 0. \\
x_1\,\delta_{1-2} \quad \cdot\, x_2\,\delta_{2-2} + \quad \delta_{3-2} + \delta_{p-2} &= 0. \\
\cdot\, x_2\,\delta_{2-3} + \quad \delta_{3-3} + \delta_{p-3} &= 0.
\end{aligned}
\qquad (11\text{-}12
$$

$$
\frac{2 \times 15}{6 \times 2I}\left(0 + 15 \times 15 + 4 \times \frac{15}{2} \times \frac{15}{2}\right) + \frac{40}{6I}(6 \times 15 \times 15
$$

$$
+ \quad 10{,}125
$$

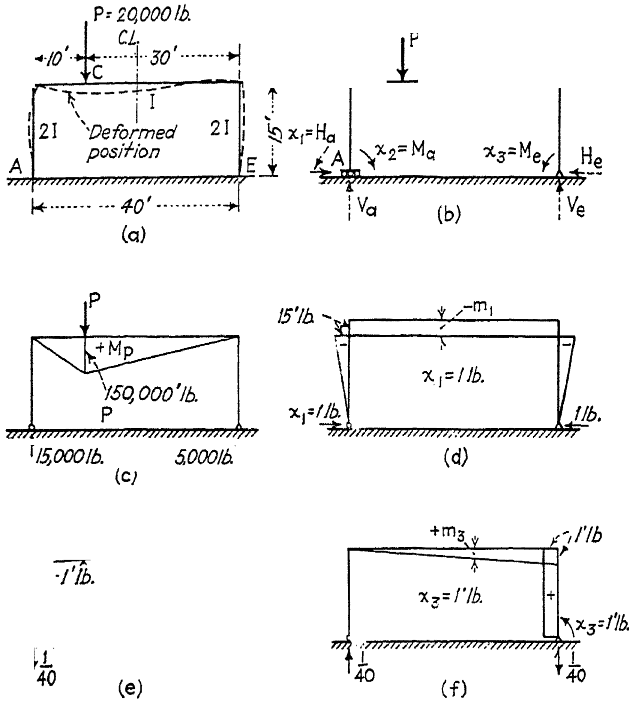


FIG. 11-18.

$$\delta_{1-2} = \delta_{2-1} = -\frac{15}{6 \times 2I}\left(0 + 1 \times 15 + 4 \times 1 \times \frac{15}{2}\right) - \frac{40}{6I}\left(1 \times 15 + 0 + 4 \times \frac{1}{2} \times 15\right) - 0 = -\frac{356}{I}$$

$$\delta_{1-3} = \delta_{3-1} = \delta_{1-2} = -\frac{356}{I}$$

$$\delta_{2-2} = +\frac{15}{6 \times 2I}(6 \times 1 \times 1) + \frac{40}{6I}(2 \times 1 \times 1) + 0 = +\frac{20.8}{I}$$

$$\delta_{2-3} = \delta_{3-2} = +0 + \frac{40}{6I}\left(0 + 0 + 4 \times \frac{1}{2} \times \frac{1}{2}\right) + 0 = +\frac{6.67}{I}$$

$$\delta_{3-3} = \delta_{2-2} = +\frac{20.8}{I}$$

$$\delta_{p-1} = -\frac{10}{6I}(0 + 150{,}000 \times 15 + 4 \times 75{,}000 \times 15) - \frac{30}{6I}(150{,}000 \times 15 + 0 + 4 \times 75{,}000 \times 15) = -\frac{45{,}000{,}000}{}$$

$$\delta_{p-2} = +\frac{10}{6I}(0 + 150{,}000 \times 0.75 + 4 \times 75{,}000 \times 0.875) + \frac{30}{6I}(150{,}000 \times 0.75 + 0 + 4 \times 75{,}000 \times 0.375) =$$

$$\delta_{p-3} = +\frac{10}{6I}(0 + 150{,}000 \times 0.25 + 4 \times 75{,}000 \times 0.125) + \frac{30}{6I}(150{,}000 \times 0.25 + 0 + 4 \times 75{,}000 \times 0.625) = +\frac{1{,}250{,}000}{I}.$$

Therefore,

$$\begin{aligned}
10{,}125x_1 - 356x_2 - 356x_3 - 45{,}000{,}000 &= 0.\\
-356x_1 + 20.8x_2 + 6.67x_3 + 1{,}750{,}000 &= 0.\\
-356x_1 + 6.67x_2 + 20.8x_3 + 1{,}250{,}000 &= 0.\\
H_a = x_1 &= +6{,}820 \text{ lb.}\\
M_a = x_2 &= +16{,}060 \text{ ft.-lb.}\\
M_e = x_3 &= +51{,}450 \text{ ft.-lb.}
\end{aligned}$$

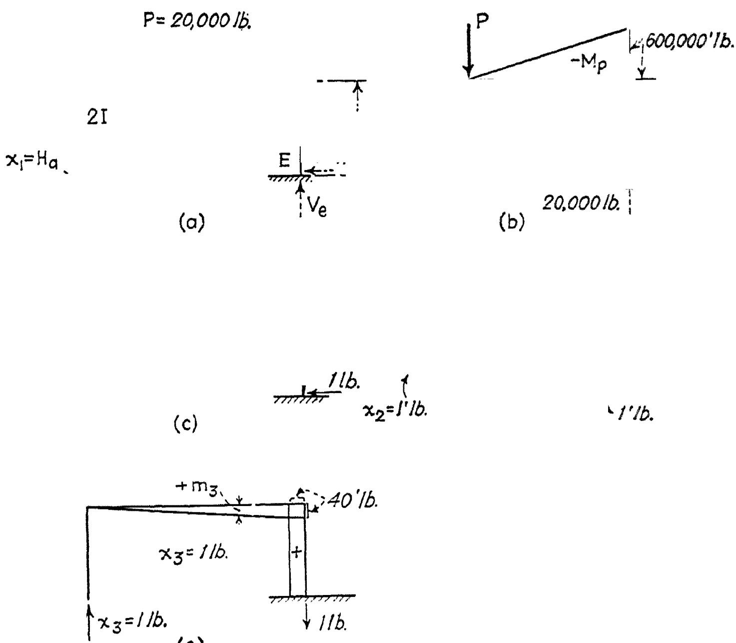


FIG. 11-19.

The bending moment at any point must equal the algebraic sum of all of the bending moments which are caused by the various forces that are involved, as shown by Fig. 11-18.

$$M_{AB} = +M_p - H_a m_1 + M_a m_2 + M_e m_3 = 0 - 0 + 16{,}060 + 0 = 16{,}060 \text{ ft.-lb.}$$

$$- H_a m_1 + M_a m_2 + M_e m_3 = 0 - 6{,}820 \times 15 + 16{,}060 + 0 = -86{,}240 \text{ ft.-lb.}$$

$$- H_a m_1 + M_a m_2 + M_e m_3 = +51{,}450 \text{ ft.-lb.}$$

$$M_{DE} = +M_p - H_a m_1 + M_a m_2 + M_e m_3 = -6{,}820 \times 15 + 51{,}450 = -50{,}850 \text{ ft.-lb.}$$

$$M_{CB} = +M_p - H_a m_1 + M_a m_2 + M_e m_3 = +150{,}000 - 6{,}820 \times 15 + 16{,}060 \times 0.75 + 51{,}450 \times 0.25 = +72{,}600 \text{ ft.-lb.}$$

2. *Second Solution.* Assume that the frame is cut entirely free at A [Fig. 11-19(a)]. The bending moments for P and for the redundants are shown in Figs. 11-19(b) to (e), inclusive. The general equations (11-12) remain as before. The integration of the coefficients follows:

$$\delta_{1-1} = +\frac{10{,}125}{I} \text{ (as for first solution).}$$

$$\delta_{1-2} = \delta_{2-1} = -\frac{2 \times 15}{6 \times 2I}\left(0 + 1 \times 15 + 4 \times 1 \times \frac{15}{2}\right) - \frac{40}{6I}(6 \times 1 \times 15) = -\frac{712.5}{I}$$

$$\delta_{1-3} = \delta_{3-1} = 0 - \frac{40}{6I}\left(0 + 15 \times 40 + 4 \times 15 \times \frac{40}{2}\right) - \frac{15}{6 \times 2I}\left(15 \times 40 + 0 + 4 \times \frac{15}{2} \times 40\right) = -\frac{14{,}250}{I}$$

$$\delta_{2-2} = +\frac{2 \times 15}{6 \times 2I}(6 \times 1) + \frac{40}{6I}(6 \times 1) = +\frac{55}{I}$$

$$\delta_{2-3} = \delta_{3-2} = +0 + \frac{40}{6I}\left(0 + 1 \times 40 + 4 \times 1 \times \frac{40}{2}\right) + \frac{15}{6 \times 2I}(6 \times 1 \times 40) = +\frac{1{,}100}{I}$$

$$\delta_{3-3} = +\frac{40}{6I}\left(0 + 40 \times 40 + 4 \times \frac{40}{2} \times \frac{40}{2}\right) + \frac{15}{6 \times 2I}(6 \times 40 \times 40) = +\frac{33{,}330}{-}$$

$$\delta_{p-1} = +\frac{30}{6I}(0 + 600{,}000 \times 15 + 4 \times 300{,}000 \times 15) + \frac{15}{6 \times 2I}\left(600{,}000 \times 15 + 0 + 4 \times 600{,}000 \times \frac{15}{2}\right) = +\frac{168{,}700{,}000}{I}$$

$$\delta_{p-2} = -\frac{30}{6I}(0 + 600{,}000 \times 1 + 4 \times 300{,}000 \times 1) - \frac{15}{6 \times 2I}(6 \times 600{,}000 \times 1) = -\frac{13{,}500{,}000}{}$$

$$-\frac{30}{6I}(0 + 600{,}000 \times 40 + 4 \times 300{,}000 \times 25) - \frac{450{,}000{,}000}{6 \times 2I}(6 \times 600{,}000 \times 40) = -$$

Therefore,

$$10{,}125x_1 - 712.5x_2 - 14{,}250x_3 + 168{,}700{,}000 = 0.$$
$$-712.5x_1 + 55x_2 + 1{,}100x_3 - 13{,}500{,}000 = 0.$$
$$-14{,}250x_1 + 1{,}100x_2 + 33{,}330x_3 - 450{,}000{,}000 = 0.$$

H_a $x_1 = +6{,}910$ lb.
M_a $x_2 = +17{,}280$ ft.-lb.
V_a $x_3 = +15{,}890$ lb.
M_{AB} $+17{,}280$ ft.-lb.
$M_{BA} = -15 \times 6{,}910 + 17{,}280 = -86{,}370$ ft.-lb.
$M_{ED} = +17{,}280 + 15{,}890 \times 40 - 600{,}000 = +52{,}880$ ft.-lb.
$M_{DE} = -15 \times 6{,}910 + 17{,}280 + 40 \times 15{,}890 - 600{,}000 = -50{,}770$ ft.-lb.
$M_{CB} = -15 \times 6{,}910 + 17{,}280 + 10 \times 15{,}890 = +72{,}530$ ft.-lb.

A comparison of the bending moments that have been found by these two solutions shows that they agree as closely as can be expected because the integration by means of the use of the prismoidal formula (method of volumes) has not been carried out with absolute exactness. The method of "cutting" the frame should not change the bending moments in the structure itself.

Now refer to Fig. 11-18(a), and notice that the deformed position of the frame is shown with points B and D displaced toward the right. The unsymmetrical loading causes this "side sway" because the frame moves until the shears in the legs balance. Notice that this action results in a smaller moment at A but a larger one at E; also note that the numerical sum of the bending moments in the legs at A and B equals that of the bending moments at E and D. Then the shear in the legs must equal the algebraic difference in the end moments divided by the length of the leg.

11-8. Members Having Varying Moment of Inertia. In practice, it is customary—as well as economical—to taper or to curve the members of a rigid frame. Figure 11-20 is a typical case. This gives a pleasing arch effect to the intrados; it provides the greatest clearance at the center where it is needed most; and it places the strongest parts where the bending moments are the largest.

In the Portland Cement Association's pamphlet entitled "Analysis of Rigid Frame Concrete Bridges" (3d ed.), it is stated, in effect, that the following thicknesses may be assumed in the preliminary proportioning of a rigid frame:

At A (Fig. 11-20), use 1 ft. 6 in. for 30-ft. spans, 2 ft. 6 in. for 60-ft. spans, and 3 ft. 4 in. for 90-ft. spans.

At B, for the leg and deck, use $L/15$, where L = the clear span.

At C, use $L/35$.

Trial sections of symmetrical and unsymmetrical rigid frames can be tentatively analyzed very rapidly by using the formulas[1] given in Figs. 1 and 2 of the Appendix. These data are for hinged-end frames. The approximate formulas enable one to

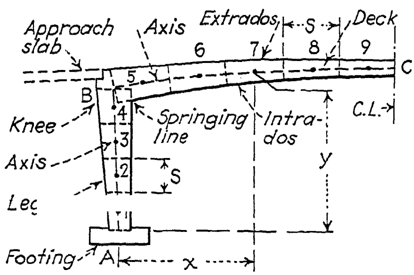


FIG. 11-20.

compute H; then the tentative bending moments in the frame can be found; and the sections can be tested before proceeding with a refined analysis.

In Fig. 11-20, the axis of the frame is assumed to be the locus of the centers of the cross sections of the members, as shown by AB and BC, when the structure has a uniform section at any given point. (Ribbed frames will not be illustrated herein.[2]) The axis of the leg is inclined, but it is sometimes assumed to be a

[1] Weiskopf and Pickworth, "Symmetrical Rigid Frames"; "Unsymmetrical Rigid Frames," American Institute of Steel Construction. For the mathematical development of these formulas, see A.S.C.E., *Trans.*, Vol. 102, p. 1, 1937.

[2] The principles of analysis are the same as for the solid-barrel type. A ribbed frame is useful in case the deck of the structure is to be used as the pavement or if a wearing surface is to be placed directly on the deck, when sharper curvature of the intrados and greater depth of spandrel face are desired than would be provided economically by a solid deck. However, such a structure is likely to have very large compressive stresses at the knee—the junction of the deck and the leg.

vertical line through the center of gravity of the leg or through point A. The height of the leg is the vertical distance from A to B; the half span is the horizontal distance from A to C or sometimes from B to C. These theoretical lengths are generally used instead of the clear height and the clear span.

The distances x and y for use in calculating the bending-moment diagrams will be measured as indicated in Fig. 11-20 for Sec. 7. For the experienced designer, there is efficiency in measuring y from the center of gravity of all of the ys/EI values for the entire frame so that $\sum \frac{ys}{EI} = 0$. (See Art. 12-8.)

In calculating the moment of inertia, a section of the frame 1 ft. wide is used. The leg is then divided into any desired number of sections as indicated in Fig. 11-20. The moments of inertia of these parts are calculated at their centers—1, 2, 3, etc. —measuring their thicknesses horizontally. Generally, I_c is computed for the full concrete section alone so that $I_c = bh^3/12 = h^3/12$ for a strip 1 ft. wide. The deck is divided into similar sections of equal horizontal length, but I is usually computed with h or t equal to the radial thickness at the center of the block. It is sufficient to assume that I is constant for any one division, but of course the accuracy is greater when the pieces are short.

The stiffness of the leg or the deck varies directly as the moment of inertia of the member and inversely as its length. The variations in I at different sections greatly affect the distribution of the moments. Therefore, since I appears in the denominator of all of the expressions for deflection [see Eq. (11-10)], the relative magnitudes of the bending moments at any points in the frame depend upon the *relative* values of I instead of their *absolute* magnitudes.

In this connection, Fig. 11-21 should be studied very carefully. BC represents the axis of an actual frame like that of Fig. 11-20, drawn as a horizontal line. Curves 1 to 4 inclusive are plotted on BC as a base and show, to scale, the moments of inertia of the corresponding sections of the frame. The depression in 3 is due to the variation of the area of the steel; its peak at F pictures the sudden jump in I where the small direct thrust in the frame produces a greater stress than that which is caused by the small bending moments near the point of contraflexure E of the bending-moment diagram HJ. The effect of this peak upon the total

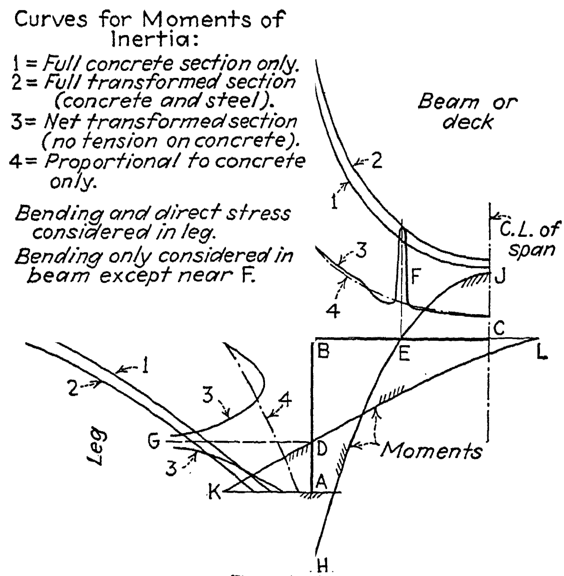


FIG. 11-21.

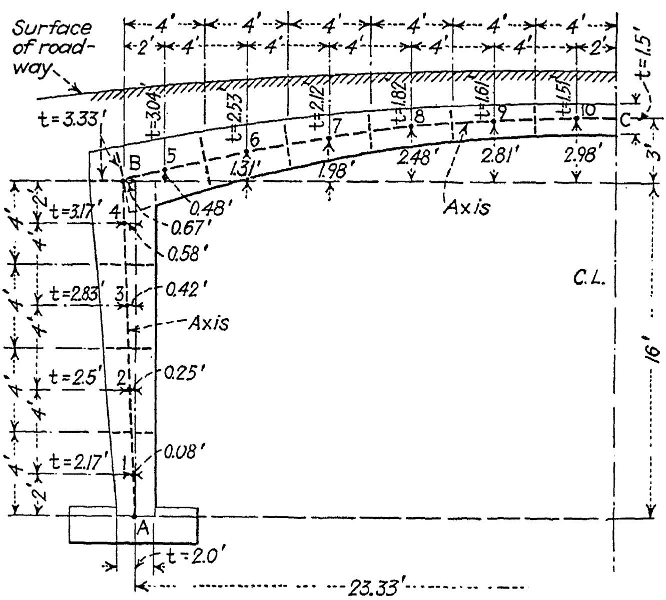


FIG. 11-22.

stiffness of the frame is negligible. However, tension in the concrete may extend its effect. The fact that curve 4 is proportional to the assumed I_c for concrete only and that it is closely proportional to I for the cracked concrete shows that the use of No. 1 is satisfactory.

The leg AB is a different matter. It has a much larger direct, longitudinal load. Curve 3 is plotted on AB as a base, and it shows, to scale, the moments of inertia of the transformed sections, computed about the neutral axis of each section. The direct load and the bending affect it. Near D, the point of

TABLE 11-1.—GENERAL DATA

Point	s	t or h	Weight, lb.	x	y	I, ft.[4]
1	4	2.17	1,300	− 0.08	2	0.85
2	4	2.5	1,500	− 0.25	6	1.30
3	4	2.83	1,700	− 0.42	10	1.89
4	4	3.17	1,900	− 0.58	14	2.65
5	4.1	3.04	1,870	+ 1.33	16.48	2.34
6	4.06	2.53	1,540	+ 5.33	17.31	1.35
7	4.03	2.12	1,280	+ 9.33	17.98	0.79
8	4.01	1.82	1,090	+13.33	18.48	0.50
9	4	1.61	970	+17.33	18.81	0.35
10	4	1.51	910	+21.33	18.98	0.29
Σ	....	....	14,060			

contraflexure of the bending-moment diagram KL, the peak is very pronounced so that No. 3 departs widely from the proportional curve, No. 4. In this particular case, part of the leg seems to be much stiffer than it was assumed to be. It will therefore drag to itself more of the bending moments than the calculations show. However, it is entirely impracticable to strive to allow for such results. The use of the full concrete section alone will be adopted.

The integration needed for the calculation of the coefficients of the work equations is found by summing up the integrals for each imaginary piece, as will be illustrated in the next problem.

Problem 11-6. Find the bending moments at B and C in the symmetrical, hinged-end, rigid frame of Fig. 11-22 due to the dead load of the frame itself.

The weights of the 10 sections into which the half frame is divided, the distances y of their centers above the hinge A, the distances x of their centers from A (plus toward the right), and the moments of inertia at these centers are given in Table 11-1.

This frame is the same general type as that of Art. 11-6. There are two vertical reactions and two horizontal ones. By cutting the frame at A and by applying the unit or dummy load, as shown in Fig. 11-23(a), the expression for the redundant horizontal reaction H_a will be

$$H_a \qquad \delta_a' = H_a \int \frac{m_1^2\,dx}{EI} + \int \frac{M_p m_1\,dx}{EI} \qquad 0$$

the same as Eq. (11-7). E is constant and cancels out. If s is used instead of dx, and if the integral signs are replaced by Σ ,then

$$H_a \tag{11-13}$$

The ordinates of the bending-moment diagram for the force $x_1 = 1$ are found by taking moments about the centers of the sections so that $m_1 = -1(y)$. This approximate diagram is pictured in Fig. 11-23(a).

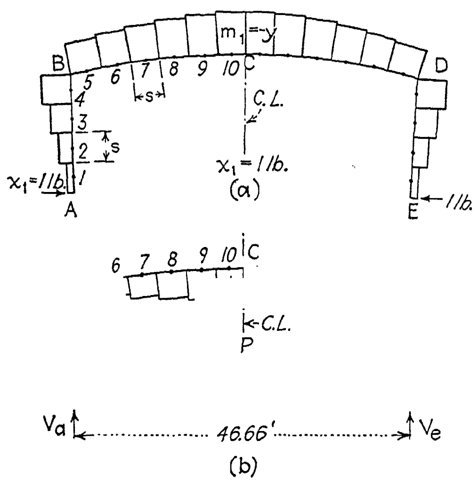


FIG. 11-23.

The ordinates of the bending-moment diagram for the external loads of the deck (the weights of the sections) are found by treating these weights as a series of concentrations upon a simply supported, horizontal beam of span AE. The net reaction (the weight of sections 5 to 10 inclusive) is assumed to be applied vertically over A because, otherwise, there would be

a negative bending moment of $V_a \times 0.67$ at B due to the eccentricity from A to B. The weights of sections 1, 2, 3, and 4 are neglected; the bending moments in these divisions are insignificant and will be omitted. The diagram is then somewhat as shown in Fig. 11-23(b).

The values in Table 11-2 are readily found, using the data which are given in Table 11-1. The summations are for half of the frame only, because it is symmetrical. If it were not so, the summations should be obtained for the entire structure.

TABLE 11-2.—CALCULATIONS FOR H_a

Point	$m_1 = -y$	y^2	M_p, ft.-lb.	s	$s \div I$	$y^2s \div I$	$M_pys \div I$
1	− 2	4	......	4	4.71	19	
2	− 6	36	......	4	3.08	111	
3	−10	100	......	4	2.12	212	
4	−14	196	......	4	1.51	296	
5	−16.48	272	10,200	4.1	1.75	476	− 294,000
6	−17.31	300	33,300	4.06	3.01	903	− 1,735,000
7	−17.98	323	50,300	4.03	5.10	1,647	− 4,612,000
8	−18.48	342	62,200	4.01	8.02	2,743	− 9,219,000
9	−18.81	354	69,700	4	11.43	4,046	−14,985,000
10	−18.98	360	73,400	4	13.79	4,964	−19,211,000
Σ	....	...	......	....	.....	15,417	−50,056,000

When the values of the summations from Table 11-2 are inserted in Eq. (11-13),

$$H_a = -\frac{(-50,056,000)}{15,417} \quad +3,250 \text{ lb.}$$

Then, referring to Figs. 11-22 and 11-23, and applying H_a at A,

$$M_{BC} = -H_a \times AB = -3,250 \times 16 = -52,000 \text{ ft.-lb.}$$
$$M_{CB} = +M_p - m_1H_a = +73,400 - 18.98 \times 3,250 = +11,700 \text{ ft.-lb.}$$
$$\text{Shear at } B = \tfrac{1}{2}\Sigma w \text{ for the deck} = 7,660 \text{ lb.}$$

Notice that M_p and m_1 for division 10 (Table 11-2) are assumed to be equal to the values at C.

11-9. Temperature Changes, Shrinkage, and Rib Shortening. Assume the rigid frame of Fig. 11-24 to have hinged ends which cannot slide. Then, if this structure is subjected to a decrease of temperature, which causes its length to shorten by an amount equal to ΔL, the frame moves into the position shown by the dotted lines. The effect is equivalent to spreading the footings

at A and E a total distance equal to ΔL. It causes horizontal reactions at A and E.

Let

$$\Delta L = \pm \omega t L \text{ (+ denotes rise in temperature)}$$

where ω = the coefficient of thermal expansion = 0.000006 to 0.0000065, t = the change of temperature in degrees Fahrenheit, and L = span length. Then, for a drop in temperature,

$$H_a \times \delta_{1-1} = -\Delta L$$

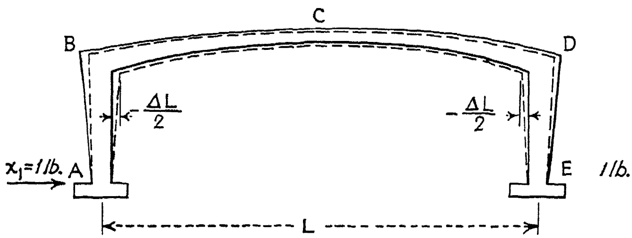


FIG. 11-24.

if δ_{1-1} expresses the deflection due to the unit load $x_1 = 1$ as pictured in Fig. 11-24. Therefore,

$$H_a = -\frac{\Delta L}{\delta_1} = -\frac{}{\sum \frac{m_1^2 s}{EI}} \tag{11-14}$$

A rise of temperature causes compression in the deck and reactions that are directed inward; a fall of temperature produces tension in the deck and reactions that are directed outward.

Notice that E must remain in Eq. (11-14) and that L and the summation are given for the entire structure. Also, it is apparent that H_a increases when I increases. In other words, the stiffer the frame becomes the greater are the resultant horizontal thrust and the bending moments.

Because I is taken as the moment of inertia of the full concrete section as pictured by curve 1 in Fig. 11-21, instead of the I for the cracked sections (curve 3), the theoretical thrust which is due to a change of temperature may be higher than the actual thrust. However, the assumption is on the side of safety.

The effect of shrinkage is similar to a drop in temperature. If no other specifications are given, it is sufficient to assume that

the shrinkage is equivalent to a decline of 40 to 80°F. Equation (11-14) will then apply.

Rib shortening,[1] which is also equivalent to a drop in temperature, is the decrease in the length of the frame that is due to the direct compression in it. If f_c is the average compressive unit stress upon each division of the frame, and if dx is the horizontal projection of the sections,

$$\Delta L = -\frac{\Sigma f_c\,dx}{E}$$

or

$$H_a = \frac{\Delta L}{\delta_{1-1}} = -\frac{\Sigma f_c\,dx}{\sum \frac{m_1^2 s}{EI}} = -\frac{\Sigma f_c\,dx}{\sum y^2 \frac{s}{I}} \tag{11-15}$$

where H_a acts in an outward direction upon the legs of the frame, and L and the summations are for the complete structure.

In reality, any force that spreads the footings of the rigid frame of Fig. 11-24 or that pushes them closer together by a distance $\mp \Delta L$ must be

$$F = \mp \frac{E\,\Delta L}{\sum y^2 \frac{s}{EI}}.$$

On the other hand, a vertical settlement of one footing of a hinged-end rigid frame merely rotates the structure as a unit, but it does not bend it. However, if the ends are fixed, such a settlement will distort and bend the frame.

Problem 11-7. Find the bending moments at B and C in the rigid frame of Fig. 11-22 for a rise in temperature of 60°F.; also, find the bending moments for a fall in temperature of 60°F. combined with an equivalent shrinkage of 40°F. Assume E = 3,000,000 lb. per sq. in. and ω = 0.000006.

From Table 11-2, $\sum \frac{y\,s}{I}$ = 15,417 for the half span. Then, using Eq. (11-14), for a rise of 60°F. (ΔL being positive),

[1] The effect can be reduced to an approximately equivalent drop in temperature by finding the temperature that will produce a shortening of the same amount.

$$H_a = + \frac{144 \times 3,000,000 \times 0.000006 \times 60 \times 46.66}{2 \times 15,417}$$

$$H_a = +235 \text{ lb.}$$
$$M_{BC} = -H_a y = -235 \times 16 = -3,760 \text{ ft.-lb. (tension outside).}$$
$$M_{CB} = -H_a y = -235 \times 19 = -4,460 \text{ ft.-lb. (tension on top).}$$

For the drop in temperature and the shrinkage (100°F.),

$$H_a = \frac{-144 \times 3,000,000 \times 0.000006 \times 100 \times 46.66}{2 \times 15,417} = -392 \text{ lb}$$
$$M_{BC} = +392 \times 16 = +6,270 \text{ ft.-lb. (compression outside).}$$
$$M_{CB} = +392 \times 19 = +7,450 \text{ ft.-lb. (compression on top).}$$

11-10. Lateral Earth Pressure. In ordinary cases, the earth of the approach to a rigid-frame bridge is filled directly against

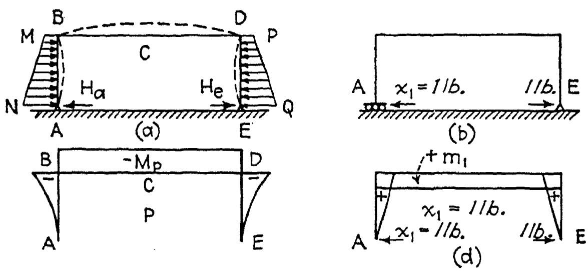


FIG. 11-25.

the backs of the legs. If so, it will exert a lateral pressure which will bend the frame somewhat as shown for the hinged structure that is pictured in Fig. 11-25(*a*).

Assume that *ABMN* and *EDPQ* represent the pressure diagrams for a case of symmetrically placed earth backfills. Then H_a may be taken as the redundant, and the frame may be cut at *A*, adding imaginary rollers as shown in (*b*). The structure will then be determinate; the legs will act as cantilever beams which are rigidly connected to the deck *BD*. Sketches (*c*) and (*d*) picture the bending-moment diagrams for the external loads and for the unit or dummy load, respectively. H_a can then be found as usual from Eq. (11-7),

$$H_a = -\frac{\delta'_a}{\delta_{1-1}}.$$

The earth pressure behind the legs of the frame tends to bend the deck as shown in Fig. 11-25(*a*). This relieves the positive

bending moment (compression on the top) in BD. The magnitude of the lateral pressure may vary from zero to the active earth pressure, or, in the case of live load or rise in temperature which bends the legs outward, it may approach the passive earth pressure. It is therefore advisable to neglect these lateral forces in calculating the bending moments in the center of the deck but to consider them in the design of the legs and in that of the knee at B. Furthermore, these pressures may be applied unsymmetrically, in which case the resultant stresses must be considered if they increase the critical stresses that are produced by other forces.

Problem 11-8. Assume that the frame of Fig. 11-22 is loaded unsymmetrically as pictured in Fig. 11-26(a). Find the bending moments at B, C, and D due to these loads.

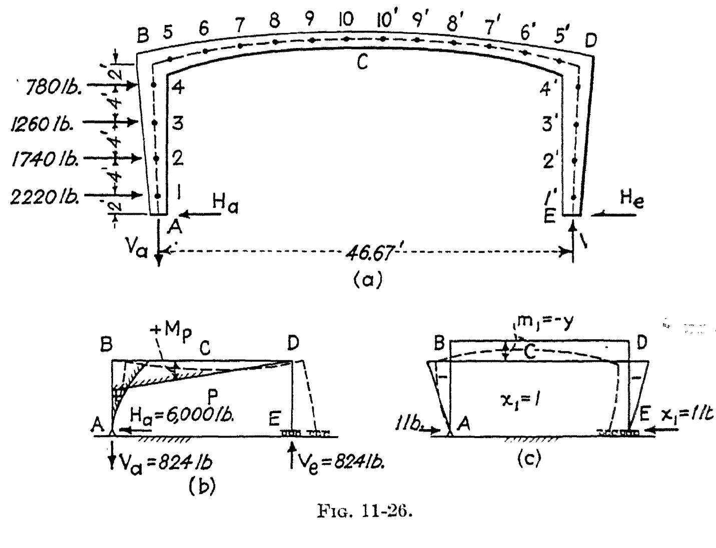


FIG. 11-26.

In this case, cut the frame at E, and add imaginary rollers, calling H_e the redundant reaction. Then, because $\Sigma H = 0$, H_a of Sketch (b) = 780 + 1 260 + 1,740 + 2,200 = 6,000 lb. By taking moments about A,

$$V_e \times 46.67 = 2{,}220 \times 2 + 1{,}740 \times 6 + 1{,}260 \times 10 + 780 \times 14$$

$$V_e = \frac{38{,}400}{46.67} = 824 \text{ lb.}$$

$$V_a = -V_e.$$

It will be sufficient to represent the bending-moment diagram for the external loads as shown in Fig. 11-26(*b*), and that for $x_1 = 1$ in Sketch (*c*). In diagrammatic pictures, it is satisfactory to represent the frame by means of straight lines. The outline of the deformed structure for each case is pictured by the dotted lines.

TABLE 11-3.—DATA FOR PROBLEM 11-8

Point	$m_1 = -y$	y^2	M_p	$s \div I$	$y^2 s \div I$	$M_p y s \div I$
1	− 2.00	4	12,100	4.71	19	− 114,000
2	− 6.00	36	27,300	3.08	111	− 505,000
3	−10.00	100	35,700	2.12	212	− 757,000
4	−14.00	196	38,900	1.51	296	− 822,000
5	−16.48	272	37,400	1.75	476	− 1,079,000
6	−17.31	300	34,100	3.01	903	− 1,777,000
7	−17.98	323	30,800	5.10	1,647	− 2,824,000
8	−18.48	342	27,500	8.02	2,743	− 4,076,000
9	−18.81	354	24,200	11.43	4,046	− 5,203,000
10	−18.98	360	20,900	13.79	4,964	− 5,470,000
10′	−18.98	360	17,600	13.79	4,964	− 4,607,000
9′	−18.81	354	14,300	11.43	4,046	− 3,075,000
8′	−18.48	342	11,000	8.02	2,743	− 1,630,000
7′	−17.98	323	7,700	5.10	1,647	− 706,000
6′	−17.31	300	4,400	3.01	903	− 229,000
5′	−16.48	272	1,100	1.75	476	− 32,000
4′	−14.00	196	......	1.51	296	
3′	−10.00	100	......	2.12	212	
2′	− 6.00	36	......	3.08	111	
1′	− 2.00	4	......	4.71	19	
Σ	....	...	......	.....	30,834	−32,906,000

Equation (11-13) may be used to find the redundant.

$$H_e = -\frac{\quad}{\sum \frac{y^2 s}{I}} \tag{11-13}$$

The data for this equation are assembled in Table 11-3, the summation covering the entire frame, and part of the terms being carried over from Table 11-2. Neglect the bending in *DE* which is due to V_e because it is so small. See Fig. 11-22 for the dimensions of the frame. Then, substituting the summations from Table 11-3 in Eq. (11-13),

$$H_e = -\frac{-(-32{,}906{,}000)}{30{,}834} \quad +1{,}070 \text{ lb.}$$

Then

$$H_a = 6{,}000 - 1{,}070 = 4{,}930 \text{ lb.}$$
$$M_{BC} = +V_e(46.67 + 0.67) - H_e(16)$$
$$= 824 \times 47.34 - 1{,}070 \times 16 = 21{,}900 \text{ ft.-lb.}$$

$$M_{CB} = \qquad -1{,}100 \text{ ft.-lb.}$$

$M_{DC} = -V_e(0.67) - H_e(16) = -17{,}600$ ft.-lb. (the same as at B when the opposite leg is loaded).

11-11. Influence-line Diagrams. An influence-line diagram (henceforth called an "influence line") for any function is a diagram that gives the magnitude of the function at one given point for a load placed anywhere upon the structure when the

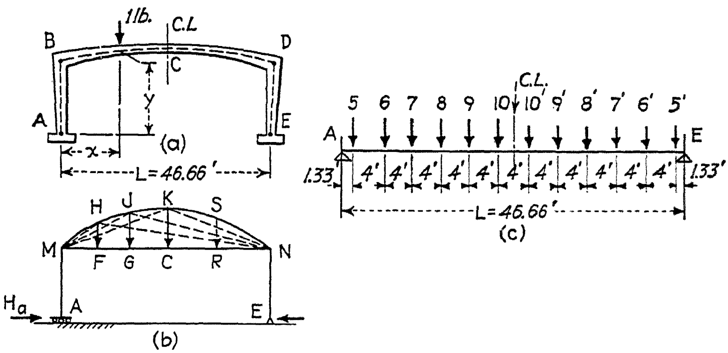


FIG. 11-27.

ordinate of the diagram at the point of application of the load is multiplied by the magnitude of the load itself. Such diagrams are of great assistance when one is dealing with moving, concentrated live loads. They may be used for a large variety of purposes.

As an illustration, construct the influence line for the horizontal reaction at A for the hinged-end rigid frame of Fig. 11-27(a). Cut the structure at A, making H_a the redundant. Then it is possible to place a unit load at varying positions, like F, G, and C, and to find the corresponding magnitudes of H_a by means of Eq. (11-13); to plot these values of H_a as the ordinates FH, GJ, and CK, respectively [Fig. 11-27(b)]. The envelope curve $MHJKN$ and the line MN will then constitute the influence line for H_a. If a load of 1,000 lb. is placed at any point R, then H_a will be 1,000 times H_a for a unit load $= 1{,}000 \times RS$.

However, the construction of the influence line by this means is laborious. A more convenient method is the use of the principle of "elastic weights" on a fictitious (conjugate) beam. An outline of the method is as follows:

Assume that AE of Fig. 11-27(*c*) is the deck of the frame. Any loads outside A and E will be omitted because they will not increase the magnitude of H_a. Next, represent each of the divisions of the deck by a force—an "elastic weight"—that equals ys/EI for that particular section, E being omitted in the denominator in this case because it is constant. Then treat these

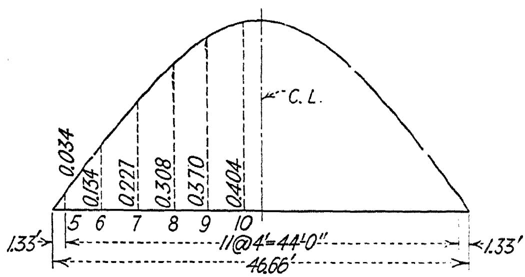
TABLE 11-4.—INFLUENCE-LINE ORDINATES FOR PROBLEM 11-9

Point	$ys \div I$	Shear at left	Lever arm	ΔM	ΣM	Ordinate $= \Sigma M \div \Sigma y^2\frac{s}{I}$
5	28.8	797.5	1.33	1,061	1,061	0.034
6	52.1	768.7	4	3,075	4,136	0.134
7	91.7	716.6	4	2,866	7,002	0.227
8	148.2	624.9	4	2,500	9,502	0.308
9	215.0	476.7	4	1,907	11,409	0.370
10	261.7	261.7	4	1,047	12,456	0.404
Σ	797.5					

elastic weights as loads upon the simply supported beam AE; compute the bending moments in the beam at the load points; and divide these moments by $\sum y^2\frac{s}{I}$. The calculations appear in Table 11-4. Plot the results in Fig. 11-28. This is the influence line for the horizontal reaction H_a, because the ordinate at any point, as 6 in Fig. 11-28, times the load gives the magnitude of H_a.

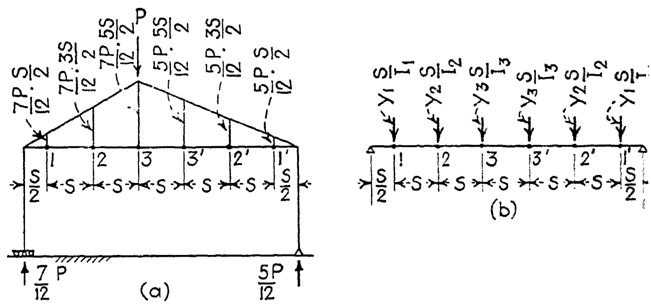
That Fig. 11-28 is an influence line may be demonstrated as shown in Fig. 11-29. Take a symmetrical frame, and divide the deck into an equal number of parts—say 6, as in Sketch (*a*); place a load P at the center of the third division; draw the bending-moment diagram; find its ordinates at each point times the corresponding $y\frac{s}{I}$; then sum up the products to get $\sum M_p y\frac{s}{I}$. Next, draw the simply supported beam of Sketch (*b*); divide it

into six equal parts; apply the elastic weights as loads; then find the bending moment at point 3. The result is the same as



Influence Line for H_a

FIG. 11-28.



$$\Sigma M_p y \frac{S}{I} = \left(\frac{7}{12}P\cdot\frac{S}{2}\cdot y_1\frac{S}{I_1} + \frac{7}{12}P\cdot\frac{3S}{2}\cdot y_2\frac{S}{I_2}\right.$$

$$+\frac{7}{12}P\cdot\frac{5S}{2}\cdot y_3\frac{S}{I_3} + \frac{5}{12}P\cdot\frac{5S}{2}\cdot y_3\frac{S}{I_3}$$

$$\left.+\frac{5}{12}P\cdot\frac{3S}{2}\cdot y_2\frac{S}{I_2} + \frac{5}{12}P\cdot\frac{S}{2}\cdot y_1\frac{S}{I_1}\right) =$$

$$P\left(\frac{S}{2}\cdot y_1\frac{S}{I_1} + \frac{3S}{2}\cdot y_2\frac{S}{I_2} + \frac{5S}{2}\cdot y_3\frac{S}{I_3}\right)$$

$$M_3 = \left(y_1\frac{S}{I_1} + y_2\frac{S}{I_2} + y_3\frac{S}{I_3}\right)\frac{5S}{2}$$

$$-y_1\frac{S}{I_1}\cdot 2S - y_2\frac{S}{I_2}\cdot S =$$

$$\frac{S}{2}\cdot y_1\frac{S}{I_1} + \frac{3S}{2}\cdot y_2\frac{S}{I_2} + \frac{5S}{2}\cdot y_3\frac{S}{I_3}$$

FIG. 11-29.

before except that the coefficient is unity instead of P. Therefore, the ordinate at 3 is the influence-line ordinate which is

desired; in other words, it is $\sum M_p y \frac{\circ}{I} \div P$. This method can be used as a means of analysis when the function to be integrated is in the form $M_p m \frac{\circ}{I}$.

An influence line for the bending moment at C [Fig. 11-27(a)] may be constructed by computing the bending moment at C as for a simply supported beam with a unit load at any given point, by finding H_a from Fig. 11-28, and then by subtracting Hy from the simple beam moment.

Problem 11-9. Find the magnitude of H_a for the hinged-end rigid frame of Fig. 11-22 due to the dead load of the frame itself, using an influence line.

The influence line is shown in Fig. 11-28; the ordinates are given in Table 11-4; the weights of the sections are given in Table 11-1; $\sum \frac{m_1^2 s}{I} = \frac{y^2 s}{I}$ is given in Table 11-2 for half of the frame; and the products of the loads times the ordinates are shown in Table 11-5. Then $H_a = 3{,}248$ lb. Compare this answer with that of Problem 11-6.

TABLE 11-5

Point	Ordinate	Load	Product
5	0.034	1,870	64
6	0.134	1,540	206
7	0.227	1,280	291
8	0.308	1,090	336
9	0.370	970	359
10	0.404	910	368
Σ for half frame	.....	7,660	1,624
Σ for entire frame	.....	.....	$H_a = 3{,}248$

Problem 11-10. By the use of the influence line of Fig. 11-28, find the magnitude of H_a, also the bending moments at B and C, for the rigid frame of Fig. 11-22, using the series of concentrated wheel loads which is pictured above the diagram of Fig. 11-28.

Multiply each load by the scaled value of the corresponding ordinate of the influence line. Then

$$H_a = (1{,}600 \times 0.227 + 4{,}000 \times 0.404 + 1{,}600 \times 0.084) = 2{,}117 \text{ lb.}$$
$$M_{BC} = -H_a y = -2{,}117 \times 16 = -33{,}900 \text{ ft.-lb.}$$
$$M_{CB} = +M_p - H_a y$$

$$M_p \text{ at } C = +(1{,}600 \times 37.33 + 4{,}000 \times 23.33 + 1{,}600 \times 3.33)\frac{23.33}{46.67}$$
$$- 1{,}600 \times 14 = 3{,}390 \times 23.33 - 1{,}600 \times 14 = 56{,}700 \text{ ft.-lb.}$$

Therefore,

$$M_{CB} = +56{,}700 - 2{,}117 \times 19 = +16{,}500 \text{ ft.-lb.}$$

Shear at B = 3,390 lb.; shear at C = 3,390 − 1,600 = 1,790 lb.

Notice that this problem could be solved in the same manner as Problem 11-6, but $M_p y \frac{s}{I}$ in Table 11-2 would be replaced by new values for the new loading conditions.

Problem 11-11. Find the bending moments at B and C that are due to the pavement and earth fill on the frame of Fig. 11-22.

In Fig. 11-22, the surface of the roadway is shown to be flatter than the top or extrados of the deck because this frame is chosen so as to emphasize the effect of curvature in the deck. It is not desirable to have such a "hump" in the roadway. Therefore, the pavement is assumed to be on a fill which is 6 in. deep at the crown and 2 ft. deep at the knee of the frame. If the pavement is 10 in. thick, then the superimposed dead loads are approximately as shown in Fig. 11-30.

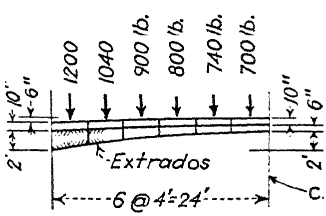


FIG. 11-30.

Use the concentrations that are indicated in Fig. 11-30 with the influence line of Fig. 11-28 to find H_a. The loads times the influence-line ordinates are shown in Table 11-6.

TABLE 11-6

Point	Ordinate	Load	Product
5	0.034	1,200	41
6	0.134	1,040	139
7	0.227	900	204
8	0.308	800	246
9	0.370	740	274
10	0.404	700	283
Σ for half frame	.....	5,380	1,187
Σ for entire frame	.....	.....	H_a = 2,374

H_a = 2,374 lb.

$M_{BC} = -H_a y = -2{,}374 \times 16 = -38{,}000$ ft.-lb.

$M_{CB} = +M_p - H_a y$

$M_p = +5{,}380 \times 23.33 - 1{,}200 \times 22 - 1{,}040 \times 18 - 900 \times 14 - 800 \times 10 - 740 \times 6 - 700 \times 2 = +54{,}000$ ft.-lb.

$M_{CB} = +54{,}000 - 2{,}374 \times 19 = +8{,}900$ ft.-lb.

11-12. Design of Members. Each portion of the rigid frame must be designed for the combination of forces and bending moments which may occur simultaneously and which will cause the maximum stresses at the point in question. Using the frame of Fig. 11-22, with the data from Problems 11-6 to 11-11, inclusive, the maximum moments, shears, and thrusts for the deck at points B and C are as shown in Table 11-7. Rib shortening has been neglected.

TABLE 11-7

	Bending moment, ft.-lb.		Shear, lb.		Thrust in deck, lb.
	B		B		B to C*
DL of frame	52,000	+11,700	7,660		−3,250
DL of fill, etc.	38,000	+ 8,900	5,380		−2,374
Live load.	33,900	+16,500	3,390	1,790	−2,117
Earth pressure.	17,600		820	820	−1,070
Rise in temp.	3,760				− 235
Fall in temp. and shrinkage. .		+ 7,450			
	−145,260	+44,550	17,250	2,610	−9,046

* Negative sign denotes compression in deck.

In the design of the deck, assume that $n = 10$ and that the allowable stresses are $f_c = 1{,}000$ lb. per sq. in. and $f_s = 20{,}000$ lb. per sq. in. Neglect

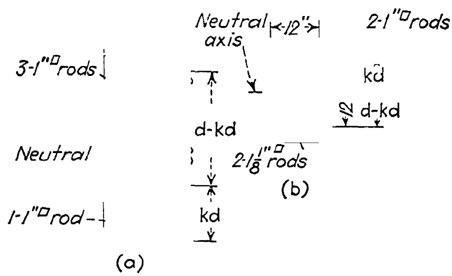


FIG. 11-31.

the effect of the thrust because it is relatively small when compared to that of the large bending moments.

1. *Section at B.* Assume the section that is shown in Fig. 11-31(a), the depth being that of the deck at a section over the center of the point A of

Fig. 11-22. Let $M = 145{,}260$ ft.-lb. $= 1{,}740{,}000$ in.-lb., from Table 11-7. The arrangement of the rods is made so as to have 6 in. between the bars horizontally in order to provide adequate space for splices, etc.

Using the transformed-section method,

$$^2 + 9 \times 1(kd - 3) = 10 \times 2(36 - kd) + 10 \times 1(33 - kd)$$
$$kd = 10.5 \text{ in.,} \qquad d - kd = 25.5 \text{ in.}$$
$$I_c = 23{,}200 \text{ in.}^4$$

Therefore,

$$\frac{1{,}740{,}000 \times 10.5}{23{,}200}$$

Therefore,

$$\frac{1{,}740{,}000 \times 25.5 \times 10}{23{,}200}$$

These results are conservative but acceptable.

2. *Section at C.* Assume the cross section to be as shown in Fig. 11-31(*b*). An analysis of it when $M = 44{,}550$ ft.-lb. (Table 11-7) gives $kd = 5.63$ in., $I_c = 3{,}070$ in.^{4}, $f_c = 980$ lb. per sq. in., and $f_s = 16{,}300$ lb. per sq. in. These stresses will be accepted, although the thrust will increase f_c slightly. There is an excess of steel in the tensile side, but it is used to keep the neutral axis farther from the top in order to relieve the concrete; it is used also in order to have less cracking of the tensile side near the crown.

11-13. Fixed-end Frame with Varying Moment of Inertia. Fixed supports at the bottom ends of the legs of the frame that is pictured in Fig. 11-22 make the analysis of the frame similar to that of the prismatic one of Problem 11-5. The work equations [Eq. (11-12)] must be solved for the three redundants, the integration of the coefficients being performed by summation. Each different condition of loading compels one to find new values for the δ_p terms, but the other coefficients, which depend upon the shape of the structure, remain constant. Of course, the equations must be solved for each different case. The general method of procedure will be illustrated more fully in the analysis of two-span frames.

Influence lines for a fixed-end frame may be obtained by placing unit loads upon various points—about $0.2L$, $0.4L$, etc.—solving the equations for the redundants, and plotting curves for the desired functions.[1]

[1] The influence lines for a fixed-end rigid frame can be drawn more easily by the use of elastic weights upon an imaginary cantilevered beam. This method is developed and illustrated for a fixed-end arch in the next

11-14. Deflections. The deflection of the deck of a rigid frame under its own dead load is often desired in order to camber the forms or "centering" sufficiently to allow for the settlement that will occur when the forms are removed. This deflection can be found by using Eq. (11-3), with M_p equal to the dead-load bending moment, and m_q equal to the moment caused by a unit load at the point where the deflection is desired and acting in the desired direction. In other words, sum up (integrate) the products of the two moment diagrams for the entire frame,

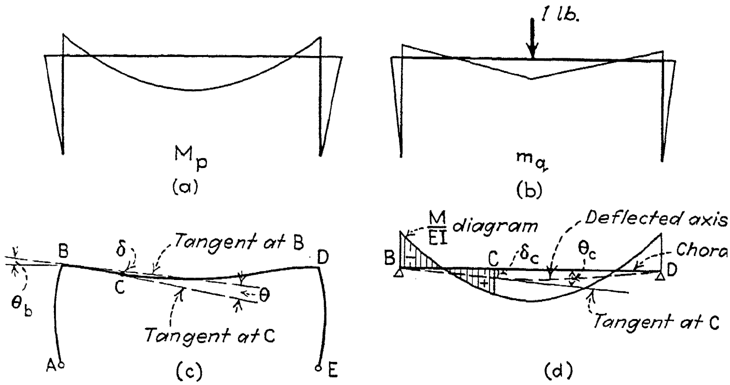


FIG. 11-32.

as shown in Figs. 11-32(*a*) and (*b*), times the lengths of the pieces, each divided by EI.

A second method of obtaining the deflection of any member is the use of "moment areas." This may be applied when the members are not curved excessively. The method may be stated as follows, referring to Figs. 11-32(*c*) and (*d*):

1. Draw the bending-moment diagram for the loading for which the deflection of the structure is desired; then divide the moment at all points by EI, plotting a new diagram which represents M/EI for the member.
2. The area of the M/EI diagram between any two points like B and C gives the *change* in the slope of the *tangent* to the elastic curve at C with respect to the tangent at $B - \theta$ of Sketch (*c*), in radians.

chapter. However, the short-cut method with constant horizontal lengths which is used therein for the preparation of the tables of cantilever moments should be replaced by direct calculation of the moments with whatever lengths of divisions may be suitable for the deck and for the legs.

3. The moment of the M/EI diagram between B and C about point C as a center gives the deflection of C with respect to the *tangent* at B — δ of Sketch (c).

The foregoing statements apply to any two points in the member. Of course, the signs of the moments must be considered also.

The method of moment areas is sometimes confusing because the angle of the tangent at such a point as B—θ_b of Sketch (c)—must be found before θ or δ can be computed.

A third means of finding the deflection is the "conjugate-beam" method. This may be stated as follows:

1. Draw the M/EI diagram for the loading that is being considered.
2. Assume that the M/EI diagram constitutes the loading diagram of a fictitious or conjugate beam which has the same span as the given member but which is simply supported—as in Sketch (d).
3. The shear in the conjugate beam at any point gives the angle between the *chord* joining the two ends and the tangent at the given point—θ_c of Fig. 11-32(d)—in radians.
4. The bending moment in the conjugate beam at any point gives the deflection of that point with respect to the *chord* joining the two ends of the conjugate beam—δ_c of Fig. 11-32(d).

Problem 11-12. Find the deflection of the center of the deck of the rigid frame of Fig. 11-22 for the dead load of the deck itself, the earth fill, and the pavement. Assume E = 3,000,000 lb. per sq. in. Use the conjugate-beam method, assuming the deck to be straight. Also find the angular deflection at the knee.

TABLE 11-8.—CALCULATIONS FOR ANGULAR AND VERTICAL DEFLECTION

Point	DL at pt.	Simple beam M	y	$-H_a y =$ $-5{,}620y$	Resultant M	I	$\frac{M}{I}$	s	$\frac{M}{I}s$
5	3,070	17,400	16.48	− 92,600	−75,200	2.34	−32,100	3.33	−107,000
6	2,580	57,200	17.31	− 97,400	−40,200	1.35	−29,800	4	−119,300
7	2,180	86,800	17.98	−101,000	−14,200	0.79	−18,000	4	− 72,000
8	1,890	107,800	18.48	−104,000	+ 3,800	0.50	+ 7,600	4	+ 30,400
9	1,710	120,900	18.81	−105,800	+15,100	0.35	+43,100	4	+172,400
10	1,610	127,300	18.98	−106,800	+20,500	0.29	+70,700	4	+282,800
Σ*	13,040	....	.....	....	....	....	....	....	+187,300

* Σ for one-half of frame.

$$\left(\frac{\;}{E}\right)(57.3) = \frac{187{,}300 \times 57.3}{432{,}000{,}000} \quad 0.025^\circ$$

$$\delta_c = \frac{1}{E}(187{,}300 \times 23.33 + 107{,}000 \times 22 + 119{,}300 \times 18 + 72{,}000 \times 14 - 30{,}400 \times 10$$
$$- 172{,}400 \times 6 - 282{,}800 \times 2)12 = 0.222 \text{ in.}$$

The loads of the deck are given in Table 11-5; those for the fill and pavement, in Table 11-6. These are combined in Table 11-8, in which the values of M/I at the centers of the divisions are computed. E is not brought into the calculations until last in order to minimize the work. The magnitude of H_a is the sum of the values that are given in Tables 11-5 and 11-6—$H_a = 3{,}248 + 2{,}374 = 5{,}620 \pm$ lb.

The details of the calculations (which are made by slide rule) are given in Table 11-8, all of the quantities being in terms of feet or pounds. The frame is symmetrical so that $\sum \frac{Ms}{I}$ for half of it automatically gives the shear at the end of the conjugate beam. The calculations for θ_b and δ_c as given under the table are self-explanatory.

11-15. Two-span Rigid Frames. The bridge shown in Fig. 11-33 is one of a number of two-span rigid frames over the New

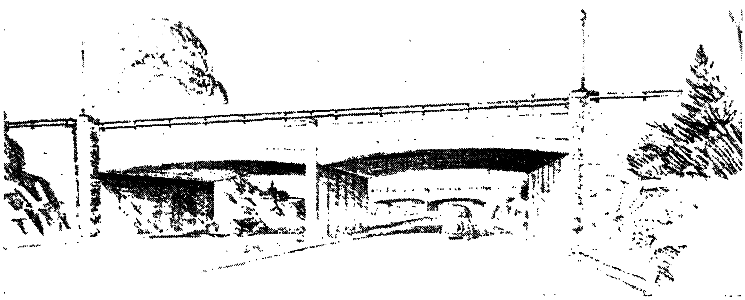


FIG. 11-33.—Sketch of two-span rigid-frame bridge, New Jersey approach to the Lincoln Tunnel, Union City, N. J.

Jersey approach to the Lincoln Tunnel at New York. This type of structure should come into extensive use along with the development of modern parkways which have a physical separation between two roadways. This particular structure has fixed ends, a narrow fixed center pier, and unsymmetrical thicknesses for the deck over each opening as pictured in Fig. 11-35.

The principles to be used in the design of such a bridge are the same as those which have been illustrated for single-span frames, but the presence of the center pier causes more redundants in

the structure—a total of six. If the ends of the legs were hinged, there would be only four unknowns. However, the fixed-end condition will be illustrated sufficiently to enable one to understand the procedures and to adapt them to other or more simple cases.

A brief outline of the procedure for the design of such a symmetrical, fixed-end structure is as follows, using Fig. 11-34 for reference:

1. *Tentative Sections.* Choose tentative thicknesses of the end legs and the deck in the same manner as for single spans, but allow a little more strength in the deck at the center pier. Make the center pier conform to architectural requirements, but remember that the heavier and stiffer the pier is the more severe will be the live-load bending moments at its junction with the deck. Test the legs and the deck by using the formulas for H as given in Figs. 1 and 2 of the Appendix.

2. *Dead Load.* After a tentative structure has been chosen, divide it into sections, choose an origin of coordinates [such as A, Fig. 11-34(a)], and compute x, y, and I for the centers of the divisions. The loading condition will be as shown in Fig. 11-34(a), but, because of symmetry, it may be reduced to that which is pictured in Sketch (b), point C being considered as fixed. Then cut the structure free at A, introducing the three redundants as shown in (b). The bending-moment diagrams for unit redundants are pictured in (c), (d), and (e); that for the dead loads of the deck, in (f). Next, compute the coefficients of the work equations (δ_{1-1}, δ_{2-1}, etc.); solve these equations for the redundants; calculate the bending moments at A, B, F and C; and compute the shears in AB and BC.

3. *Uniform Live Load on One Span.* Load the structure as shown in Sketch (g), cutting it free at A and E. The bending-moment diagrams for the six redundants, if each is unity, are shown in Sketches (h), (i), (j), (k), (m), (n), and (o). Assume an origin of coordinates at O, Fig. 11-34(g); compute the coefficients δ_{1-1}, δ_{2-1}, etc.; and substitute them in the six work equations as illustrated in Problem 11-14; solve for the redundants; then compute the bending moments and shears at the desired points. (Approximate but more readily usable methods of analysis will be shown in Chap. 13, because the solution of six simultaneous equations is very tedious.)

4. *Full, Uniform Live Load.* In Sketch (*g*), one span only is loaded. It causes certain bending moments and shears in the left-hand span and others in the right-hand one. If *CD* were loaded alone, all of these values would be switched to the opposite-hand point in the frame; i.e., the magnitude of the bending moment at *D* which is caused by the load on *BC* is the same as the moment at *B* which is caused by the same load on *CD*.

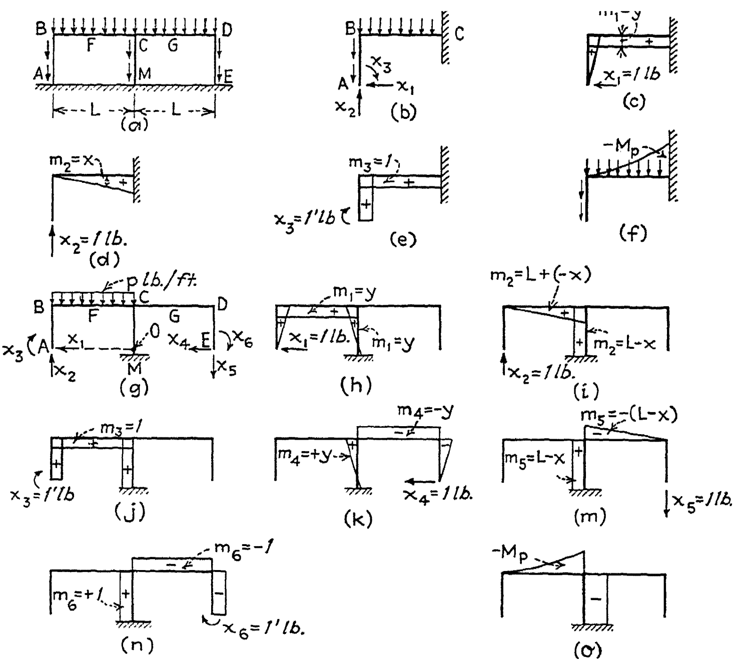


FIG. 11-34.

Therefore, the bending moments and shears at all points for full live load are the algebraic summations of the values for the corresponding left-hand and right-hand points of the frame which are due to the uniform load on one span only (multiplied by a constant factor if the specifications require the intensity of the live load to change with its length).

5. *Concentrated Live Loads on One Span.* The procedure is the same as for item 3 except that the M_p terms are different. The moment diagram for the live loads must be that which applies to the particular loading condition that is assumed.

6. *Temperature and Shrinkage.* The bending moments and shears which are due to temperature changes and shrinkage can be computed by assuming that the structure is fixed at C, as shown in Fig. 11-34(b), because of symmetry. The magnitude of the change in length ΔL is to be computed for the full span BC. The M_p terms in the work equations are zero, but Δx is replaced by ΔL. The coefficients of the three redundants are the same as those which are computed under item 2 because they depend only upon the elastic properties of the structure.

7. *Earth Pressure.* If the pressure of the earth is assumed to be applied symmetrically at both ends of the structure, the frame may be assumed to be fixed at C. The analysis is then similar to that for the dead load except that there is a new set of M_p terms.

Problem 11-13. Assuming the rigid frame of Fig. 11-35, find H_a, V_a, and M_a for the dead load of the frame itself.

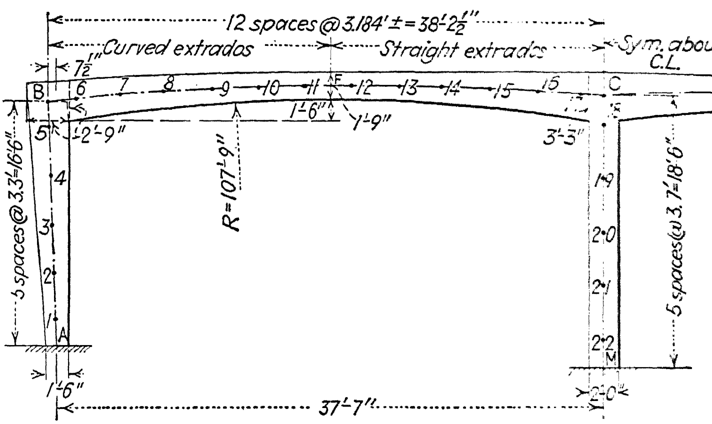


FIG. 11-35.

The springing line near C is 4 in. higher than near B. This causes an even greater difference in the levels of the points B and C on the axis of the deck. Therefore, in order to simplify the calculations, the leg will be assumed to be raised until B is level with C. The data in Table 11-9 have been recorded upon this basis, since there will be no appreciable effect upon the structure. The magnitudes of w are computed from scaled or calculated thicknesses at the centers of the divisions. The origin of coordinates is at A. The tilt of the axis of the leg gives small negative values for x which could be assumed equal to zero without appreciable error but which are

TABLE 11-9

Given data					*DL* moment as cantilever				Properties of frame					M_p terms		
Point	$w = s/I$ δ_{3-3}	x	y	Wt. of frame, kips	Shear at left	Lever arm	ΔM_p, ft.-kips	M_p, ft.-kips	xw δ_{3-2}	x^2w δ_{2-2}	yw δ_{1-3}	y^2w δ_{1-1}	xyw δ_{1-2}	M_pw δ_{p-3}	M_pxw δ_{p-2}	M_pyw δ_{p-1}
1	9.43	−0.06	1.65	0.80	0	......	0	0	− 0.6	......	15.6	26	− 1			
2	6.00	−0.19	4.95	0.93	0.80	−0.13	+ 0.1	+ 0.1	− 1.1	......	29.7	147	− 6	+ 0.6	....	+ 3
3	4.12	−0.31	8.25	1.05	1.73	−0.13	+ 0.2	+ 0.3	− 1.3	......	34.0	280	− 11	+ 1.2	....	+ 10
4	2.95	−0.44	11.55	1.18	2.78	−0.13	+ 0.4	+ 0.7	− 1.3	1	34.1	394	− 15	+ 2.1	− 1	+ 24
5	2.18	−0.56	14.85	1.30	3.96	−0.13	+ 0.5	+ 1.2	− 1.2	1	32.4	481	− 18	+ 2.6	− 1	+ 39
6	1.94	0.97	16.63	1.29	5.26	1.53	− 8.1	− 6.9	+ 1.9	2	32.4	540	+ 31	− 13.4	− 13	− 224
7	2.82	4.15	17.00	1.14	6.55	3.18	−20.8	− 27.7	+ 11.7	49	47.9	815	+ 199	− 78.1	− 324	− 1,328
8	3.93	7.34	17.25	1.02	7.69	3.18	−24.5	− 52.2	+ 28.8	211	67.8	1,169	+ 498	− 205.1	− 1,505	− 3,538
9	5.22	10.52	17.44	0.93	8.71	3.18	−27.7	− 79.9	+ 54.9	578	91.0	1,588	+ 958	− 417.1	− 4,388	− 7,274
10	6.37	13.70	17.57	0.87	9.64	3.18	−30.6	−110.5	+ 87.3	1,196	111.9	1,966	+ 1,533	− 703.9	− 9,643	− 12,368
11	7.08	16.89	17.63	0.84	10.51	3.18	−33.4	−143.9	+119.6	2,020	124.8	2,201	+ 2,108	−1,018.8	− 17,208	− 17,961
12	7.08	20.07	17.63	0.84	11.35	3.18	−36.1	−180.0	+142.1	2,852	124.8	2,201	+ 2,505	−1,274.4	− 25,577	− 22,468
13	6.00	23.26	17.57	0.89	12.19	3.18	−38.7	−218.7	+139.6	3,246	105.4	1,852	+ 2,452	−1,312.2	− 30,522	− 23,055
14	4.49	26.44	17.44	0.98	13.08	3.18	−41.6	−260.3	+118.7	3,138	78.3	1,366	+ 2,070	−1,168.7	− 30,900	− 20,382
15	3.03	29.62	17.25	1.11	14.06	3.18	−44.7	−305.0	+ 89.7	2,658	52.3	902	+ 1,548	− 924.2	− 27,375	− 15,942
16	1.93	32.81	17.00	1.30	15.17	3.18	−48.2	−353.2	+ 63.3	2,078	32.8	558	+ 1,076	− 681.7	− 22,367	− 11,589
17	1.18	35.99	16.68	1.52	16.47	3.18	−52.4	−405.6	+ 42.5	1,528	19.7	328	+ 708	− 478.6	− 17,225	− 7,983
C	.....	......	.....	....	17.99	1.59	−28.6	−434.2								
Σ	75.75	......	.....	....	17.99	......	......	−434.2	+894.6	19,558	1,034.9	16,814	+15,635	−8,269.7	−187,049	−144,036

(Horiz. deflection) $\delta_{1-1} + \delta_{2-1} + \delta_{3-1} = -\delta_{p-1}$.

(Vert. deflection) $\delta_{1-2} + \delta_{2-2} + \delta_{3-2} = -\delta_{p-2}$.

(Angular rotation) $\delta_{1-3} + \delta_{2-3} + \delta_{3-3} = -\delta_{p-3}$.

$$16{,}814H_a + 15{,}635V_a + 1{,}035M_a = 144{,}036. \quad (1)$$

$$15{,}635H_a + 19{,}558V_a + 895M_a = 187{,}049. \quad (2)$$

$$1\,035H_a + 895V_a + 76M_a = 8{,}270. \quad (3)$$

whence

$$H_a = -2.04 \text{ kips}$$
$$V_a = 10.72 \text{ kips}$$
$$M_a = 10.31 \text{ ft.-kips}$$

TABLE 11-10

Given data				Properties of frame					M_p terms			
Point	$(L-x)$	y	$w = s/I$	$(L-x)w$	$(L-x)^2w$	yw	y^2w	$(L-x)yw$	M_p ft.-k.	M_pw	$M_p(L-x)w$	M_pyw
$\Sigma 1-5$	.....	......	24.68									
6	0.97	16.68	1.94	....	......	....	......	......	− 0.13	− 0.3	− 0.3	− 5.0
7	4.15	17.00	2.82	....	......	....	......	......	− 2.32	− 6.5	− 27.0	− 110.5
8	7.34	17.25	3.93	....	......	....	......	......	− 7.27	− 28.6	− 209.9	− 493.4
9	10.52	17.44	5.22	....	......	....	......	......	− 14.94	− 78.0	− 820.6	− 1,360.3
10	13.70	17.57	6.37	....	......	....	......	......	− 25.34	− 161.4	− 2,211.2	− 2,835.8
11	16.89	17.63	7.03	....	......	....	......	......	− 38.51	− 272.7	− 4,605.9	− 4,807.7
12	20.07	17.63	7.03	....	......	....	......	......	− 54.38	− 385.0	− 7,727.0	− 6,787.6
13	23.26	17.57	6.00	....	......	....	......	......	− 73.04	− 438.2	− 10,192.5	− 7,699.2
14	26.44	17.44	4.49	....	......	....	......	......	− 94.37	− 423.7	− 11,202.6	− 7,389.3
15	29.62	17.25	3.03	....	......	....	......	......	−118.44	− 358.9	− 10,630.6	− 6,191.0
16	32.81	17.00	1.93	....	......	....	......	......	−145.33	− 280.5	− 9,203.2	− 4,768.5
17	35.99	16.68	1.18	....	......	....	......	......	−174.86	− 206.3	− 7,424.7	− 3,441.1
$\Sigma 1-17$	.....	......	75.75	894.6	19,558	1,034.9	16,814	15,635	....	−2,640.1	− 64,255.5	−45,889.4
18	37.58	14.65	5.52	207.4	7,796	80.9	1,185	3,038	−190.66	−1,052.4	− 39,549.2	−15,417.7
19	37.58	10.95	5.52	207.4	7,796	60.4	662	2,271	−190.66	−1,052.4	− 39,549.2	−11,523.8
20	37.58	7.25	5.52	207.4	7,796	40.0	290	1,504	−190.66	−1,052.4	− 39,549.2	− 7,629.9
21	37.58	3.55	5.52	207.4	7,796	19.6	70	736	−190.66	−1,052.4	− 39,549.2	− 3,736.0
22	37.58	−0.15	5.52	207.4	7,796	− 0.8	0	− 31	−190.66	−1,052.4	− 39,549.2	+ 157.9
$\Sigma 18-22$	.....	.....	27.60	1,037.0	38,980	200.1	2,207	7,518	....	−5,262.0	−197,746.0	−38,149.5
Σ	.....	.....	103.35	1,931.6	58,538	1,235.0	19,021	23,153	....	−7,902.1	−262,001.5	−84,038.9

$\delta_{1-1} = \delta_{4-4} = \Sigma y^2w$
$\delta_{1-2} = \delta_{2-1} = \Sigma(L-x)yw$
$\delta_{1-3} = \delta_{3-1} = \Sigma yw$
$\delta_{1-4} = \delta_{4-1} = \Sigma y^2w$ for 18-22
$\delta_{1-5} = \delta_{5-1} = \Sigma(L-x)yw$ for 18-22
$\delta_{1-6} = \delta_{6-1} = \Sigma yw$ for 18-22

$\delta_{2-2} = \delta_{5-5} = \Sigma(L-x)^2w$
$\delta_{2-3} = \delta_{3-2} = \Sigma(L-x)w$
$\delta_{2-4} = \delta_{4-2} = \delta_{1-5}$
$\delta_{2-5} = \delta_{5-2} = \Sigma(L-x)^2w$ for 18-22
$\delta_{2-6} = \delta_{6-2} = \Sigma(L-x)w$ for 18-22
$\delta_{3-3} = \delta_{6-6} = \Sigma w$

$\delta_{3-4} = \delta_{4-3} = \delta_{1-6}$
$\delta_{3-5} = \delta_{5-3} = \delta_{2-6}$
$\delta_{3-6} = \delta_{6-3} = \Sigma w$ for 18-22
$\delta_{4-5} = \delta_{5-4} = \delta_{1-2}$
$\delta_{4-6} = \delta_{6-4} = \delta_{1-3}$
$\delta_{5-6} = \delta_{6-5} = \delta_{2-3}$

$\delta_{p-1} = 84{,}039$
$\delta_{p-2} = -262{,}002$
$\delta_{p-3} = -7{,}902$
$\delta_{p-4} = -38{,}150$
$\delta_{p-5} = -107{,}746$
$\delta_{p-6} = 5{,}262$

included here for the purpose of demonstration. The symbol w is used for s/I.

The sketches and the calculations that are given with Table 11-9 show the complete solution of the problem. The fact that H_a is negative shows that it acts toward the right instead of toward the left.

Problem 11-14. Assume that the bridge of Fig. 11-35 carries a uniformly distributed live load of 270 lb. per sq. ft. on the left-hand span—loaded length = 37.58 ft. Find the magnitudes of the six redundants.

If the structure is cut at A and E, the condition will be that which is pictured in Fig. 11-34(g). The origin will be taken at O [Fig. 11-34(g)]. It is on the axis of the center pier and at the level of A and E. The summations that have been made in Table 11-9 for the elastic properties of the leg and the deck can be used again. Notice that y is unchanged. If the center pier is omitted for the time being, $(h) = (c)$, $(i) = (d)$, $(j) = (e)$, $(k) = -(c)$, $(m) = -(d)$, and $(n) = -(e)$. The effect of the center pier can be included easily as a separate item. Therefore, in Table 11-10 the new calculations are shown, but the other summations are copied from Table 11-9.

The directions of the redundants are chosen as shown in Fig. 11-34(g) for the following reasons:

1. All six redundants bend the center pier in the same way, causing a positive bending moment in it because the sign convention that is used in this case assumes a moment to be positive if it causes compression in the outside of the legs, the top of the deck, and the right side of the center pier.
2. There will be less difficulty in combining the numerical values if the sign of the moment in the center pier is always positive.
3. The signs of all of the coefficients in the work equations will be positive.

Table 11-10 shows the detailed calculations for the coefficients of the work equations. The values of M_p are calculated as for a cantilever beam which is fixed at M. $M_p = 0$ for the legs.

From the summations in Table 11-10, the work equations can be set up as shown in Table 11-11, using the typical form as shown in Eqs. (11-11), and using the key to the coefficients as given under Table 11-10. The magnitudes of the redundants are given under Table 11-11. The diagram also shows the bending moments in the frame.

Problem 11-15. Find H_a, V_a, and M_a for the frame of Fig. 11-35 for a fall in temperature of 50°F. and for an equivalent shrinkage of 50°F. if $E = 432{,}000{,}000$ lb. per sq. ft. and $\omega = 0.000006$.

$$E\Delta L = E\omega tL = 432{,}000{,}000 \times 0.000006 \times 100 \times 37.58 = 9{,}700{,}000.$$

From Figs. 11-34(c), (d), and (e), and from Table 11-9, the work equations can be set up as follows:

$$\begin{aligned} 16{,}814H_a + 15{,}635V_a + 1{,}035M_a &= +E\Delta L = +9{,}700{,}000 \\ 15{,}635H_a + 19{,}558V_a + 895M_a &= 0 \\ 1{,}035H_a + 895V_a + 76M_a &= 0 \end{aligned}$$

$H_a = +6{,}920$ lb., $V_a = -2{,}640$ lb., $M_a = -63{,}100$ ft.-lb.

TABLE 11-11.—WORK EQUATIONS

Δx at A:

$$19{,}021H_a+23{,}153V_a+1{,}235M_a+2{,}207H_e+7{,}518V_e+200.1M_e = 84.039$$

Δy at A:

$$+58{,}538V_a+1{,}931.6M_a+7{,}518H_e+38{,}980V_e+1{,}037M_e = 262.002$$

$$= 7.902$$

Δx at E:

$$+1{,}235M_e = 38.150$$

Δy at E:

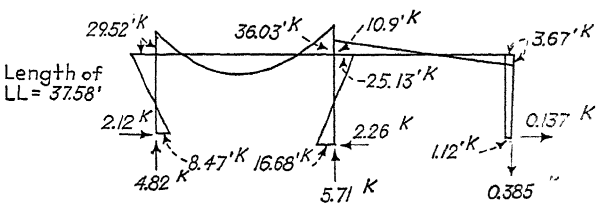
$$7{,}518H_a+38{,}980V_a+1{,}037M_a+23{,}153H_e+58{,}538V_e+1{,}931.6M_e = 197.746$$

$\Delta\theta$ at E:

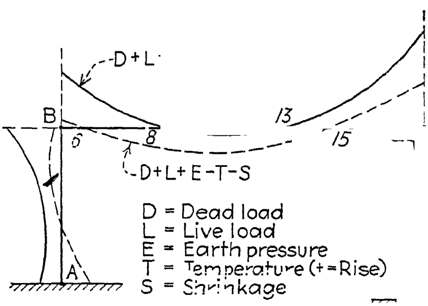
$$\ldots 3.35M_e = 5{,}262$$

$H_a = -2.117$ kips, $V_a = +4.815$ kips, $M_a = +8.470$ ft.-kips,

$H_e = -0.137$ kip, $V_e = +0.385$ kip, $M_e = -1.118$ ft.-kips



11-16. Variations in Bending Moments in Two-span Rigid Frames Due to Loading Conditions. Figure 11-36 pictures the



Diagrams of Max. M.

FIG. 11-36.

bending moments in the bridge of Fig. 11-35 for the extreme conditions. The full lines give the critical conditions at B and C, with live load on both spans; the dotted lines give the maximum moment at the crown, for which the live load is on span BC only.

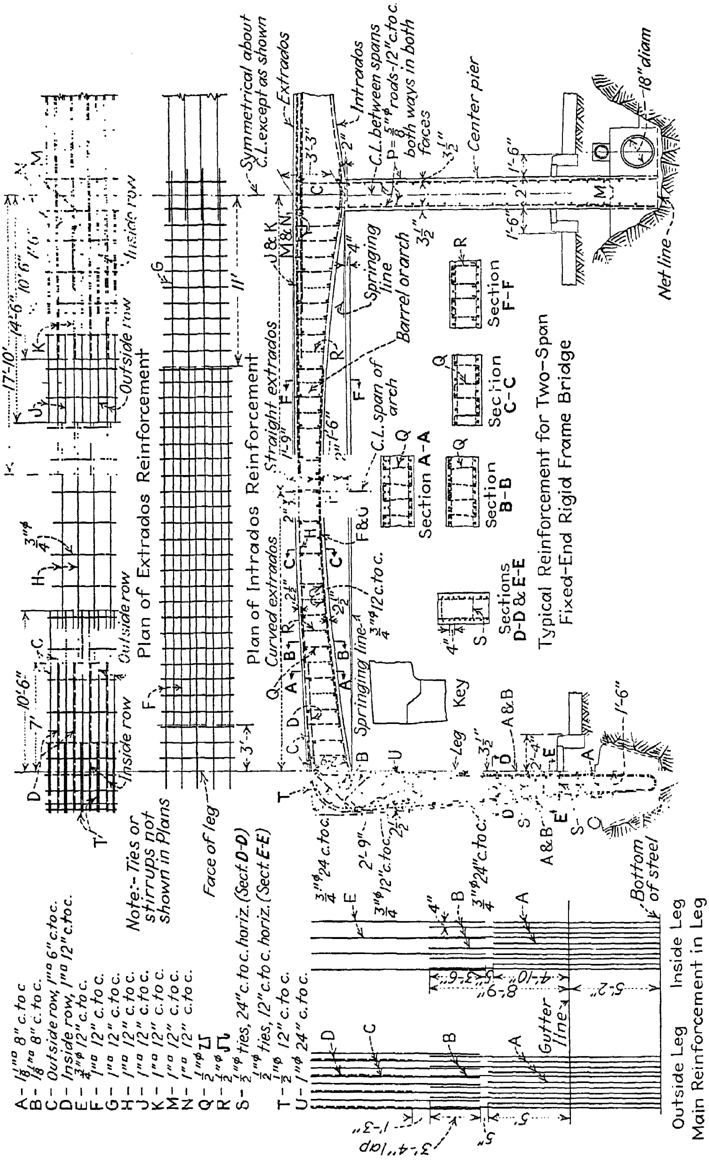

A- 1 1/8"□ 8" c. to c.
B- 1 1/8"□ 8" c. to c.
C- Outside row, 1"□ 6" c. to c.
D- Inside row, 1"□ 12" c. to c.
E- 3/4"φ 12" c. to c.
F- 1"□ 12" c. to c.
G- 1"□ 12" c. to c.
H- 1"□ 12" c. to c.
J- 1"□ 12" c. to c.
K- 1"□ 12" c. to c.
M- 1"□ 12" c. to c.
N- 1"□ 12" c. to c.
Q- 1/2"φ
R- 1/2"φ
S- 1/2"φ ties, 24" c. to c. horiz. (Sect. D-D)
1/2"φ ties, 12" c. to c. horiz. (Sect. E-E)
T- 1/2"φ 12" c. to c.
U- 1"φ 24" c. to c.
Note:- Ties or stirrups not shown in Plans
Face of leg
Plan of Extrados Reinforcement
Inside row
Outside row
Plan of Intrados Reinforcement
Curved extrados
Straight extrados
Symmetrical about C.L. except as shown
Extrados
Intrados
C.L. between spans
P = 5/8"φ rods-12" c. to c. both ways in both faces
Center pier
Springing line
Barrel or arch
C.L. span of arch
Key
Leg
Section A-A
Section B-B
Section C-C
Section F-F
Sections D-D & E-E
Typical Reinforcement for Two-Span Fixed-End Rigid Frame Bridge
18" diam
Net line
Gutter line
Bottom of steel
3'-4" lap
Outside Leg
Inside Leg
Main Reinforcement in Leg

Fig. 11-37.

The alternation of the stresses in the lower part of the leg, portions of the deck, and the center pier should be noted. The first is relatively severe. It is especially undesirable to have a real reversal of stress at *B*.

One should remember that the heavier and stiffer he makes a rigid frame the more severe will be the stresses in it due to temperature and shrinkage. Rapid changes in the moment of inertia of a structure are likely to cause cracking. There is a reasonable proportion for the members which one learns through experience.

11-17. Discussion of Practical Details. There are a great many practical details in rigid frames which are very important. A few such things will be included in a brief outline of the reasons for some of the details that are pictured in Fig. 11-37. The two-span frame is used because it involves more difficulties.

1. The pouring procedure probably would be as follows:

a. First Pour. The footing of the center pier below *M*, with key at *M*.

b. Second Pour. The footing of the leg below *A*, with a strong keyway to resist the horizontal thrust and provision to resist twisting.

c. Third Pour. The center pier from *M* to *C*, with key at *C*.

d. Fourth Pour. The leg from *A* to *B* with a special keyway at *B* to resist all thrusts (vertical and horizontal). This joint is shown in Fig. 11-38.

e. Fifth Pour. The entire deck. If absolutely necessary, a vertical construction joint could be made over the middle of the center pier so as to pour the decks of the two spans separately.

2. The fillet at *B* is small, but it helps to decrease the stresses in the concrete at the reentrant corner.

3. Rods *A* and *B* are alternated so as to extend the latter to lap with rods *C* and *E*.

4. Rods *D* are placed inside or under rods *C* so as to keep the spacing in multiples of 6 in. and to get a better arrangement with more clear space for placing of concrete than could be done with closer spacing. This is shown in Fig. 11-39.

5. Rods *E* and *P* are hooked into the deck so as to reinforce the reentrant corners against opening.

6. Rods *F* and *G* are staggered so as to reduce the lengths of individual rods, yet to provide 6- or 12-in. spacing at all points.

7. Rods *H* are placed in the middle of the 6-in. spaces between rods *C* so as to get adequate bond and not to interfere with the regularity of the spacing of *J*, *K*, *M*, and *N*, which are arranged so as to increase the area of steel as required.

8. Rods *Q* should be about $\frac{1}{2}$-in. round. They are looped under the bottom rods and over the top ones, as pictured in Figs. 11-39 and 11-40, for the following purposes:

a. To provide web reinforcement with proper anchors.

b. To provide struts to support the weight of the top rods by bearing upon the bottom ones, these last being held off from the forms by means of wire "chairs" or precast concrete blocks.

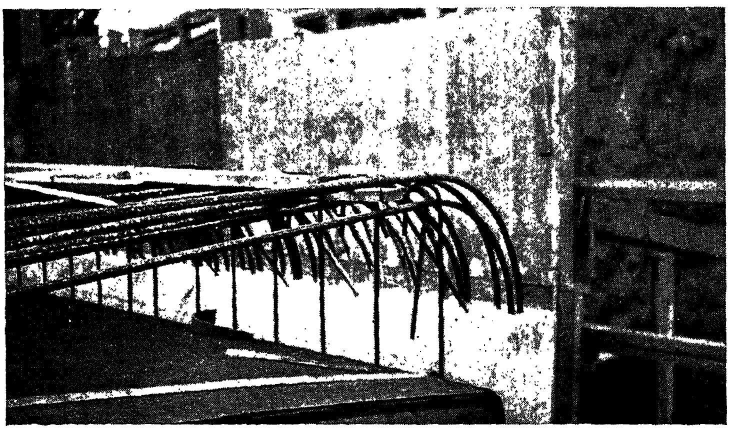


FIG. 11-38.—Keyway and tensile steel at knee of rigid-frame bridge.

c. To prevent the tension that is carried by the curved bottom rods from straightening them out and spalling the concrete, also to prevent compression in the curved rods *C* and *D* from causing them to buckle.

9. Rods *R* are used as extra web reinforcement. A glance at Fig. 11-36 will show that the location of the point of contraflexure or inflection varies widely for the two extreme conditions of loading. This means that, between points 6 and 8, also 13 and 15, the concrete sometimes tends to crack on top; at other times it tends to crack on the bottom. This is a possible cause of serious trouble. Therefore, the web reinforcement in these regions is made capable of withstanding the entire shear at about 25,000 lb. per sq. in. in the rods.

Rods S in the leg serve somewhat the same purpose.

10. Rods T are used merely to avoid excessive cracking of the outer corner, with rupturing of the waterproofing.

11. Rods U are special web reinforcement for the keyway at B.

12. The longitudinal steel for temperature and shrinkage reinforcement is about 0.25 per cent of the cross section of the frame, about two-thirds being placed near the inside face.

13. At first glance, the footing for the leg may not seem to be fixed, but it is really locked in position by the shearing forces which can be developed at the junction between the concrete and the rock. The center footing is fixed similarly because the concrete is poured against the rock even when the latter breaks badly as a result of the blasting.

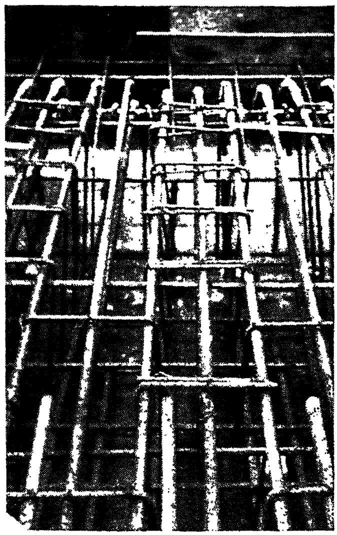


FIG. 11-39.—Deck reinforcement near knee of rigid-frame bridge.

Other footings are shown in Fig. 11-41. Sketch (*a*) is one type of hinged end on rock. Beveled steel bearing plates are used in order to insure an effective hinge action. Sketch (*b*) shows a simple bearing on earth. The bottom of the leg rests in a keyway with some yielding material at its bottom. This keyway might be semicircular, but such refinement is questionable. The dowels are primarily for the purpose of anchors to which the leg reinforcement can be wired during construction to avoid its displacement during the pouring of the concrete. These dowels provide only a slight restraint which can do little harm. Sketch (*c*) shows a simple footing of the center pier on rock.

The parapets of rigid-frame bridges need special attention. Figure 11-42(*a*) shows the construction that was used in the bridge of Fig. 11-33. The parapet was poured after the centering was "struck." It is reinforced as a vertical cantilevered beam, but it also utilizes heavy prestressed rods whose purpose is the avoidance of cracks due to temperature, shrinkage, and partici-

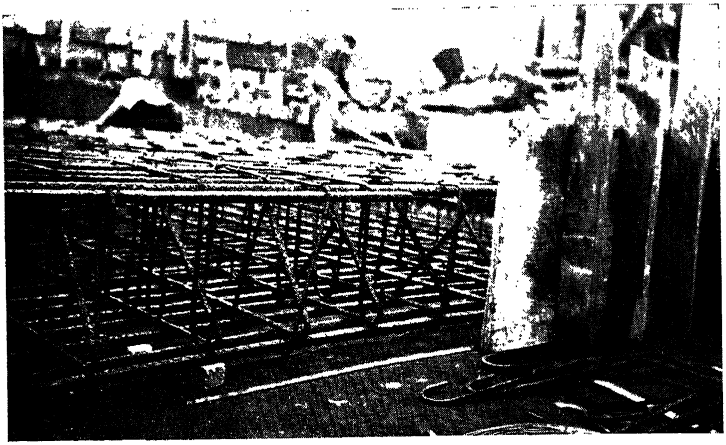


FIG. 11-40.—Details of reinforcement in deck of rigid-frame bridge.

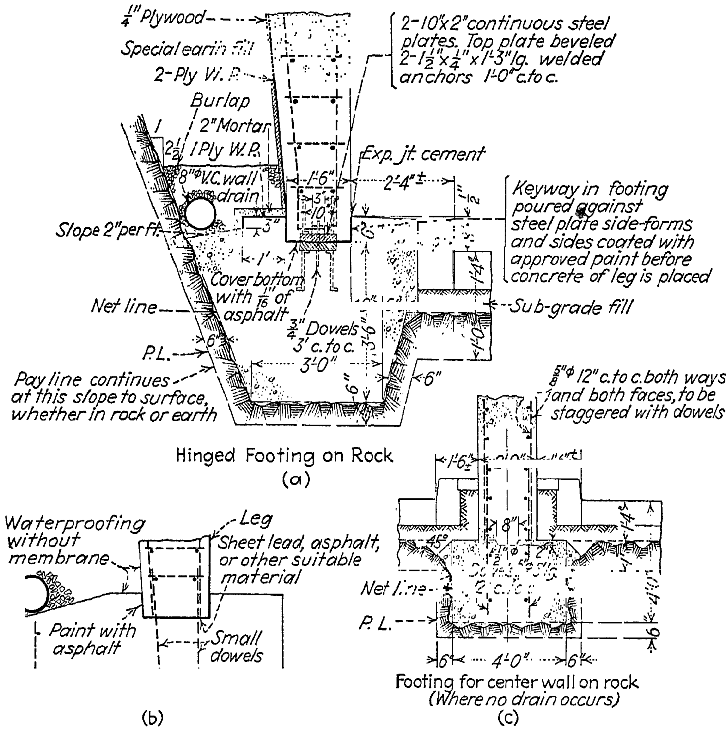


FIG. 11-41.

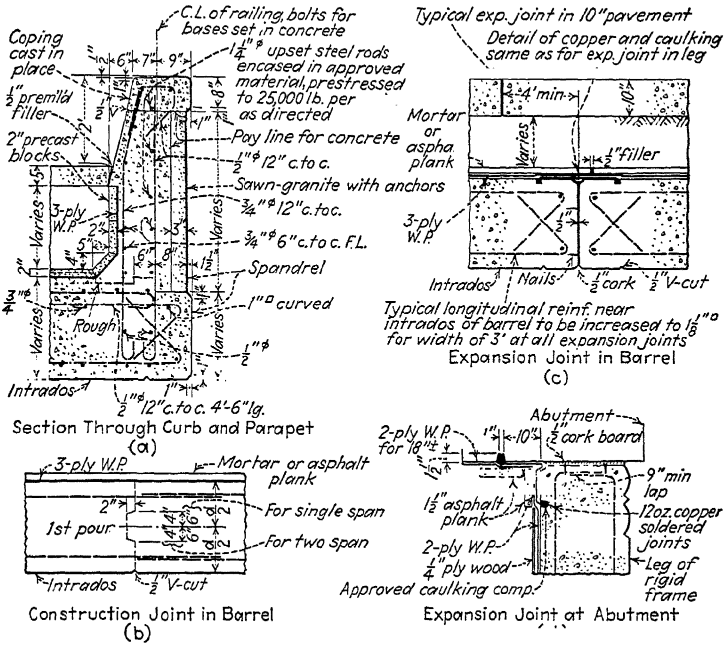


FIG. 11-42.

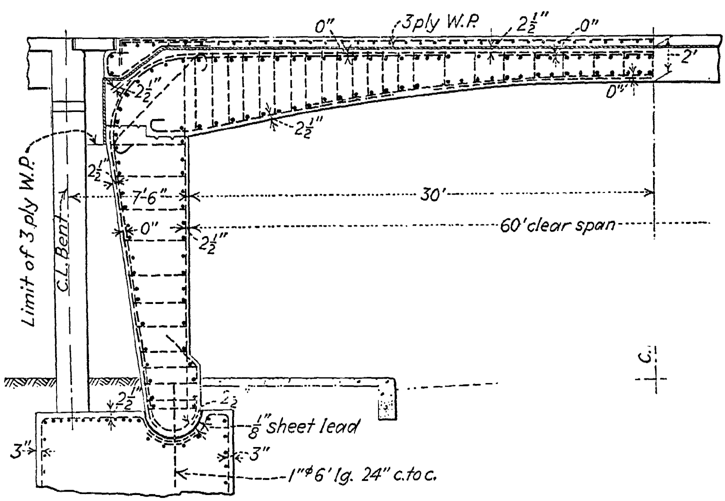


FIG. 11-43.—Rigid-frame bridge, part of approach to Meeker Ave. Bridge, New York City. (*Courtesy of Dept. of Public Works, New York City.*)

pation of the parapet in the frame action of the structure. The parapets of the bridge in Fig. 11-1*A* are built upon a different principle. The posts are cantilevered from the deck, but the panels are isolated units which are held in keyways along the sides of the posts but have expansion joints which prevent the parapets from acting with the frame.

Figures 11-42(*b*), (*c*), and (*d*) picture construction and expansion joints which were used in the bridge of Fig. 11-33. Figure 11-43 pictures a single-span bridge which is part of a grade-separation project that has been designed by the Department of Public Works of New York City. It illustrates a case in which there is no backfill behind the legs of the structure.

CHAPTER 12

ARCHES

12-1. Introduction. Arches possess inherent beauty as well as practical utility. Figures 12-1 and 15-2 (Chap. 15) are illustrations of two such structures. Although they are widely different in character, each possesses a beauty of its own. Figure 12-1 is a picture of an arch over Riverside Drive in New York

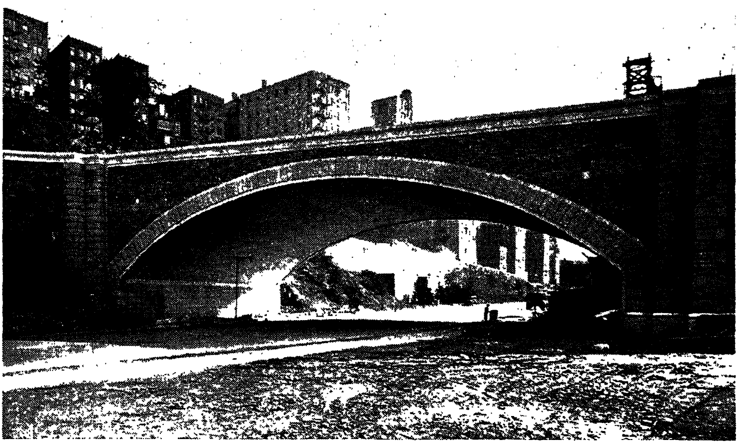


FIG. 12-1.—Arch over Riverside Drive, New York approach to the George Washington Bridge.

City, the photograph being taken when the bridge was nearing completion. This structure is part of the approach to the George Washington Bridge. Figure 15-2 shows the Dykman Street Bridge at New York City. This is part of the Henry Hudson Parkway.

The design of reinforced-concrete arches is based upon the same general principles as those which have been demonstrated in the preceding chapter on rigid frames, viz., the theory of work.

There are variations in the details of the methods that may be used, but it is the purpose of this chapter to develop a simple, understandable procedure which will facilitate practical designing.

12-2. The Fundamental Action of an Arch. The fundamental nature of an arch is illustrated in Fig. 12-2 as follows:

1. Sketch (*a*) shows a fixed-end beam which has varying depths of cross section but whose neutral plane (or axis AB) is horizontal. The load P will cause bending moments in this beam; it also causes vertical reactions.

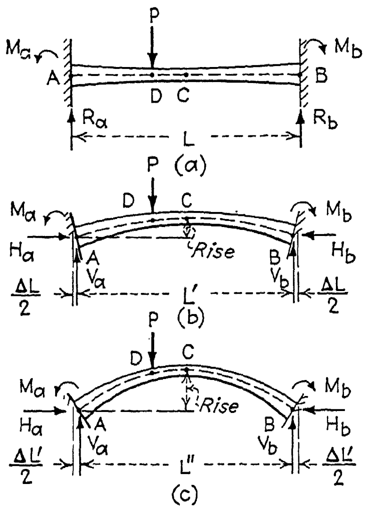


FIG. 12-2.

2. Next, suppose that this beam is bowed upward so that the point C rises, AB is curved, and the supports A and B are rotated. Then the horizontal projection of the arch is shortened by a distance ΔL because the length of ACB is unchanged. When the load P is applied now, AB bends, but it also tends to flatten out again, causing horizontal pressures at A and B along with the vertical reactions. The structure is therefore subjected to a longitudinal compressive force which did not exist before. If the curvature is very slight, the member is still primarily a beam; but when the rise is large, as in (*c*), the effect of the horizontal thrust becomes very important. Then such a structure is called an "arch."

The great economy of an arch is due to the fact that, because of its curvature, the horizontal components of the reactions cause bending moments which relieve those that would exist in the structure if it were an ordinary beam. Of course, an ideal arch is one in which the loads cause little or no bending moments. In such a case, the concrete of the entire cross section is available to resist compression instead of having the design controlled by the large compressive stresses at one edge, as in a beam.

The differences in the deflections of a simply supported, curved beam and a hinged-end arch are pictured in Fig. 12-3. It is easy to see that the strength of (*a*) depends upon its ability to

resist bending but that that of (*b*) depends upon both its beam action and the resistance that is due to the direct compressive forces. The dotted lines show the deformed positions of both structures.

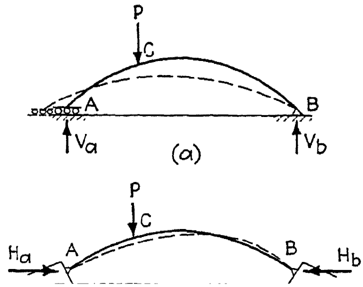


FIG. 12-3.

12-3. Parts and Kinds of Arches. Some of the terms that denote parts of an arch are shown in Fig. 12-4. The ends may be fixed or hinged at the abutments, depending upon the arrangement of the details. When both ends are fixed and the arch is continuous, it is called "hingeless"; when both ends are hinged and the arch is continuous, it is "two hinged"; when there are hinges at both ends and at the center, it is "three hinged."

When an arch has columns or cross walls which carry the roadway structure and which rest upon the extrados of the barrel, as pictured in Fig. 12-4, it is called an "open-spandrel" arch; when the pavement is supported upon earth fill which bears

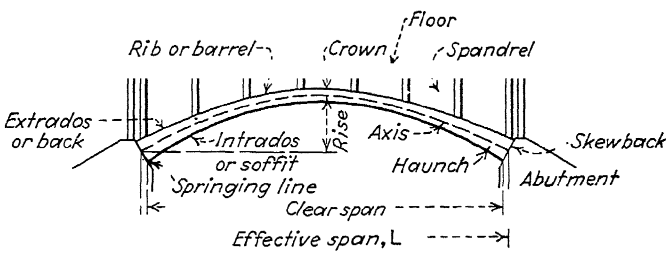


FIG. 12-4.

directly against the extrados, the structure is called a "filled-spandrel" arch.

Furthermore, some engineers refer to arches in terms that denote the kind of curve to which the intrados is laid out, such as semicircular, segmental, three-centered, parabolic, elliptical.

12-4. Determination of Curvature of Arch Axis. Most arches have variable depths of section as shown in Fig. 12-4. The resultant variation in I causes complications in the calculations

which are similar to those that occur in the analysis of rigid frames. Therefore, since the design of an arch is really the analysis of a tentative structure, it is very important to choose the right shape for the axis—also to find satisfactory thicknesses for the sections—before a refined calculation is made.

The following formulas will be useful in assisting in the choice of the proper curve for the axis:[1]

For open-spandrel arches:

$$y = \frac{8rL}{6 + 5r}(3c^2 + 10c^4 r)$$

$$\tan \theta = \frac{8r}{6 + 5r}(3 + 5r).$$

For filled-spandrel arches:

$$y = \frac{4rL}{1 + 3r}(c^2 + 24c^5 r)$$

$$\tan \theta = \frac{4r}{1 + 3r}(1 + 7.5r).$$

The meanings of the terms used in these formulas are given in Fig. 12-5. The rise ratio r is h/L.

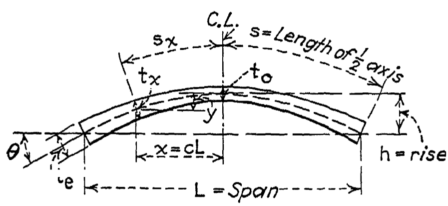


FIG. 12-5.

Of course, the rise and the span of any arch are generally controlled, or at least greatly influenced, by the conditions at the site; but when r is small the horizontal thrusts and the bending moments are much greater than when r is large. This affects the economy of construction in many cases.

For full uniform loads alone, a parabolic curve will result in an arch that has no bending in it. However, partial loadings gen-

[1] Taken from Victor H. Cochrane, Design of Symmetrical Hingeless Concrete Arches, Engineers' Society of Western Pennsylvania, *Proc.*, vol. 32, No. 8.

erally cause greater bending moments than full loadings of equal intensity.

12-5. Determination of Thickness of Arch Ring. At best, any formula for arch thicknesses is only a guide for the designer. The strength of the concrete, the length and rise of the arch, the magnitudes of the loads and their distribution—all of these things must be considered. Various formulas have been developed by engineers; but of course the best guides are the data that can be secured from some other arch that is safe, attractive, economical, and also comparable to the proposed structure. In any case, good judgment is needed.

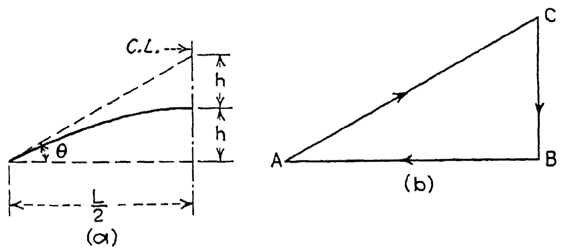


FIG. 12-6.

A seemingly rational approximation for a symmetrical, fixed-end arch can be made as follows:

1. Assume a longitudinal strip of the arch 1 ft. wide. Make a rough estimate of the total dead load and live load for one-half of this strip.
2. Assume that the arch axis is a parabola, in which case the tangent of the inclination of the axis at the skewback is twice the rise divided by half of the span, as shown in Fig. 12-6(*a*).
3. Lay off the entire load from step 1 as the vertical line CB of Fig. 12-6(*b*); draw AB horizontally; draw AC parallel to the tangent as determined in step 2.
4. Assume the safe working stress f_c of the concrete to be used; scale off AB and AC from Fig. 12-6(*b*); divide AB by about $0.4f_c$ to find the trial area at the crown; divide AC by about $0.3f_c$ to determine the trial area at the skewback; then divide both by 12 in. to obtain the tentative depths of the two sections t_o at the crown and t_e at the end (in inches).

After t_o and t_e are chosen, the depths of t_x of Fig. 12-5 at intermediate points may be tentatively determined by using the curves of Fig. 12-7 which have been prepared by using Cochrane's data. The axis, intrados, and extrados of the arch should then

be drawn very carefully to large scale, making sure that the curves are smooth.

12-6. Preliminary Test of an Arch. After the trial shape of an arch has been chosen, it is advisable to test it in a preliminary way before making a detailed analysis. To do this, draw the outline of the rib as in Fig. 12-8(a), which pictures half of a

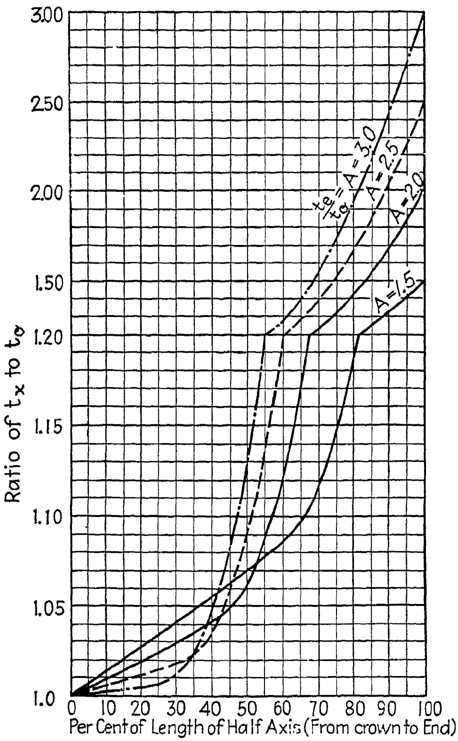


FIG. 12-7.—Variation in thickness of half of the barrel of a symmetrical arch.

symmetrical arch. Then test it by constructing the line of pressure somewhat as for a stone-masonry arch. The procedure for a filled-spandrel arch is as follows:

1. Assuming a 1-ft. strip, draw the half rib as in Fig. 12-8(a), using a large scale; convert the earth fill into an equivalent depth of concrete, where $CF = \frac{100}{150} \times$ depth of earth; convert the pavement into concrete also; do likewise for one-half of the average uniform live load over the entire structure; IJ is then the "reduced-load contour."

2. Divide the half span into any reasonable number of equal spaces; then project lines upward to divide the figure into sections like $MRQK$; find the area of each division; locate each center of gravity, laying off $KL = LM$, $QP = PR$, $MN = RQ$, $QS = KM$, and intersecting NS with LP at O; find the weight of each division at 150 lb. per cu. ft.

3. Construct the force diagram for the forces as in Sketch (*b*), using a trial pole at O'; starting at the center of the arch in Sketch (*c*), draw the funicular polygon from A to B, then to C which locates the resultant of the forces; since B is outside the arch ring, draw DE from the center of the crown, parallel to CB, locating E on the resultant; in Sketch (*b*), draw a new ray SO parallel to EA of Sketch .(*c*), locating a new pole at O; draw a new

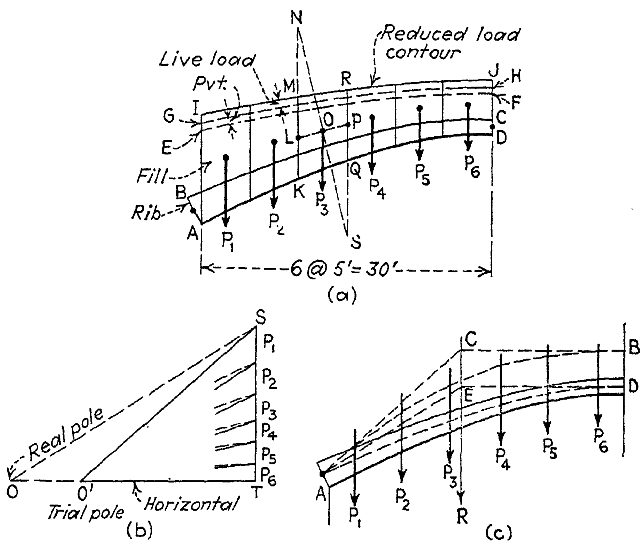


FIG. 12-8.

force diagram and funicular polygon. If the new polygon closely follows the arch axis, it generally indicates that the assumed arch is reasonably satisfactory; otherwise a change of shape or of depths is necessary.

4. Scale OT of Fig. 12-8(*b*); apply it as a direct thrust on the crown section as plain concrete. If the unit stress does not exceed 0.4 of the allowable working stress, there should be enough reserve strength to take care of the bending moment at the crown.

12-7. Theory of Arch Analysis in General. The elastic theory will be used in the study of the action of an arch when it is subjected to bending. This theory assumes that the stresses and strains are proportional to each other; i.e., E is a constant, and

the stress at any point in the cross section of the arch equals $\frac{P}{A} \pm \frac{Mc}{I}$. Furthermore, it is assumed that the distribution of stresses in the arch itself due to direct forces and bending moments will be the same as in straight members.

If the fixed-end arch of Fig. 12-2(c) has a load P applied at D, there will be horizontal and vertical reactions at A and B; there will be restraining moments at these points also. Therefore, the structure will be statically indeterminate to the third degree. In fact, the arch may be looked upon as a rigid-frame bridge with large curvature and with legs of zero length.

There are various ways of "cutting" the structure so as to solve for the three redundants. The method of attack, if the calculations are correct, should have no material effect upon the results. It must be remembered also that any method of analysis is merely an attempt, through mathematical processes, to determine in advance what the stresses in the actual structure will be for any given conditions of loading, doing so with sufficient accuracy for practical purposes.

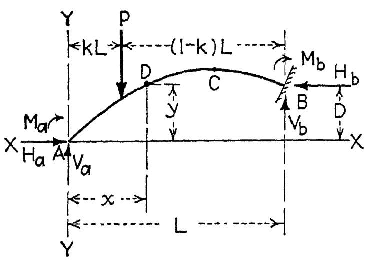


FIG. 12-9.

As a very general case, assume an unsymmetrical arch as in Fig. 12-9; cut it at the left support; call H_a, V_a, and M_a the three redundants. The structure is now statically determinate, being a curved cantilever beam which is fixed at B. For convenience and clarity, assume the origin of coordinates at point A.

By applying a unit or dummy load at A, acting horizontally toward the right, Eq. (11-3) of the preceding chapter gives

$$\Delta x = \int \frac{M_p m_q \, ds}{EI} = \frac{Mys}{EI} \tag{12-1}$$

where Δx = the horizontal deflection of point A with respect to B, M = the combined bending moment at D due to the loads and redundants, $y = -1(y)$, or the bending moment at D due to the dummy load at A, the sign being minus because it causes tension

in the top fibers; and $\sum \frac{s}{EI}$ is used instead of $\int \frac{ds}{EI}$. Similarly,

$$Mxs \tag{12-2}$$

Then, using Eq. (11-6) of the preceding chapter and a unit moment at A,

$$= \sum \frac{Ms}{EI}. \tag{12-3}$$

The general expression for M at any point is

$$M = M_a + V_a x - H_a y - M_p \tag{12-4}$$

where $M_p = P(x - kL)$, or similar cantilever moments which are caused by any external loads. The minus sign indicates tension in the top fibers.

When Eq. (12-4) is substituted in Eqs. (12-1), (12-2), and (12-3), they become

$$\Delta x = -\sum (M_a + V_a x - H_a y - M_p)\frac{ys}{EI}. \tag{12-5}$$

$$\Delta y = \sum (M_a + V_a x - H_a y - M_p)\frac{xs}{EI}. \tag{12-6}$$

$$\Delta \theta = \sum (M_a + V_a x - H_a y - M_p)\frac{s}{EI}. \tag{12-7}$$

These are three fundamental equations for the arch in terms of the left reactions, the left end moments, the cantilever beam moments, and the properties of the structure as represented by ys/EI, xs/EI, and s/EI. They can be solved for H_a, V_a, and M_a for any given condition of loading by placing Δx, Δy, and $\Delta \theta$ each equal to zero and summing up from A to B, because there is actually no change in x, y, or θ at A with respect to B, since the arch is fixed. Therefore, the bending moments at any point in the structure can be calculated for this particular loading. However, the numerical work involved in this method of analysis is very tedious.

12-8. Elastic Center. The analysis of an arch can be greatly simplified by using a special origin of coordinates.

If the rib is divided into any given number of sections, if E is canceled out because it is constant, if the s/I values (elastic

weights) of these divisions are computed, if the center of gravity of these elastic weights is calculated with respect to any axes by assuming each elastic weight to be concentrated at the center of gravity of its particular division; then this center of gravity of the entire structure is called the "elastic center," or "centroid." For instance, Fig. 12-10(a) shows an unsymmetrical arch. If the original axes are X'-X' and Y'-Y' through A, the elastic center will be found to be at O, a distance $n = \Sigma y'w/\Sigma w$ above X'-X' and a distance $m = \Sigma x'w/\Sigma w$ to the right of Y'-Y'. Of course, any axes can be assumed for determining O. ($w = s/I$.)

There are various ways in which an arch can be divided into sections for the purposes of analysis, but it seems to be best

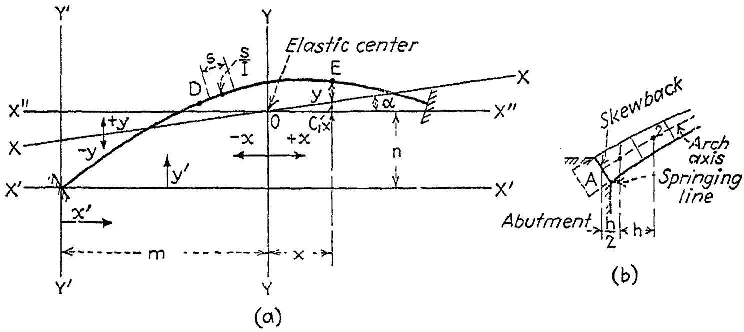


FIG. 12-10.

to use portions that have equal horizontal projections. The end sections should start at the skewback as shown in Fig. 12-10(b) in which A = the intersection of the arch axis with this skewback. Obviously, the real effective span of the structure will start from a point that lies between A and the springing line so that this assumption for the first division is more correct than would be the use of point A as the center of a section, the left part of which is shown by the dotted lines. Furthermore, it is sufficient to assume that the center of gravity of a section is at the mid-point of the horizontal projection of its axis.

If the arch is a semicircular one, or if the intrados is elliptical with a relatively large rise, it may be advantageous to divide the arch into sections along its axis. In such a case, the general procedure for the analysis will be similar to that which will be illustrated herein. (This also applies to rigid frames.)

Next, by using the elastic center as the origin of coordinates for new axes X''-X'' and Y-Y, $\Sigma ys/I = 0$ and $\Sigma xs/I = 0$. It is now possible to use these two axes in the analysis of the arch, but it is desirable to find the special axes about which $\Sigma xys/I$ will also be zero. To do this, assume the Y-Y axis to be vertical through O, with $+x$ measured to the right; assume the X-X axis to be inclined at some angle α, with $+y$ measured vertically upward, as in Fig. 12-10. The new ordinate to any point E from X-X will be

$$y = y' - n - C_1x$$

where y' = the original ordinate from X'-X', and $C_1 = \tan\alpha$. Furthermore, the new abscissa to E will be

where x' = the original abscissa from Y'-Y'. Then, substituting w for s/I and the new values of x and y in the expression $\Sigma xys/I$ and equating it to zero gives

$$\begin{aligned} \Sigma xyw &= 0 \\ \Sigma(x' - m)(y' - n - C_1x)w &= 0 \\ \Sigma(x' - m)[y' - n - C_1(x' - m)]w &= 0. \end{aligned}$$

Expanding this and grouping the terms gives

$$= -\Sigma x'y'w + m\Sigma y'w +$$

However, the last two terms cancel out, since the sum of the moments of each of the separate elastic weights about Y'-Y' = the sum of all of the elastic weights times m, the distance to the center of gravity (elastic center). Therefore,

$$\Sigma x'y'w - m\Sigma y'w \qquad (12\text{-}8)^1$$

This enables one to find C_1 in terms of the original coordinates.

Obviously, for a symmetrical arch, $m = L/2$ and $C_1 = 0$.

[1] Equation (12-8) may be simplified to the form

$$C \qquad \Sigma x'y'w - m\Sigma y'w$$

Problem 12-1. Assume the unsymmetrical arch of Fig. 12-11. Find the elastic center and the location of the inclined axis X-X, using the data that are given in columns 2 to 5, inclusive (Table 12-1).

TABLE 12-1.—CALCULATIONS FOR ELASTIC CENTER

Given data					Derived data				
								10	11
Point	s, ft.	t, ft.			$x'w$	$y'w$	$x'y'w$		y'^2w
	7.53	3.64		.34	5.6	4.4	13	17	10
	7.26	3.00		.66	29.1	21.5	194	262	143
	7.00	2.54	15	.50	77.0	53.9	808	,154	566
	6.76	2.26	21	.86	147.6	97.4	2,046	,100	1,350
	6.56	2.12	27	16.74	223.0	138.3	3,733	,022	2,315
	6.38	2.07	33	19.	284.8	165.2	5,451	,398	3,162
	6.23	2.03	39	21	348.3	188.1	7,335	,583	3,961
	6.12	2.01	45	22	406.8	203.4	9,153	,306	4,577
	6.04	2.00	51	23	462.1	212.5	10,840	,565	4,985
10	6.00	2.01	57	23	505.6	212.3	12,104	,819	5,084
11	6.00	2.02	63	23	550.6	209.2	13,182	,689	5,009
12	6.04	2.05	69	23	580.3	197.3	13,614	,040	4,629
13	6.12	2.10	75	22	594.8	178.4	13,382	,606	4,015
14	6.23	2.20	81	21	568.6	147.8	11,975	,058	3,114
15	6.38	2.35	87	19.14	513.3	112.9	9,825	,657	2,161
				108.05	5,297.5	2,142.6	113,655	314,276	45,081

$$- = \frac{5{,}298}{108.05} \quad 49.03 \text{ ft.} \qquad n \quad \frac{\Sigma y'w}{\Sigma w} = \frac{2{,}142.6}{108.05} \quad 19.83 \text{ ft.}$$

C_1 $\quad \Sigma x'y'w -$

$$\frac{113{,}655 - 49.03 \times 2{,}142.6}{49.03^2 \times 108.05 - 2 \times 49.03 \times 5{,}297.5 + 314{,}276} = 0.1577.$$

Take the initial origin of coordinates at A in order to make the work as clear as possible and because it will be useful in subsequent problems. Divide the span of the arch into 15 equal parts 6 ft. long. By projecting these vertically upward to the axis, the arch will be divided into 15 pieces of equal horizontal projection. The lengths of these pieces, their average thicknesses, and the coordinates of their centers may be scaled from a large drawing, or else they may be calculated. These data for this case are recorded in columns 2 to 5, inclusive, Table 12-1.

The magnitudes of the elastic weights of the divisions are the s/I values. Calculate them for a strip of the arch 1 ft. wide. These w values are in terms of 1/ft.³ and are given in column 6 of Table 12-1.

The moments of the elastic weights about the axes through A of Fig. 12-11 must be computed accurately. The results are given in columns 7 and 8

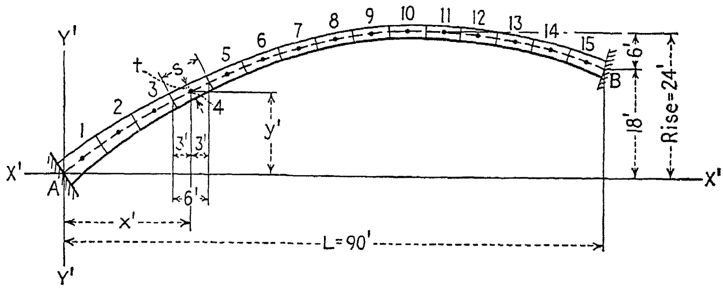


FIG. 12-11.

of Table 12-1. Each set is added up and divided by Σw to find m and n. The elastic center is then located as in Fig. 12-10.

The effect of the reinforcement upon I is neglected. The justification for doing so is shown in Fig. 12-12A which gives curves for the moments of

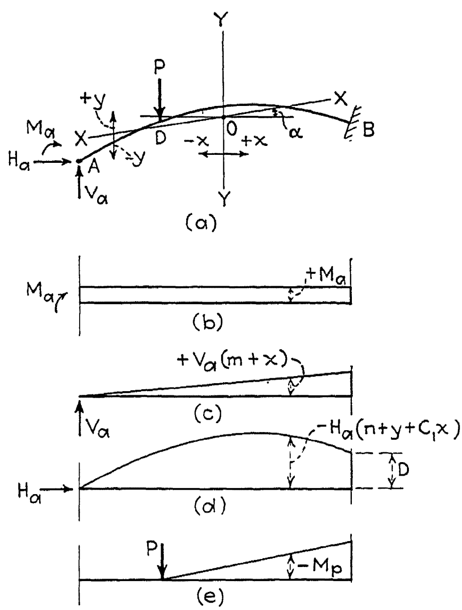


FIG. 12-12.

inertia of the arch of Fig. 12-1 with and without the rods. The proportionality of the two is obvious.

The calculations for m and n should be made with considerable accuracy, because these quantities will be multiplied by numbers of relatively large magnitude.

The next step is the calculation of $C_1 = \tan \alpha$, using Eq. (12-8). The magnitudes of $x'y'w$ and x'^2w are given in columns 9 and 10 of Table 12-1. The value of C_1 is then calculated as shown under the table.

Column 11 gives the magnitudes of y'^2w. These values will be used in problems in Arts. 12-11 and 12-12.

12-9. Influence Lines for Redundants. Equations (12-1), (12-2), and (12-3), which are fundamental ones, should be modi-

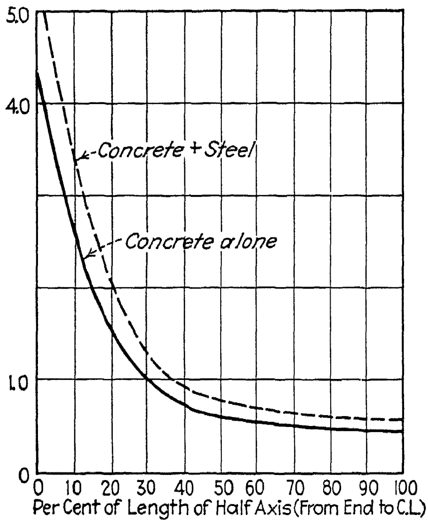


FIG. 12-12A.—Magnitude of I for half of the barrel of a symmetrical arch.

fied by shifting the axes of coordinates to the elastic center O [Fig. 12-12(a)], the axes being X-X and Y-Y, about which $\Sigma xw = 0$, $\Sigma yw = 0$, and $\Sigma xyw = 0$.

The arch is still "cut" at A. A load P is applied at D. The bending-moment diagrams for M_a, V_a, H_a, and P in terms of the new coordinates are pictured in Figs. 12-12(b) to (e), inclusive. The new equations for Δx, Δy, and $\Delta\theta$ for any point can be found from Eqs. (12-1), (12-2), and (12-3) as follows, omitting E:

$\Delta x = -\Sigma My'w$, $\Delta y = \Sigma Mx'w$, and $\Delta\theta = \Sigma Mw$ when the axes pass through A. For the new axes through O,

$x' = m + x$, and $y' = n + y + C_1x$, when x and y are measured as shown in Fig. 12-12(a).

The actual deformations of A with respect to B are zero. Then

$$\begin{aligned}
\Delta x &= -\Sigma M(n + y + C_1 x)w = n\Sigma Mw + \Sigma Myw + C_1\Sigma Mxw = 0.^*\\
\Delta y &= \Sigma M(m + x)w = m\Sigma Mw + \Sigma Mxw = 0.\\
\Delta\theta &= \Sigma Mw = 0.
\end{aligned}$$

Since $\Sigma Mw = 0$ in the preceding equations, they become

$$\left.\begin{aligned}
\Delta x &= \Sigma Myw + C_1\Sigma Mxw = 0.\\
\Delta y &= \Sigma Mxw = 0.\\
\Delta\theta &= \Sigma Mw = 0.
\end{aligned}\right\} \tag{12-9}$$

Then, using Fig. 12-12(b), etc.,

$$M = +M_a + V_a(m + x) - H_a(n + y + C_1 x) - M_p. \tag{12-10}$$

Substituting this value of M in Eqs. (12-9) and solving them gives

$$M_a = \qquad nH_a - mV_a. \tag{12-13}$$

For a symmetrical arch, the foregoing equations remain unchanged except for the omission of C_1H_a of Eq. (12-12).

Equations (12-11), (12-12), and (12-13) are well adapted to the use of influence lines. The quantities Σy^2w, Σx^2w, and Σw are constants for any given structure. It then remains to find ΣM_pyw, ΣM_pxw, and ΣM_pw.

The principle of a beam with loads equal to the elastic weights can be used in the construction of the influence lines for the foregoing quantities, somewhat as for the rigid frame of the preceding chapter. It should be noted that w, xw, or yw may be used as elastic weights, depending upon the particular problem.

In this case, the "cut" arch is a cantilever beam which is supported at B [Fig. 12-13(a)]. Therefore, assume a straight cantilever beam of span L which is fixed at the opposite end, as in Sketch (b); load this beam with the elastic weights yw, xw, or w, as the case may be, considering their sign; calculate the cantilever bending moment at any point in this imaginary beam

* The negative sign can be dropped because $\Delta x = 0$.

for the loads on the right-hand side of that point; then this moment, divided by the proper constant, is the ordinate of the desired influence line at that particular point, this ordinate giving the magnitude of H, V, or M at the elastic center. It is important to remember that M_p causes tension in the top fibers of the cut arch. Therefore, any elastic weight with a positive sign must be acting downward.

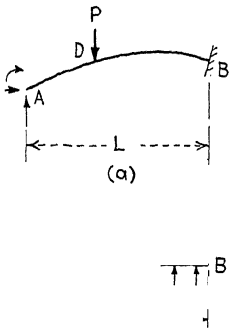


FIG. 12-13.

This principle of a beam with elastic weights is so important, and it is so useful, that it will be demonstrated further.

Assume that H_a is required for the arch of Fig. 12-14(a). By Eq. (12-11),

$$H_a = -$$

The bending moment due to the load P at D is shown in Sketch (b) where h = the length of the equal horizontal projections of the divisions. The bending-moment diagram is cut into strips corresponding to the arch sections; the bending moments at the centers of these pieces are calculated; then $\Sigma(M_p)(yw) \div \Sigma y^2 w$ may be written thus:

$$H_a = -$$

$$\Big]_8 \div$$

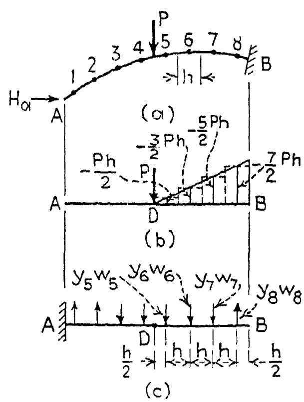


FIG. 12-14.

Of course, the integral is zero beyond the limits of the diagram for M_p.

Taking the fictitious beam of Fig. 12-14(c) and calculating the bending moment at D gives

$$M = -\Big| - \qquad +$$

$$5 \qquad + \frac{3}{2} \qquad + \frac{1}{2} \qquad \Big].$$

When this expression for M is divided by $\Sigma y^2 w$, it is identical with that for H_a except for the substitution of unity instead of P.

Therefore, the value of $M/\Sigma y^2w$ is the magnitude of the ordinate at D of the influence line for H_a.

The procedure in the case of xw and w is similar to that for yw.

It must be noticed that, if the bending moment for M_p is that of a simply supported beam, the imaginary beam with its elastic weights is also a simply supported beam of the same span; if the bending moment for M_p is that of a cantilever beam, then the imaginary one is another cantilever beam of equal span but fixed at the opposite end.

Problem 12-2. Draw the influence lines for H_a, V_a, and M_a for the arch of Fig. 12-11.

This means that influence lines are to be constructed by means of which Eqs. (12-11), (12-12), and (12-13) can, in effect, be solved for any condition of loading.

The mathematical work should be done with considerable precision so as to avoid errors which may affect the influence-line diagrams seriously. The calculations for this problem are tabulated in Table 12-2. The division numbers are listed in the order in which they appear in Fig. 12-11 so that their relationship with respect to the arch and to the fictitious cantilever beam will not be confused and so that the mathematical work will proceed in the same order as that which is indicated in the pictures. The magnitudes of w, x', y', m, n, and C_1 are taken from Table 12-1. The formulas for x and y are given in the left margin.

Lines 8 to 12, inclusive, give the computations for H_a. In line 9, the yw values are summed up from right to left. These figures are really the shears which exist in the imaginary beam at a point on the left side of the elastic weight at the center of each division. The fact that Σyw is zero (or practically so) when summed up in this way is a valuable check upon the calculations. If the calculations were made with extreme exactness, Σyw would be zero, but such refinement is not necessary. Line 10 is the summation of the various Σyw values from point 15 to point 1, working right to left. In reality, this gives the cantilever bending moments at all points from the right end of Fig. 12-11 to the left end in terms of the equal horizontal projections of the divisions—6 ft. This is why the line is labeled $M_{yw} \div 6$, meaning "the cantilever moments $\div$ the length of the horizontal divisions." Lines 11 and 12 are self-explanatory, but it must be noticed that the real horizontal length of the sections must be included. This is done most conveniently by dividing it (6 ft.) by Σy^2w, then using the result as the factor by which the quantities in line 10 are multiplied. It is advisable to carry out the values of the ordinates for H_a to three decimal places because of the fact that they will be multiplied by large numbers in determining V_a and M_a. The sign of H_a becomes positive because of the minus sign in the formula for H_a as shown under Table 12-2.

TABLE 12-2.—CALCULATION FOR ORDINATES OF INFLUENCE LINES FOR H_a, V_a, AND M_a

Line	Point	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	Σ
1	x'	3	9	15	21	27	33	39	45	51	57	63	69	75	81	87	
2	$x = x' - m$	−46.03	−40.03	−34.03	−28.03	−22.03	−16.03	−10.03	−4.03	+1.97	+7.97	+13.97	+19.97	+25.97	+31.97	+37.97	
3	y'	2.34	6.66	10.50	13.86	16.74	19.14	21.06	22.50	23.46	23.94	23.94	+23.46	22.50	21.06	19.14	
4	$y' - n$	−17.49	−13.17	−9.33	−5.97	−3.09	−0.69	+1.23	+2.67	+3.63	+4.11	+4.11	+3.63	+2.67	+1.23	−0.69	
5	C_1x	−7.26	−6.31	−5.37	−4.42	−3.47	−2.53	−1.58	−0.64	+0.31	+1.26	+2.20	+3.15	+4.10	+5.04	+5.99	
6	$y = (y' - n) - C_1x$	−10.23	−6.86	−3.96	−1.55	+0.38	+1.84	+2.81	+3.31	+3.32	+2.85	+1.91	+0.48	−1.43	−3.81	−6.68	
7	w	1.87	3.23	5.13	7.03	8.26	8.63	8.93	9.04	9.06	8.87	8.74	8.41	7.93	7.02	5.90	108.05
8	yw	−19.1	−22.2	−20.3	−10.9	+3.1	+15.9	+25.1	+29.9	+30.1	+25.3	+16.7	+4.0	−11.3	−26.8	−39.4	
9	Σyw	+0.1	+19.2	+41.4	+61.7	+72.6	+69.5	+53.6	+28.5	−1.4	−31.5	−56.8	−73.5	−77.5	−66.2	−39.4	
10	$M_{yw} \div 6$	+0.2	−19.0	−60.4	−122.1	−194.7	−264.2	−317.8	−346.3	−344.9	−313.4	−256.6	−183.1	−105.6	−39.4	0	
11	y^2w	195.7	152.0	80.4	16.9	1.2	29.2	70.5	99.0	99.9	72.0	31.9	1.9	16.2	101.9	263.3	1,232.0
12	H_a	0	0.093	0.294	0.595	0.948	1.287	1.548	1.687	1.680	1.526	1.250	0.892	0.514	0.192	0	
13	xw	−86.1	−129.3	−174.6	−197.1	−182.0	−138.3	−89.6	−36.4	+17.8	+70.7	+122.1	+168.0	+205.9	+224.4	+224.0	
14	Σxw	−0.5	+85.6	+214.9	+389.5	+586.6	+768.6	+906.9	+996.5	+1,032.9	+1,015.1	+944.4	+822.3	+654.3	+448.4	+224.0	
15	$M_{xw} \div 6$	9,090.0	9,004.4	8,789.5	8,400.0	7,813.4	7,044.8	6,137.9	5,141.4	4,108.5	3,093.4	2,149.0	1,326.7	672.4	224.0	0	
16	x^2w	3,962	5,176	5,941	5,523	4,009	2,218	898	147	35	563	1,706	3,354	5,348	7,175	8,506	54,561
17	$M_{xw} \div \Sigma x^2w$	1.0	0.990	0.967	0.924	0.859	0.775	0.675	0.565	0.452	0.340	0.236	0.146	0.074	0.025	0	
18	C_1H_a	0	0.015	0.046	0.094	0.149	0.203	0.244	0.266	0.265	0.241	0.197	0.141	0.081	0.030	0	
19	V_a	1.0	1.005	1.013	1.018	1.008	0.978	0.919	0.831	0.717	0.581	0.433	0.287	0.155	0.055	0	
20	Σw	108.05	106.18	102.95	97.82	90.79	82.53	73.93	64.97	55.93	46.87	38.00	29.26	20.85	12.92	5.90	108.05
21	$M_w \div 6$	828.87	722.69	619.74	521.92	431.13	348.60	274.70	209.73	153.80	106.93	68.93	39.67	18.82	5.90	0	
22	$M_w \div \Sigma w$	46.03	40.13	34.41	28.98	23.94	19.36	15.25	11.65	8.54	5.94	3.83	2.20	1.05	0.33	0	
23	nH_a	0	1.84	5.83	11.80	18.80	25.52	30.70	33.45	33.31	30.26	24.79	17.69	10.19	3.81	0	
24	$[M_w \div \Sigma w] + nH_a$	46.03	41.97	40.24	40.78	42.74	44.88	45.95	45.10	41.85	36.20	28.62	19.89	11.24	4.14	0	
25	mV_a	49.03	49.28	49.62	49.91	49.42	47.95	45.06	40.74	35.15	28.49	21.23	14.07	7.60	2.70	0	
26	M_a	−3.00	−7.31	−9.38	−9.13	−6.68	−3.07	+0.89	+4.36	+6.70	+7.71	+7.39	+5.82	+3.64	+1.44	0	

$$H_a = -\frac{M_{yw}}{\Sigma y^2 w}. \qquad V_a = \frac{M_{xw}}{\Sigma x^2 w} + C_1 H_a. \qquad M_a = \frac{M_w}{\Sigma w} + nH_a - mV_a.$$

The plotting of the influence line for H_a in Fig. 12-15 should be done immediately because the smoothness and the shape of the curve will give one a good idea as to the accuracy of the mathematical work.

Lines 13 to 19, inclusive, of Table 12-2 give the calculations for V_a, using xw as the elastic weights. A check of the calculations for Σxw is obtained through the fact that $\Sigma xw = 0$ (or practically so) for the entire arch. The

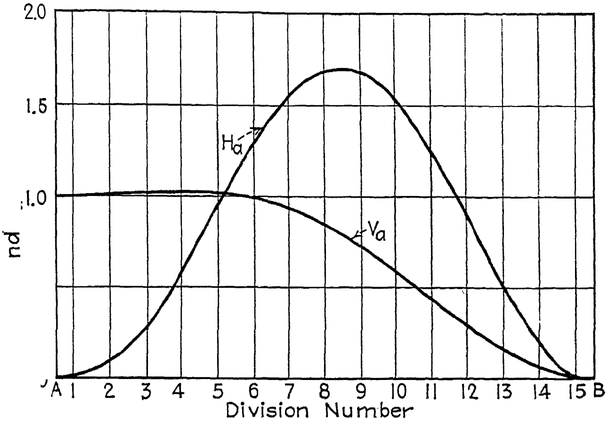


FIG. 12-15.—Influence lines for H_a and V_a.

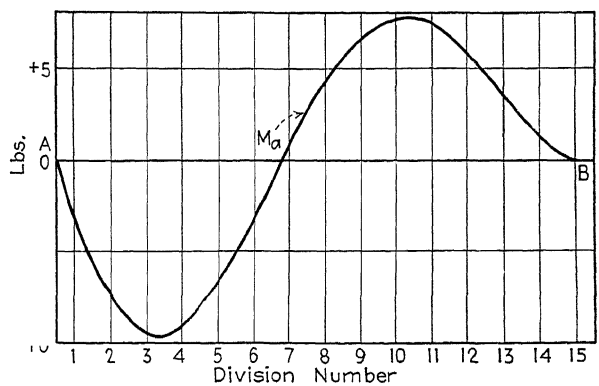


FIG. 12-16.—Influence line for M_a.

summations are to be made from right to left. The work is similar to that for H_a. Again, the plotting of the influence line, as V_a in Fig. 12-15, will usually reveal any glaring errors. The fact that V_a slightly exceeds unity at the divisions that are near the left end of the arch shows the effect of the end moments at A and B.

The computations for M_a are given in lines 20 to 26, inclusive, of Table 12-2. They are self-explanatory. The computations for Σw and $M_w \div 6$

are also made from the right toward the left, just as one would calculate the bending moments in the imaginary cantilever beam, because experience has shown that it is advisable to arrange the points in the tables in the same order as that in which they appear in the drawings, so as to avoid errors. The magnitude of M_a for point 1 should be $-V_a$ times the horizontal projection of half of the first division (-1×3), because $H_a y' = 0$ (or practically so). This is a valuable check on the computations. The influence line for M_a is plotted in Fig. 12-16.

12-10. Influence Lines for H, V, and M at Intermediate Points in an Arch. After the influence lines for the redundants H_a, V_a, and M_a are found, the construction of the influence lines for H, V, and M at any point in the arch is very simple. For instance, assume the arch of Fig. 12-17(*a*). The bending-moment diagrams for the redundants and for the external load are shown in the sketches of Fig. 12-17(*b*) to (*e*), inclusive.

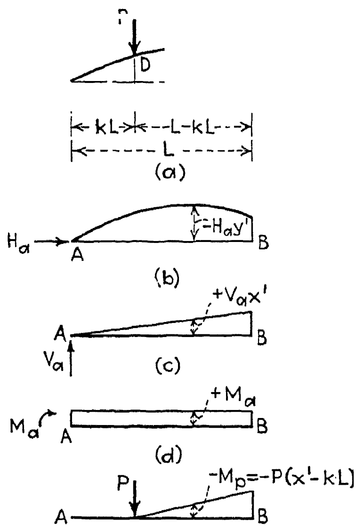


FIG. 12-17.

Obviously, H is constant throughout the arch so that the influence line for H_a applies equally well for the entire structure.

The shear at any point is V_a minus the load when that load is at the left of the given point; it is V_a alone when the load is at the right of the point. Therefore, the influence line for V_a is the basis from which the influence line for V is constructed.

The bending moment at any point C in the arch is the resultant of the effects of the various forces so that, considering the forces at the left of C [Fig. 12-17(*a*)],

$$M_c = M_a + V_a x' - H_a y' - P(x' - kL). \qquad (12\text{-}14)$$

This can be found readily by placing a unit load at each division; by scaling M_a, V_a, and H_a from their respective influence-line diagrams and multiplying them by the coefficients that are shown in Eq. (12-14); by multiplying unity by $(x' - kL)$; then by com-

bining the results and plotting the total as the ordinate at the point of application of that particular unit load.

It is obvious that, by the use of such influence lines, H, V, and M at any point can be found easily for any conditions of loading. Furthermore, the influence lines themselves show where the live loads should be applied to produce the maximum positive or negative moments and shears.

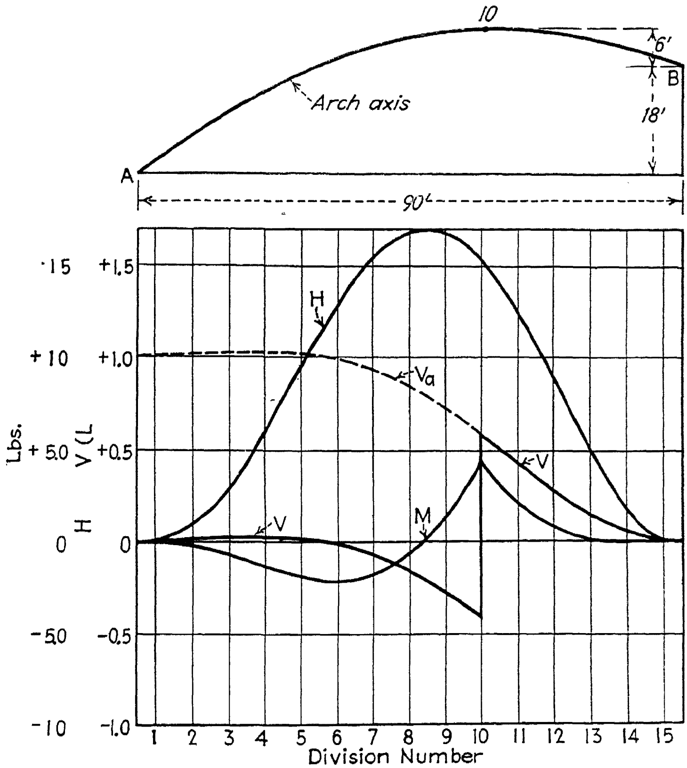


FIG. 12-18.—Influence lines for H, V, and M at point 10.

Problem 12-3. Construct the influence lines for H, V, and M at point 10 of the arch of Fig. 12-11, and find the effect of a load of 10,000 lb. placed at point 5.

The influence lines for H_a, V_a, and M_a have already been shown in Figs. 12-15 and 12-16. The diagram of the arch and the desired influence lines are given in Fig. 12-18. Table 12-3 shows the calculations that are required to obtain the ordinates of these influence lines. It is often suffi-

ciently accurate to calculate the ordinates at alternate points rather than at every one, but the latter is used in this illustration.

Placing a load of 10,000 lb. at point 5 and multiplying the ordinates of the influence lines of Fig. 12-18 at point 5 by 10,000 gives

$H_{10} = 10{,}000 \times 0.948 = 9{,}480$ lb.
$V_{10} = 10{,}000 \times 0.008 = 80$ lb. (upward).
$M_{10} = 10{,}000(-1.92) = -19{,}200$ ft.-lb. (tension on top).

12-11. Effects of Temperature, Shrinkage, and Rib Shortening. Using the same symbols as those which are given in Art. 11-9 of the preceding chapter, the change in the horizontal length of an arch for a change in temperature is

$$\Delta x = \qquad (12\text{-}15)$$

This is pictured in Fig. 12-19 for a rise in temperature, assuming the arch to be free to take its desired position as shown by the dotted lines. It is important to notice also that for a rise in temperature this unsymmetrical arch will tend to push point A downward as well as toward the left to A'. In general, for any change of temperature,

$$\Delta y = \mp \omega t D. \qquad (12\text{-}16)$$

On the other hand, the direction of the tangent to the arch axis at A will not change unless the abutment yields. Therefore,

$$\Delta \theta = 0. \qquad (12\text{-}17)$$

Line	Point	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	V_a	1.0	1.005	1.012	1.018	1.008	0.978	0.919	0.831	0.717	0.581	0.433	0.287	0.155	0.055	0
2	$x'V_a = 57V_a$	57	57.28	57.68	58.03	57.46	55.75	52.38	47.37	40.87	33.12	24.68	16.36	8.84	3.25	0
3	M_a	−3	−7.31	−9.38	−9.13	−6.68	−3.07	+0.89	+4.36	+6.70	+7.71	+7.39	+5.82	+3.64	+1.44	0
4	$M_a + V_a x'$	54	49.97	48.30	48.90	50.78	52.68	53.27	51.73	47.57	40.83	32.07	22.18	12.48	4.69	0
5	H_a	0	0.093	0.294	0.595	0.948	1.287	1.548	1.687	1.680	1.526	1.250	0.892	0.514	0.192	0
6	$y'H_a = 23.94H_a$	0	2.23	7.04	14.24	22.70	30.81	37.06	40.39	40.22	36.53	29.92	21.35	12.31	4.60	0
7	$M_a + V_a x' - H_a y'$	54	47.74	41.26	34.66	28.08	21.87	16.21	11.34	7.35	4.30	2.15	0.83	0.17	0.09	0
8	kL	3	9	15	21	27	33	39	45	51	57					
9	$x' - kL = 57 - kL$	54	48	42	36	30	24	18	12	6	0					
10	$M_{10} = M_a + V_a x' - H_a y' - (x' - kL)$	0	−0.26	−0.74	−1.34	−1.92	−2.13	−1.79	−0.66	+1.35	+4.30	+2.15	+0.83	+0.17	+0.09	0

Inasmuch as point A is not free to move at all, the effects of the rise in temperature are equivalent to the effects that would result from forcing the point A' back to A, keeping the angle θ constant. It is then apparent that Eqs. (12-15), (12-16), and (12-17) give the quantities that should be substituted in Eqs. (12-5), (12-6), and (12-7). Of course, $M_p = 0$ because no

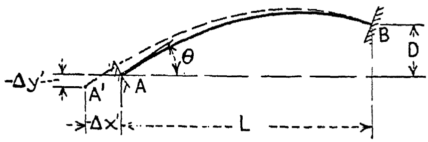


FIG. 12-19.

external loads are considered to be on the structure. It then is possible to rewrite the general equations as follows:

$$ys$$

$$xs \qquad (12\text{-}18)$$

Notice that Eqs. (12-18) are in terms of axes through the left support A. When these equations are expanded, when x' and y' are substituted for x and y so as to show that they are measured from the left support, and when w is used instead of s/I,

$$\begin{aligned} &= M_a\Sigma y'w + V_a\Sigma x'y'w - H_a\Sigma \\ +\omega tDE &= M_a\Sigma x'w + \\ 0 &= M_a\Sigma w + \end{aligned} \qquad (12\text{-}19)$$

The coefficients of M_a, V_a, and H_a are constants for any given structure. All except $\Sigma y'^2w$ have already been computed in locating the elastic center and the axis X-X. Very little extra work is required in finding $\Sigma y'^2w$. Then Eqs. (12-19) can be solved simultaneously for M_a, V_a, and H_a.

A more simple method of computing the effects of temperature changes can be found by changing the foregoing equations to apply for axes through the elastic center. By substituting $y' = y + n + C_1x$ and $x' = x + m$ in Eqs. (12-19) and simplifying them, the following equations will result:

$$
\left.
\begin{aligned}
H_a &= \frac{\pm \omega tLE \pm}{\Sigma y^2 \imath} \\
V_a &= \frac{\pm \omega tDE}{\Sigma x^2 w} + C_1 H_a. \\
M_a &= H_a n - V
\end{aligned}
\right\} \qquad (12\text{-}19a)
$$

In Eqs. (12-19a), the positive signs are for use with a rise of temperature; the negative, for a fall in temperature.

The effects of shrinkage, which may be assumed as equivalent to a drop in temperature of about 30 to 80°F., can be calculated in the same manner as for a decline in temperature. The magnitudes of M_a, V_a, and H_a may be derived directly from the solution of Eqs. (12-19a) by proportion from the relationships of the equivalent changes in temperature.

Rib shortening is the decrease in the span of the arch that would occur because of the compression of its material due to the thrust. After H_a has been determined, the total shortening of the horizontal projection of the arch because of these direct thrusts can be approximated as follows: divide H_a by the average of the areas of the sections to find the average f_c; compute the shortening of the arch, remembering that $\Delta L = f_cL/E$; find $\Delta D = \Delta L(D/L)$; substitute ΔL for ωtL and ΔD for ωtD in Eqs. (12-19a), treating them as a decline in temperature. Notice that the final values to substitute in Eqs. (12-19a) are $-(\Delta L)E = -f_cL$ and $-(\Delta D)E = -f_cD$. In fact, the effects of this rib shortening can be found by proportion from the previous solution of Eqs. (12-19a), having the relationship $(f_cL/E) \div \omega tL$.

Problem 12-4. Find H_a, V_a, and M_a for the arch of Fig. 12-11 for the following conditions:

1. A rise in temperature of 40°F.
2. A shrinkage equivalent to 35°F.
3. A rib shortening due to an H of 100 kips.

Assume $\omega = 0.000006$, $E_c = 3{,}000{,}000$ lb. per sq. in. $= 432 \times 10^6$ lb. per sq. ft.

Part 1:

$$
\begin{aligned}
\omega tL &= 0.000006 \times 40 \times 90 = 0.0216 \text{ ft.} \\
&= 0.0216 \times 432{,}000{,}000 = 9{,}331{,}200 \\
&= 0.000006 \times 40 \times 18 \times 432{,}000{,}000 = 1{,}866{,}240.
\end{aligned}
$$

Using Eqs. (12-19a), with values taken from Table 12-2,

$$
H_a = \frac{+9{,}331{,}200 + 0.1577 \times 1{,}866{,}240}{1{,}232} = 7{,}813 \text{ lb.}
$$

$$V_a = \frac{+1,866,240}{54,565} + 0.1577 \times 7,813 = 1,266 \text{ lb.}$$
$$M_a = 7,813 \times 19.83 - 1,266 \times 49.03 = 92,860 \text{ ft.-lb.}$$

Part 2: By proportion from the values that were obtained in the preceding calculations, using the suffix s to denote shrinkage,

$$\begin{aligned} H_{as} &= -\tfrac{35}{40}H_a = -0.875 \times 7,813 = -6,836 \text{ lb.} \\ V_{as} &= -\tfrac{35}{40}V_a = -0.875 \times 1,266 = -1,108 \text{ lb.} \\ M_{as} &= -\tfrac{35}{40}M_a = -0.875 \times 92,860 = -81,250 \text{ ft.-lb.} \end{aligned}$$

Part 3: The average of the areas of the divisions of the arch can be found by adding the quantities in column 3 of Table 12-1, by dividing this sum by the number of sections, and by multiplying by 1 ft.

$$\text{Average area} = \frac{34.4}{15} = 2.3 \text{ sq. ft.}$$
$$f_c = \frac{H}{2.3} = \frac{100,000}{2.3} = 43,500 \text{ lb. per sq. ft.}$$
$$\Delta T = f_cL = \frac{43,500 \times 90}{432,000,000} \quad 0.00906 \text{ ft.}$$

Then the coefficient that gives the relation between the results of the calculations of Part 1 and Part 3 is

$$K = -\frac{\Delta L}{\omega tL} = -\frac{0}{0.0216} \quad -0.42.$$

Therefore, using the suffix r to denote rib shortening,

$$\begin{aligned} H_{ar} &= -0.42H_a = -0.42 \times 7,813 = -3,281 \text{ lb.} \\ V_{ar} &= -0.42V_a = -0.42 \times 1,266 = -532 \text{ lb.} \\ M_{ar} &= -0.42M_a = -0.42 \times 92,860 = -39,000 \text{ ft.-lb.} \end{aligned}$$

12-12. Displacement of Foundations. It is dangerous to build an arch upon unreliable foundations. However, if a displacement of foundations is discovered, or if one may occur, it is important to compute its effects upon the stresses in the structure.

Equations (12-5), (12-6), and (12-7) cannot be equated to zero when any real (or assumed) deformation takes place. They are restated below with the use of w for s/I, with the omission of M_p, and with coordinates through point A:

$$\left.\begin{aligned} E(\Delta x') &= -M_a\Sigma y'w - V_a\Sigma x'y'w + H_a\Sigma y'^2w. \\ E(\Delta y') &= M_a\Sigma x'w + V_a\Sigma x'^2w - H_a\Sigma x'y'w. \\ E(\Delta\theta) &= M_a\Sigma w + V_a\Sigma x'w - H_a\Sigma y'w. \end{aligned}\right\} \qquad (12\text{-}19b)$$

In order to assist in the determination of the proper signs to use in Eqs. (12-19*b*), Fig. 12-20 pictures separately the proper signs and values for the left-hand side of the equations for each of the three displacements of the abutment at A with respect to that at B. In words, the following rules apply:

1. If A is displaced horizontally toward the left, $E\Delta x$ is negative.
2. If A settles vertically downward, $E\Delta y$ is negative.
3. If A rotates in a clockwise direction, the sign of $E\Delta\theta$ is positive.

Of course, the signs reverse when the displacements are reversed.

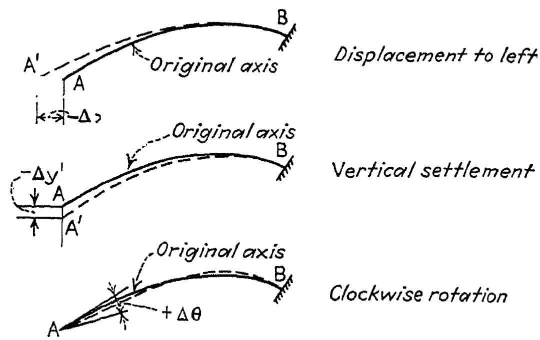


Fig. 12-20.

A further simplification for cases where only one of these displacements occurs is found by transferring the axes to the elastic center so as to avoid the solution of simultaneous equations. The resulting formulas are as follows:

1. If A is displaced horizontally toward the left,

$$H_a = -E(\Delta x) \qquad V_a = \qquad M_a = H_a n -$$

2. If A settles vertically downward,

$$H_a = -\frac{C_1 E(\Delta y)}{\Sigma y^2 w}, \qquad V_a = H_a C_1 - \frac{E(\Delta y)}{\Sigma x^2 w}, \qquad M_a = H_a n - V_a m.$$

3. If A rotates in a clockwise direction,

$$H_a = \frac{(n - C_1 m)E(\Delta\theta)}{\Sigma y^2 w}, \qquad V_a = H_a C_1 - \frac{mE(\Delta\theta)}{\Sigma x^2 w},$$

$$M_a = H_a n - V_a m + \frac{E(\Delta\theta)}{\Sigma w}.$$

For a symmetrical arch, $C_1 = 0$, and $m = L/2$ in the foregoing equations.

Problem 12-5. By the use of Eq. (12-19*b*), find H, V, and M at point A of the arch of Fig. 12-11 if the horizontal displacement of A is $\frac{1}{2}$ in. toward the left, the vertical displacement is 1 in. downward, and the angular rotation is 0.01 radian in a clockwise direction. Assume $E_c = 3{,}000{,}000$ lb. per sq. in. $= 432 \times 10^6$ lb. per sq. ft.

The coefficients for Eq. (12-19*b*) can be taken from Table 12-1. Then

$$
\begin{aligned}
2{,}143M_a + 113{,}655V_a - 45{,}081H_a &= +\tfrac{1}{24} \times 432{,}000{,}000.\\
5{,}298M_a + 314{,}276V_a - 113{,}655H_a &= -\tfrac{1}{12} \times 432{,}000{,}000.\\
108.05M_a + 5{,}298V_a - 2{,}143H_a &= +0.01 \times 432{,}000{,}000.
\end{aligned}
$$

The solution of the preceding equations gives

$$
\begin{aligned}
H_a &= +23{,}230 \text{ lb.}\\
V_a &= -880 \text{ lb.}\\
M_a &= +543{,}700 \text{ ft.-lb. (excessive).}
\end{aligned}
$$

12-13. Two-hinged Arch. If the arch of Fig. 12-21 has hinges at A and B, there can be no bending moments at these ends; the

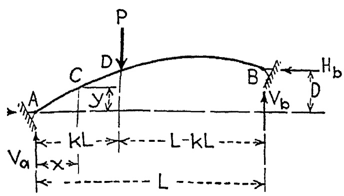


FIG. 12-21.

structure is indeterminate to the first degree. Therefore, the equation for H_a, the assumed redundant, can be found by placing imaginary rollers at A, applying a dummy load at A in the direction of H_a, and writing the work equation; thus,

1. For a unit load at A, in place of H_a,

$$\Delta x = \sum M_p m_q \frac{s}{EI} = \sum \left(-y + \frac{D}{L}x\right) \frac{s}{EI}$$

because the bending moment M_p at any point C due to this dummy load $= \left(-y + \frac{D}{L}x\right)$; the bending moment m_q due to the other unit load (directional one) is also $\left(-y + \frac{D}{L}x\right)$. Since

the ends are not level, these forces cause an upward reaction at A which is D/L.

2. For the load P,

where M_p = the bending moment in a simple beam of span L, carrying the load P.

3. For the combined loads, using $w = s/I$, omitting E, and considering that H_a will cause H_a times the displacement due to a unit horizontal thrust,

$$H_a \times \Delta x - \Delta x_p = 0$$

$$H_a = \frac{\Sigma M_p y w}{\sum y^2 w - 2\frac{D}{L}\sum xyw + \frac{D^2}{L^2}\sum} . \qquad (12\text{-}20)$$

For any given structure, the denominator of Eq. (12-20) is a constant.

The influence line for H_a can be constructed by using the elastic weights yw as loads on a simply supported beam of span L, computing the bending moments in this beam at the various points (under the centers of the arch sections), and dividing these values by the constant denominator of Eq. (12-20), the results being the ordinates of the influence line at these particular points. The procedure is the same as for the rigid-frame bridge of Art. 11-11 and Problem 11-9 of the preceding chapter.

If the arch is symmetrical, the last two terms of the denominator of Eq. (12-20) are zero.

It is difficult to make reinforced-concrete arches that have truly frictionless hinges. On this account, their use is not general.

12-14. Symmetrical Arch. The symmetrical arch is really a special case, but such structures are used more generally than are unsymmetrical ones. The procedure in the analysis of such an arch is practically the same as that which has been given in the previous articles. However, in order to illustrate the details of the calculations, a symmetrical arch will be analyzed.

Problem 12-6. Draw the influence-line diagrams for the redundants at the left support of the arch which is shown in Fig. 12-1; also draw the

influence line for the bending moment at the crown. The axis is pictured in Fig. 12-22, and the dimensions for the left half are given in columns 2 to 5, inclusive (Table 12-4).

Briefly stated, the procedure is as follows:

1. Refer to Table 12-4 and Fig. 12-22:

a. "Cut" the structure at A, cantilevering it from B, and divide it into 20 sections with horizontal projections equal to 6.15 ft.

b. Calculate $w = s/I$ for a strip 1 ft. wide, recording the results in column 6; get Σw for the left half, and double it for the entire arch.

c. To locate the elastic center, which must lie on the vertical axis of symmetry, calculate $y'w$, using A as the origin of coordinates; record the results in column 8; divide $\Sigma y'w$ by Σw to get n.

d. Calculate the quantities in columns 7, 9, 10, and 11 for use in future problems.

e. Compute the values of x and y measured from an origin at the elastic center, and record them in columns 12 and 13.

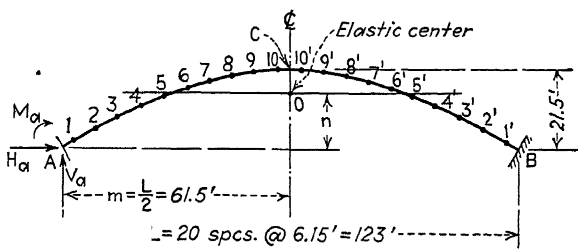


FIG. 12-22.

2. Refer to Table 12-5 and Fig. 12-22 for the right half of the arch:

a. Compute H_a as shown in lines 1 to 7, inclusive; note that $\Sigma yw = 0$ at Sec. 10′ and that the calculations proceed from right to left; notice that the quantities in line 5 must be multiplied by $6.15 \div 2\Sigma y^2 w = 6.15 \div 4{,}196$ because the figure 2,098 is for half of the arch.

b. Compute V_a as indicated in lines 8 to 13, inclusive; note that Σxw at 10′ cannot be zero because it applies to half of the arch; multiply line 11 by $6.15 \div 162{,}112$ to get line 13.

c. Calculate M_a as shown in lines 14 to 20, inclusive; notice that line 15 is multiplied by $6.15 \div 194.92$ to get line 16.

3. Refer to Table 12-6 and Fig. 12-22 for the left half of the arch:

a. Place values of H_a in line 7 that are equal to those of Table 12-5 for symmetrically placed points.

b. Place values of V_a in line 13 that are 1 minus the values of Table 12-5 for symmetrically placed points. The values of V_a can be checked by exterpolating from point 10′ to find the ordinate for V_a at the center of the span where it should be 0.5. If desired, the summations in lines 8 to 13 of Table 12-5 can be carried through in Table 12-6. If so, V_a at point 1 should be unity.

TABLE 12-4.—CALCULATIONS FOR ELASTIC CENTER

Given (scaled)					Derived							
1	2	3	4	5	6	7	8	9	10	11	12	13
Point	s, ft.	t, ft.	x'	y'	w	$x'w$	$y'w$	$x'y'w$	x'^2w	y'^2w	x*	y
1	7.75	3.35	3.08	2.5	2.47	7.6	6.18	19	23	15	−58.43	−14.706
2	7.32	2.72	9.23	6.8	4.36	40.2	29.65	273	371	202	−52.28	−10.406
3	6.98	2.30	15.38	10.4	6.88	105.8	71.55	1,100	1,627	744	−46.13	− 6.806
4	6.71	2.05	21.53	13.4	9.35	201.3	125.29	2,697	4,334	1,679	−39.98	− 3.806
5	6.52	1.94	27.68	15.8	10.72	296.7	169.38	4,688	8,213	2,676	−33.83	− 1.406
6	6.38	1.88	33.83	17.7	11.52	389.7	203.90	6,898	13,184	3,609	−27.68	+ 0.494
7	6.27	1.83	39.98	19.2	12.27	490.6	235.58	9,420	19,614	4,523	−21.53	+ 1.994
8	6.21	1.79	46.13	20.3	12.99	599.2	263.70	12,164	27,641	5,353	−15.38	+ 3.094
9	6.17	1.77	52.28	21.05	13.35	697.9	281.02	14,691	36,486	5,915	− 9.23	+ 3.844
10	6.15	1.76	58.43	21.45	13.55	791.7	290.65	16,982	46,259	6,234	− 3.08	+ 4.244
Σ for half arch........................					97.46	3,620.7	1,676.9	68,932	157,752	30,950		
Σ for entire arch......................					194.92	....	3,353.8					

* Using x' in reverse for symmetry

$$m = \frac{L}{2} = 61.5 \text{ ft.}$$

$$n = \frac{\Sigma y'w}{\Sigma w} = \frac{3,353.8}{194.92} = 17.206 \text{ ft.}$$

TABLE 12-5.—CALCULATION FOR ORDINATES OF INFLUENCE LINES FOR H_a, V_a, AND M_a
(Right half)

Line	Point	10′	9′	8′	7′	6′	5′	4′	3′	2′	1′	Σ	2Σ
1	y	+4.244	+3.844	+3.094	+1.994	+0.494	−1.406	−3.806	−6.806	−10.406	−14.706		
2	w	13.55	13.35	12.99	12.27	11.52	10.72	9.35	6.88	4.36	2.47		
3	yw	+57.5	+51.3	+40.2	+24.5	+5.7	−15.1	−35.6	−46.8	−45.4	−36.3		
4	Σyw	0	−57.5	−108.8	−149.0	−173.5	−179.2	−164.1	−128.5	−81.7	−36.3	0	
5	$M_{yw} \div 6.15$	−1,078.6	−1,021.1	−912.3	−763.3	−589.8	−410.6	−246.5	−118.0	−36.3	0		
6	y^2w	244	197	124	49	3	21	135	319	472	534	2,098	4,196
7	$H_a = -M_{yw} \div \Sigma y^2w$	1.581	1.497	1.337	1.119	0.864	0.602	0.361	0.173	0.053	0		
8	x	3.08	9.23	15.38	21.53	27.68	33.83	39.98	46.13	52.28	58.43		
9	xw	41.7	123.2	199.8	264.2	318.9	362.7	373.8	317.4	227.9	144.3		
10	Σxw	2,373.9	2,332.2	2,209.0	2,009.2	1,745.0	1,426.1	1,063.4	689.6	372.2	144.3		
11	$M_{xw} \div 6.15$	11,991	9,659	7,450	5,441	3,696	2,270	1,206	516	144	0		
12	x^2w	128	1,137	3,073	5,688	8,827	12,270	14,945	14,642	11,915	8,431	81,056	162,112
13	$V_a = M_{xw} \div \Sigma x^2w$	0.4549	0.3664	0.2826	0.2064	0.1402	0.0861	0.0458	0.0196	0.0055	0		
14	Σw	97.46	83.91	70.56	57.57	45.30	33.78	23.06	13.71	6.83	2.47	97.46	194.92
15	$M_w \div 6.15$	337.19	253.28	182.72	125.15	79.85	46.07	23.01	9.30	2.47	0		
16	$M_w \div \Sigma w$	10.64	7.99	5.76	3.95	2.52	1.45	0.73	0.29	0.08	0		
17	$nH_a = 17.206H_a$	27.20	25.76	23.00	19.25	14.87	10.36	6.21	2.98	0.91	0		
18	$[M_w \div \Sigma w] + nH_a$	37.84	33.75	28.76	23.20	17.39	11.81	6.94	3.27	0.99	0		
19	$mV_a = 61.5V_a$	27.98	22.53	17.38	12.69	8.62	5.30	2.82	1.21	0.34	0		
20	$M_a = [M_w \div \Sigma w] + nH_a - mV_a$	+9.86	+11.22	+11.38	+10.51	+8.77	+6.51	+4.12	+2.06	+0.65	0		

TABLE 12-6.—CALCULATION FOR ORDINATES OF INFLUENCE LINES FOR H_a, V_a, AND M_a
(Left half)

Line	Point	1	2	3	4	5	6	7	8	9	10
7	H_a	0	0.053	0.173	0.361	0.602	0.864	1.119	1.337	1.497	1.581
13	V_a	1	0.9945	0.9804	0.9542	0.9139	0.8598	0.7936	0.7174	0.6336	0.5451
14	Σw	194.92	192.45	188.09	181.21	171.86	161.14	149.62	137.35	124.36	111.01
15	$M_w \div 6.15$	1,851.74	1,659.29	1,471.20	1,289.99	1,118.13	956.99	807.37	670.02	545.66	434.65
16	$M_w \div \Sigma w$	58.42	52.35	46.42	40.70	35.28	30.19	25.47	21.14	17.22	13.71
17	nH_a	0	0.91	2.98	6.21	10.36	14.87	19.25	23.00	25.76	27.20
18	$[M_w \div \Sigma w] + nH_a$	58.42	53.26	49.40	46.91	45.64	45.06	44.72	44.14	42.98	40.91
19	mV_a	61.50	61.16	60.29	58.68	56.20	52.88	48.81	44.12	38.97	33.52
20	$M_a = [M_w \div \Sigma w] + nH_a - mV_a$	−3.08	−7.90	−10.89	−11.77	−10.56	−7.82	−4.09	+0.02	+4.01	+7.39

TABLE 12-7.—CALCULATION FOR ORDINATES OF INFLUENCE LINE FOR MOMENT AT CROWN

Line	Point	1	2	3	4	5	6	7	8	9	10	Crown
1	V_a	1	0.9945	0.9804	0.9542	0.9139	0.8598	0.7936	0.7174	0.6336	0.5451	0.5
2	$x'V_a = 61.5V_a$	61.5	61.16	60.29	58.68	56.20	52.88	48.81	44.12	38.97	33.52	30.75
3	M_a	−3.08	−7.90	−10.89	−11.77	−10.56	−7.82	−4.09	+0.02	+4.01	+7.39	+8.8*
4	$M_a + V_a x'$	58.42	53.26	49.40	46.91	45.64	45.06	44.72	44.14	42.98	40.91	39.55
5	H_a	0	0.053	0.173	0.361	0.602	0.864	1.119	1.337	1.497	1.581	1.60*
6	$y'H_a = 21.5\ H_a$	0	1.14	3.72	7.76	12.94	18.58	24.06	28.75	32.19	33.99	34.40
7	$M_a + V_a x' - H_a y'$	58.42	52.12	45.68	39.15	32.70	26.48	20.66	15.39	10.79	6.92	5.15
8	kL	3.08	9.23	15.38	21.53	27.68	33.83	39.98	46.13	52.28	58.43	61.5
9	$x' - kL = 61.5 - kL$	58.42	52.27	46.12	39.97	33.82	27.67	21.52	15.37	9.22	3.07	0
10	$M_c = M_a + V_a x' - H_a y' - (x' - kL)$	0	−0.15	−0.44	−0.82	−1.12	−1.19	−0.86	+0.02	+1.57	+3.85	+5.15

* Scaled.

c. Calculate M_a by continuing the computations of Table 12-5 toward the left; notice the check on the computations when M_a at point 1 is found to be -3.08.

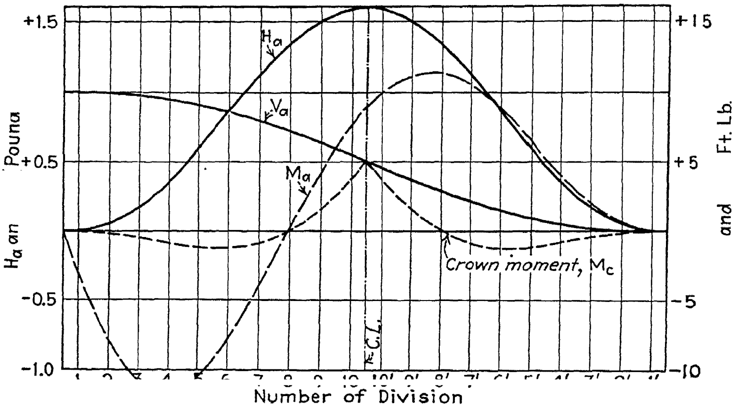


FIG. 12-23.—Influence lines for H_a, V_a, M_a, and M_c—symmetrical arch.

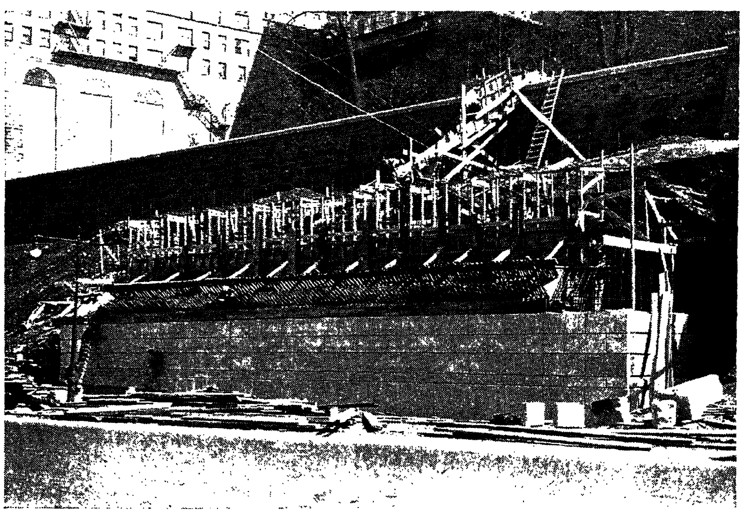


FIG. 12-24.—Abutment of arch over Riverside Drive, New York approach to the George Washington Bridge.

d. Plot the influence lines for H_a, V_a, and M_a in Fig. 12-23.

4. Refer to Table 12-7 and Fig. 12-22:

a. Since the arch is symmetrical, the influence line for the bending moment at the crown will be symmetrical also. Therefore compute M_c for the left half as shown in Table 12-7.

b. Plot the influence line for M_c in Fig. 12-23.

Problem 12-7. Using the arch of Fig. 12-22 and the influence lines of Fig. 12-23, find H, V, and M at A for the dead load of the arch rib if concrete weighs 150 lb. per cu. ft.

The calculations are self-explanatory and are shown in Table 12-8; values of s and t are taken from Table 12-4; values of H_a, V_a, and M_a for unit loads as given in Tables 12-5 and 12-6 or in Fig. 12-23 are multiplied by column 4 to obtain the quantities listed in columns 5, 6, and 7.

The abutment, skewback, and reinforcement at the east end of this arch are shown in Fig. 12–24.

Problem 12-8. Find the maximum positive bending moment at the crown of the arch of Fig. 12-22 due to a uniform live load of 200 lb. per sq. ft. Use the influence line of Fig. 12-23.

Examination of Fig. 12-23 shows that only loads between points 8 and 8′ will cause positive bending moments at the crown of the arch. Therefore,

TABLE 12-8.—CALCULATIONS FOR H_a, V_a, AND M_a FOR DEAD LOAD OF ARCH RIB

Col.	1	2	3	4	5	6	7
Point	s, ft.	t, ft.	Vol., ft.^{3}	Wt. W, kips	WH_a, kips	WV_a, kips	WM_a, ft.-kips
1	7.75	3.35	25.96	3.894	0	3.89	−11.99
2	7.32	2.72	19.91	2.986	0.16	2.97	−23.59
3	6.98	2.30	16.05	2.408	0.42	2.36	−26.22
4	6.71	2.05	13.76	2.064	0.75	1.97	−24.29
5	6.52	1.94	12.65	1.898	1.14	1.73	−20.04
6	6.38	1.88	11.99	1.798	1.55	1.55	−14.06
7	6.27	1.83	11.47	1.720	1.92	1.36	− 7.03
8	6.21	1.79	11.12	1.668	2.23	1.20	+ 0.03
9	6.17	1.77	10.92	1.638	2.45	1.04	+ 6.57
10	6.15	1.76	10.82	1.623	2.57	0.88	+11.99
10′	....	....	.....	1.623	2.57	0.74	+16.00
9′	....	....	.....	1.638	2.45	0.60	+18.38
8′	....	....	.....	1.668	2.23	0.47	+18.98
7′	....	....	.....	1.720	1.92	0.36	+18.08
6′	....	....	.....	1.798	1.55	0.25	+15.77
5′	....	....	.....	1.898	1.14	0.16	+12.36
4′	....	....	.....	2.064	0.75	0.09	+ 8.50
3′	....	....	.....	2.408	0.42	0.05	+ 4.96
2′	....	....	.....	2.986	0.16	0.02	+ 1.94
1′	....	....	.....	3.894	0	0	0
Σ	....	....	.....	.....	26.38	21.69	+ 6.34

a short live load is placed upon this portion, causing a concentrated load of 200×6.15 lb. at 9, 10, 10′, and 9′ and a load of 200×3.08 lb. at 8 and 8′. Then the resultant bending moment at the crown is

$$M_c = 200 \times 6.15(1.57 + 3.85)2 + 200 \times 3.08 \times 0.02 \times 2$$
$$M_c = +13{,}360 \text{ ft.-lb.}$$

12-15. Deflection of an Arch. When an arch is loaded, it must certainly deform under these loads. In order to find the vertical deflection of any point of the arch rib due to flexure under the action of these loads, apply a unit vertical force at the point where the deflection is desired. Then the downward deflection of E with respect to A [Fig. 12-25(a)] will be the same magnitude as that of a cantilevered beam which is cut at E and fixed at A by the three redundants. Then from Eq. (12-2),

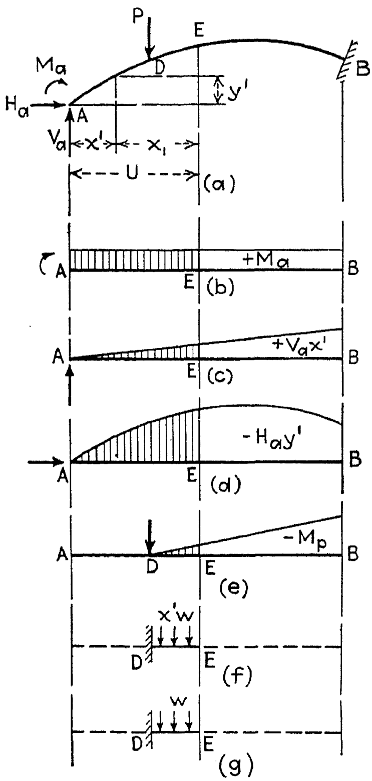


FIG. 12-25.

$$\Delta y = \sum Mx\frac{s}{EI}$$

The moment M is the total bending moment at any point, and x is measured from the cut point (E in this case) as an origin. The distortions that are caused by shear are usually relatively small, and they are neglected herein.

Similarly, when the horizontal deflection of a point is required, apply a unit horizontal force at that point. Hence,

$$\Delta x = \frac{\quad}{EI}. \tag{12-22}$$

In order to find the angular rotation of any point, apply a unit moment at that point; then

It is necessary to consider only the portion of the arch from the given point to either abutment. For instance, if the vertical deflection of E of Fig. 12-25(a) is desired, consider the section from E to A. Sketches (b) to (e), inclusive, show the bending-moment diagrams that are caused by the three redundants at A and by the external load P. The shaded portions of these diagrams are the parts that produce bending in AE, each one causing a definite distortion and, therefore, a certain vertical deflection at E. The resultant of all of these separate deflections is the actual deflection at E. Hence, using Eq. (12-21),

$$E\Delta y = \sum Mx\frac{s}{I} = \Sigma(M_a + V_a x' - H_a y'$$

Since $x_1 = x' - U$, Eq. (12-24) expands to

$$E\Delta y = M_a(\Sigma x'w - U\Sigma w) + V_a(\Sigma x'^2 w - U\Sigma x'w) + H_a(U\Sigma y'w - \Sigma x'y'w) - \Sigma M_p x'w + U\Sigma M_p w. \quad (12\text{-}25)$$

The first six terms of Eq. (12-25) are composed of constants which are multiplied by the redundants, these last being found from the influence lines for any position of loading. The last two terms of this equation are similar to expressions that were involved in the equations for V_a and M_a. Therefore, they can be integrated as follows:

1. $\Sigma M_p x'w$ is equivalent to P times the bending moment at D in the cantilever beam DE [Fig. 12-25(f)] with the elastic weights $x'w$.

2. $U\Sigma M_p w$ is equivalent to P times the bending moment at D in the cantilever beam DE [Fig. 12-25(g)] with the elastic weights w, multiplied by U.

Naturally, when the load P is at the right of E, the terms that involve M_p are zero.

The moments of inertia that must be used in the calculations for deflections should be those of the transformed sections—steel and concrete. However, it is satisfactory to use the concrete sections alone and to adjust the calculated deflections by a correction factor which is based upon the weighted average relationship between I_c and I for the transformed sections.

From Eq. (12-25), it is apparent that an influence line can be drawn for the deflection at any point by using unit loads at various locations and by solving for the ordinate at each point. This will be illustrated in Problem 12-10.

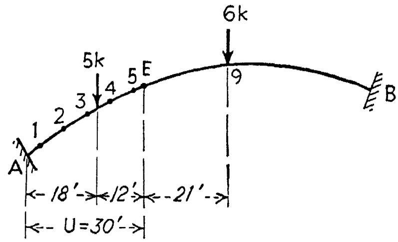


FIG. 12-26.

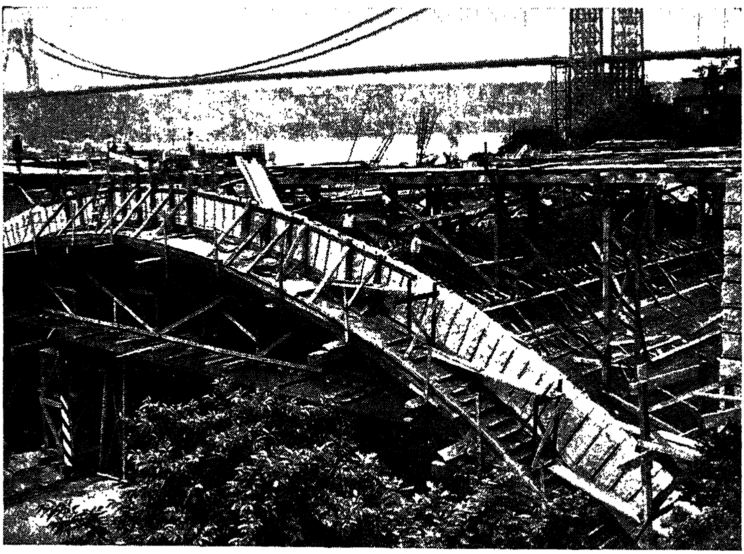


FIG. 12-27.—Construction of barrel of arch over Riverside Drive, New York approach to the George Washington Bridge.

Problem 12-9. Find the deflection at point E of the arch of Fig. 12-26 if the structure is loaded as shown. The data for the arch are the same as for Fig. 12-11. $E = 432{,}000{,}000$ lb. per sq. ft.

From the influence lines of Figs. 12-15 and 12-16,

$$H_a = 5 \times 0.42 + 6 \times 1.68 = 12.18 \text{ kips.}$$
$$V_a = 5 \times 1.015 + 6 \times 0.717 = 9.38 \text{ kips.}$$
$$M_a = 5(-9.7) + 6(+6.7) = -8.3 \text{ ft.-kips.}$$

Then, by summing up $x'w$, w, etc., for points 1 to 5, inclusive, of Table 12-1, and solving Eq. (12-25),

$$432{,}000 \times \Delta y = -8.3(482.3 - 30 \times 25.52) + 9.38(10{,}555 - 30 \times 482.3) + 12.18(30 \times 315.5 - 6{,}794) - 5(223 \times 9 + 147.6 \times 3) + 30 \times 5(8.26 \times 9 + 7.03 \times 3).$$

$$\Delta y = \frac{236.35}{432{,}000} = 0.00055 \text{ ft.}$$ (Not adjusted for the transformed section.)

This shows that the effects of the two loads practically neutralize each other. (See Fig. 12-30 for the deflection at the quarter point of a symmetrical arch.)

FIG. 12-28.—Barrel of the arch over Riverside Drive, New York approach to the George Washington Bridge.

Problem 12-10. Figure 12-27 shows the centering under the arch of Fig. 12-1 before the pouring of the concrete. Figure 12-28 shows the rib after the centering was removed. Find the vertical deflection of the crown of the arch when the centering was "struck"; in other words, find the allowance that must be made in the centering to counteract this deflection, using the influence lines of Fig. 12-23. E = 432,000,000 lb. per sq. ft.; concrete = 150 lb. per cu. ft.; and the adjusted I, considering the steel, $= 1.25I_c$.

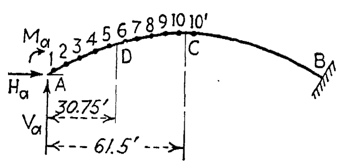


FIG. 12-29.

Since the arch is symmetrical, consider the left side only, from the left end to the crown (as pictured in Fig. 12-29). Therefore, using the data in Table 12-4, Eq. (12-25) for half of the arch becomes

$$1.25 \times 432{,}000{,}000\Delta y = M_a(3{,}620.7 - 61.5 \times 97.46) + V_a(157{,}752 - 61.5 \times 3{,}620.7) + H_a(61.5 \times 1{,}676.9 - 68{,}932) - \Sigma M_p x'w + 61.5\Sigma M_p w.$$

TABLE 12-9.—CALCULATION FOR ORDINATES OF INFLUENCE LINE FOR VERTICAL DEFLECTION AT CROWN

Line	Point	1	2	3	4	5	6	7	8	9	10
1	$x'w$	7.6	40.2	105.8	201.3	296.7	389.7	490.6	599.2	697.9	791.7
2	$\Sigma x'w$	3,620.7	3,613.1	3,572.9	3,467.1	3,265.8	2,969.1	2,579.4	2,088.8	1,489.6	791.7
3	$M_{x'w} \div 6.15$	23,837.5	20,224.4	16,651.5	13,184.4	9,918.6	6,949.5	4,370.1	2,281.3	791.7	0
4	$M_{x'w}$	146,601	124,380	102,407	81,084	60,999	42,739	26,876	14,030	4,869	0
5	w	2.47	4.36	6.88	9.35	10.72	11.52	12.27	12.99	13.35	13.55
6	Σw	97.46	94.99	90.63	83.75	74.40	63.68	52.16	39.89	26.90	13.55
7	$M_w \div 6.15$	539.95	444.96	354.33	270.58	196.18	132.50	80.34	40.45	13.55	0
8	M_w	3,321	2,737	2,179	1,664	1,207	815	494	249	83	0
9	M_a	−3.075*	−7.90	−10.89	−11.77	−10.56	−7.82	−4.09	+0.02	+4.01	+7.39
10	V_a	1	0.9945	0.9804	0.9542	0.9139	0.8598	0.7936	0.7174	0.6336	0.5451
11	H_a	0	0.053	0.173	0.361	0.602	0.864	1.119	1.337	1.497	1.581
12	$-2{,}373.1M_a$	+7,297	+18,747	+25,843	+27,931	+25,060	+18,558	+9,706	−47	−9,516	−17,537
13	$-64{,}921\ V_a$	−64,921	−64,564	−63,649	−61,948	−59,331	−55,819	−51,521	−46,574	−41,134	−35,388
14	$+34{,}197\ H_a$	0	+1,812	+5,916	+12,345	+20,587	+29,546	+38,266	+45,721	+51,193	+54,065
15	$-M_{x'w}$	−146,601	−124,380	−102,407	−81,084	−60,999	−42,739	−26,876	−14,030	−4,869	0
16	$+61.5M_w$	+204,242	+168,326	+134,009	+102,336	+74,231	+50,123	+30,381	+15,314	+5,105	0
17	Σ	+17	−59	−288	−420	−452	−331	−44	+384	+779	+1,140
18	Δy	....	−0.00131	−0.00640	−0.00933	−0.01004	−0.00735	−0.00098	+0.00853	+0.01731	+0.02533

* More accurate than −3.08.

$$\Delta y = \frac{12\Sigma \times 1000}{E \times 1.25} = \frac{12\Sigma \times 1{,}000}{432{,}000\,000 \times 1.25} = 0.00002222\Sigma.$$

Notice that the magnitudes of the redundants depend upon the relative values of I for the various divisions of the arch, but the deflections depend upon the actual values of I. Hence, since $I = 1.25I_c$, the right-hand side of the foregoing equation is to be divided by 1.25. Therefore, this figure appears as a multiplier on the left-hand side of this equation.

In order to solve this equation, proceed as follows: Use the elastic weights $x'w$ and w from Table 12-4; find the cantilever moments in the imaginary beams which are assumed to be fixed at A; record the calculations as shown in Table 12-9, lines 1 to 8, inclusive; copy the values of H_a, V_a, and M_a from Table 12-6, and place them in lines 9, 10, and 11; complete the calculations in lines 12 to 16, inclusive; sum up the terms, and record them in line 17; multiply these sums by 12 to transform them into inches, then by 1,000 to convert them into results for a load of 1 kip; divide these products by

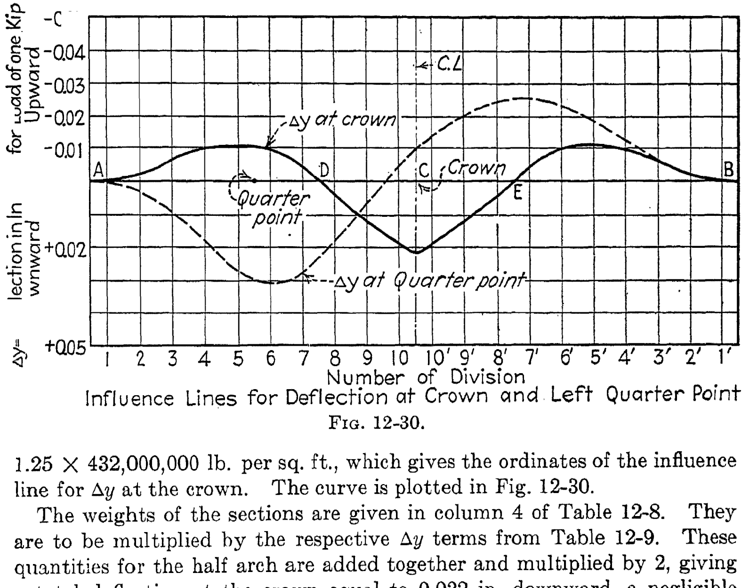


FIG. 12-30.

1.25 × 432,000,000 lb. per sq. ft., which gives the ordinates of the influence line for Δy at the crown. The curve is plotted in Fig. 12-30.

The weights of the sections are given in column 4 of Table 12-8. They are to be multiplied by the respective Δy terms from Table 12-9. These quantities for the half arch are added together and multiplied by 2, giving a total deflection at the crown equal to 0.022 in. downward, a negligible quantity in this case. Rib shortening is not included.

Figure 12-30 gives one an excellent idea of how the arch moves when loads are placed upon it at various points. Any loads from the left abutment to D or from B to E make the structure rise at the center; those between D and E cause it to sag downward; and, strange as it may seem, a load at D or E does not produce any deflection at C.

Problem 12-11. Draw the influence-line diagram for vertical deflections at the left quarter point of the arch of Fig. 12-1, with $E = 432{,}000{,}000$ lb. per sq. ft. and $I = 1.25I_c$.

This is the same type of problem as the preceding one. The method of procedure is similar to that of Problem 12-10, but the origin of coordinates

TABLE 12-10.—CALCULATION FOR ORDINATES OF INFLUENCE LINE FOR VERTICAL DEFLECTION AT LEFT QUARTER POINT

Line	Point	1	2	3	4	5	6	7	8	9	10
1	$x'w$	7.6	40.2	105.8	201.3	296.7					
2	$\Sigma x'w$	651.6	644.0	603.8	498.0	296.7					
3	$M_{x'w} \div 6.15$	2,042.5	1,398.5	794.7	296.7	0					
4	$M_{x'w}$	12,561	8,601	4,887	1,825	0					
5	w	2.47	4.36	6.88	9.35	10.72					
6	Σw	33.78	31.31	26.95	20.07	10.72					
7	$M_w \div 6.15$	89.05	57.74	30.79	10.72	0					
8	M_w	547.7	355.1	189.4	65.9	0					
9	M_a	−3.075*	−7.90	−10.89	−11.77	−10.56	−7.82	−4.09	+0.02	+4.01	+7.39
10	V_a	1	0.9945	0.9804	0.9542	0.9139	0.8598	0.7936	0.7174	0.6336	0.5451
11	H_a	0	0.053	0.173	0.361	0.602	0.864	1.119	1.337	1.497	1.581
12	$-387\ M_a$	+1,190	+3,057	+4,214	+4,555	+4,087	+3,026	+1,583	−8	−1,552	−2,860
13	$-5.469\ V_a$	−5,469	−5,439	−5,362	−5,219	−4,998	−4,702	−4,340	−3,923	−3,465	−2,981
14	$+3.586\ H_a$	0	+190	+620	+1,295	+2,159	+3,098	+4,013	+4,794	+5,368	+5,669
15	$-M_{x'w}$	−12,561	−8,601	−4,887	−1,825						
16	$+30.75\ M_w$	+16,842	+10,919	+5,824	+2,026						
17	Σ	+2	+126	+409	+832	+1,248	+1,422	+1,256	+863	+351	−172
18	Δy	0	+0.0028	+0.0091	+0.0185	+0.0277	+0.0316	+0.0279	+0.0192	+0.0078	−0.0038

* More accurate than −3.08.

$$\Delta y = \frac{12\Sigma \times 1{,}000}{432{,}000{,}000 \times 1.25} = 0.00002222\Sigma.$$

TABLE 12-10.—CALCULATION FOR ORDINATES OF INFLUENCE LINE FOR VERTICAL DEFLECTION AT LEFT QUARTER POINT.—(*Continued*)

Line	Point	10′	9′	8′	7′	6′	5′	4′	3′	2′	1′
1	$x'w$										
2	$\Sigma x'w$										
3	$M_{x'w} \div 6.15$										
4	$M_{x'w}$										
5	w										
6	Σw										
7	$M_w \div 6.15$										
8	M_w										
9	M_a	+9.86	+11.22	+11.38	+10.51	+8.77	+6.51	+4.12	+2.06	+0.65	0
10	V_a	0.4549	0.3664	0.2826	0.2064	0.1402	0.0861	0.0458	0.0196	0.0055	0
11	H_a	1.581	1.497	1.337	1.119	0.864	0.602	0.361	0.173	0.053	0
12	$-387\ M_a$	−3,816	−4,342	−4,404	−4,067	−3,394	−2,519	−1,594	−797	−252	0
13	$-5{,}469\ V_a$	−2,488	−2,004	−1,546	−1,129	−767	−471	−250	−107	−30	0
14	$+3{,}586\ H_a$	+5,669	+5,368	+4,794	+4,013	+3,098	+2,159	+1,295	+620	+190	0
15	$-M_{x'w}$										
16	$+30.75 M_w$										
17	Σ	−635	−978	−1,156	−1,183	−1,063	−831	−549	−284	−92	0
18	Δy	−0.0141	−0.0217	−0.0257	−0.0263	−0.0236	−0.0185	−0.0122	−0.0063	−0.0020	0

is at D (Fig. 12-29). The redundants at A and the elastic weights can again be taken from the influence lines of Fig. 12-23 or from Tables 12-4, 12-5, and 12-6.

Equation 12-25 becomes

$$1.25 \times 432{,}000{,}000\Delta y = M_a(651.6 - 30.75 \times 33.78) + V_a(14{,}568 - 30.75 \times 651.6) + H_a(30.75 \times 402.05 - 8{,}777) - \Sigma M_p x'w + 30.$$

$$1.25 \times 432{,}000{,}000\Delta y = -387M_a - 5{,}469V_a + 3{,}586H_a - 30.$$

The last two terms of the foregoing equation are solved in lines 1 to 8, inclusive, 15, and 16 (Table 12-10), considering the imaginary cantilever beams to be fixed at A. The balance of Table 12-10 is self-explanatory, and the influence-line ordinates in line 18 are plotted in Fig. 12-30.

Rib shortening often causes appreciable deflection of an arch when the centering is removed. In order to determine its magnitude (or the deflections due to changes in temperature), find H_a, V_a and M_a for the particular conditions, then solve Eq. (12–25), remembering that M_p is zero.

12-16. Practical Details. So many of the practical details of construction of arches are similar to those which have been explained for rigid-frame bridges in the preceding chapter that it is not necessary to consider them again. However, in the case of long arches, it is frequently advisable to minimize the effects of shrinkage by pouring the barrels or ribs in alternate, short sections, proceeding in a symmetrical manner in order to avoid detrimental and unequal settlements of the centering. The intermediate portions can be poured later on, and the central or some other key section can be poured last.

CHAPTER 13

ANALYSIS OF RIGID FRAMES BY MOMENT DISTRIBUTION

13-1. Introduction. The analysis of indeterminate structures by the method of work that is illustrated in the two preceding chapters involves considerable labor. It is therefore desirable to find another method which is easier to use and which is sufficiently exact for practical purposes. Such a method is "moment distribution."[1] It is based upon the theory of work, but it employs certain short cuts and approximations which facilitate the solution of many problems. This method is so exceedingly useful that every engineer should understand it thoroughly.

Although the demonstrations used in this chapter are confined to rigid-frame bridges, the principles that are developed are general. They will be used in the analysis of other structures in the next chapter.

The method of moment distribution assumes that the axes of the members are either straight or only slightly curved so that arch action does not exist—or at least it is negligible. When the curvature of a member is large, the method of work should be resorted to. The joints of the structure are also assumed to remain fixed in position. If they are displaced (as in side sway), special adjustments must be made.

13-2. Fundamental Principles of Moment Distribution. Basically, the method of moment distribution is founded upon the fact that any structure under load must be in equilibrium as a whole and in all of its parts. The ends of all members are assumed first to be fixed against rotation. By releasing one joint at a time and by making successive adjustments, it is possible to find the conditions that must prevail when all joints are in equilibrium under the interaction that must exist between them.

[1] Presented by Hardy Cross in A.S.C.E., *Proc.*, May, 1930.

Moment distribution utilizes the fixed-end moments of beams, the stiffness of structural members, and the moments that are "carried over" from one joint to another when the first one is rotated. All of these matters will be explained, showing their mathematical derivation and meaning, before proceeding with their application.

13-3. Calculation of Fixed-end Moments for Tapered Members. Assume any beam AB, as in Fig. 13-1(a), the ends of

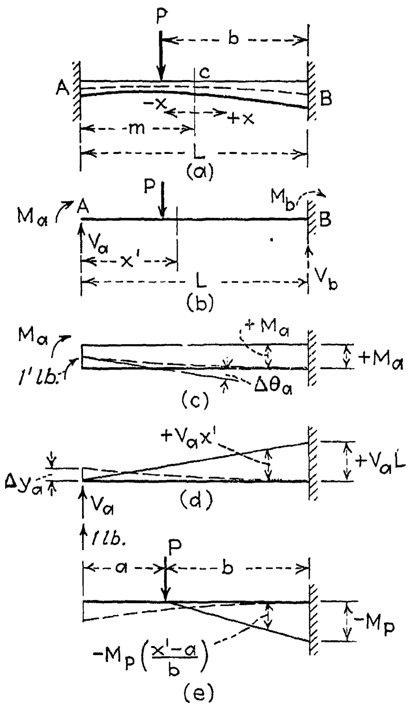


FIG. 13-1.

which are fixed against rotation. It is obvious that such a member is statically indeterminate to the second degree. However, it is necessary to find the moments at A and B—called "fixed-end" moments and denoted "f.e.m."

To do this by the method of work, the beam can be cut free at A. M_a and V_a are the redundants. Using an origin of coordinates at A and placing a unit moment at A, the work equa-

tion for the angular rotation at A is

$$= \int \frac{Mm\,dx'}{\quad} \qquad \int \frac{M\,dx'}{\quad}$$

Similarly, using a unit vertical force at A, the work equation for the vertical displacement at A is

$$y_a = \int \frac{Mm\,dx'}{EI} = \int \frac{Mx'\,dx'}{EI}. \tag{13-2}$$

The moment at any point in the cantilevered beam is a combination of the moments shown in the diagrams of Figs. 13-1(c), (d), and (e); that is,

$$M = M_a + V_a x' - M_p. \tag{13-3}$$

Substitute this value of M in Eqs. (13-1) and (13-2); omit E; use $w = s/I$ for dx'/I; and place θ_a and y_a equal to zero because point A is fixed. Therefore,

$$) = 0 \tag{13-4}$$

and

$$y_a = M_a \Sigma x'w + V_a \Sigma x'^2 w - \Sigma M_p x'w = 0. \tag{13-5}$$

These equations can be solved for M_a and V_a after the coefficients and the M_p terms are evaluated. However, such a solution would have to be made for each condition of loading, causing considerable labor. It is therefore advisable to develop influence lines for the fixed-end moments, computing M_b after M_a and V_a are calculated.

These influence lines can be obtained by the use of the imaginary cantilevered beam, fixed at A and loaded with the elastic weights w or xw, as illustrated for arches in the preceding chapter.

Assume C of Fig. 13-1(a) to be the elastic center of AB. It is a distance m from A so that

$$x' = m + x. \tag{13-6}$$

When this value of x' is substituted in Eqs. (13-4) and (13-5) and they are simplified,

$$- \Sigma M_p w = 0. \tag{13-7}$$

$$y_a = V_a \Sigma x^2 w - \Sigma M_p xw = 0. \tag{13-8}$$

Moments in beams of constant section and with fixed ends

$M = m \times W \times l$

m = Coefficient taken from diagram
W = Total load on beam
l = Length of beam
a = Length in terms of l

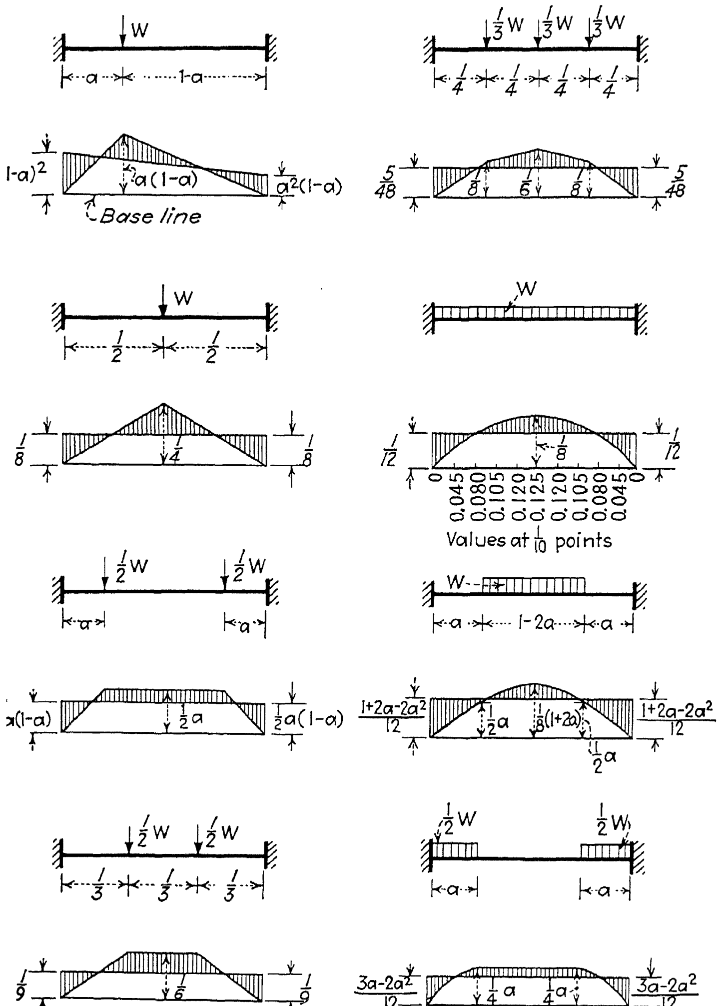
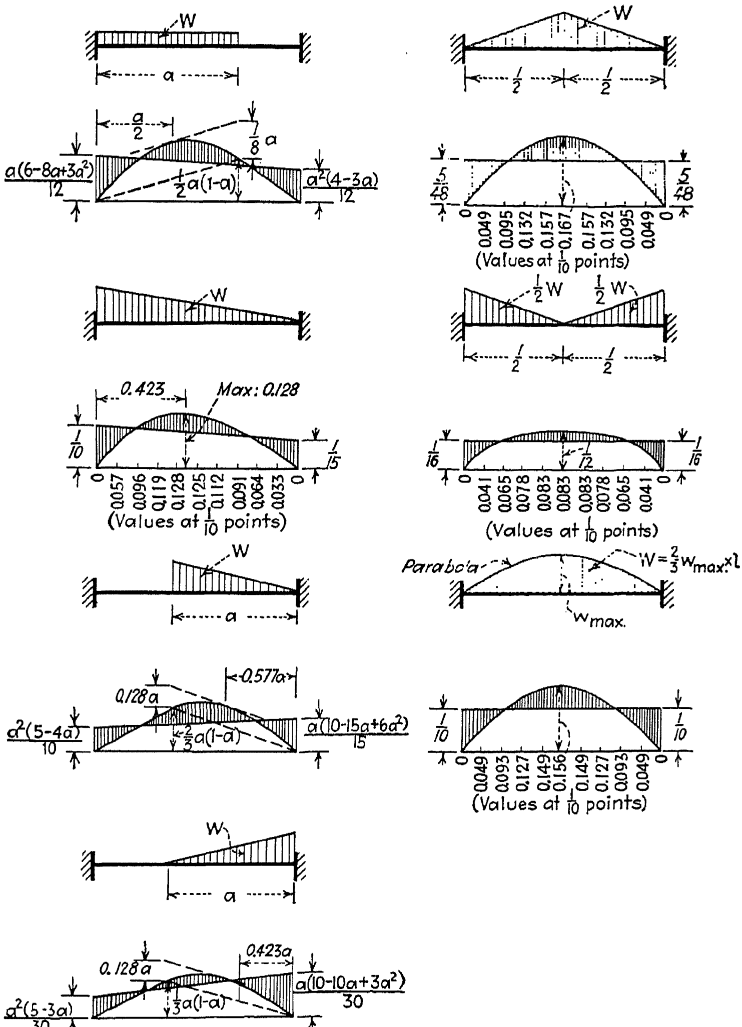


FIG. 13-2.—(Courtesy of

Moments in beams of constant section and with fixed ends

$M = m \times W \times l$

m = Coefficient taken from diagram
W = Total load on beam
l = Length of beam
a = Length in terms of l



Portland Cement Association.)

TABLE 13-1.—INFLUENCE-LINE ORDINATES FOR FIXED-END MOMENT

Line	Point	1	2	3	4	5	6	6′	5′	4′	3′	2′	1′	Σ
1	t, (ft.)	3.04	2.53	2.12	1.82	1.61	1.51							
2	I	2.34	1.35	0.795	0.503	0.347	0.287							
3	$w = s \div I = 4 \div I$	1.71	2.96	5.03	7.95	11.53	13.95	13.95	11.53	7.95	5.03	2.96	1.71	
4	x	−22	−18	−14	−10	−6	−2	+2	+6	+10	+14	+18	+22	
5	xw	−37.6	−53.3	−70.3	−79.5	−69.2	−27.9	+27.9	+69.2	+79.5	+70.3	+53.3	+37.6	
6	Σxw	0	+37.6	+90.9	+161.2	+240.7	+309.9	+337.8	+309.9	+240.7	+161.2	+90.9	+37.6	0
7	$M_{xw} \div 4$	2,018.4	1,980.8	1,889.9	1,728.7	1,488.0	1,178.1	840.3	530.4	289.7	128.5	37.6	0	
8	M_{xw}	8,070	7,920	7,560	6,910	5,950	4,710	3,360	2,120	1,160	510	150	0	
9	x^2	484	324	196	100	36	4	4	36	100	196	324	484	
10	x^2w	828	959	986	795	415	56	56	415	795	986	959	828	8,078
11	$V_b = M_{xw} \div \Sigma x^2w$	1	0.98	0.935	0.855	0.736	0.583	0.416	0.262	0.144	0.063	0.019	0	
12	Σw	86.26	84.55	81.59	76.56	68.61	57.08	43.13	29.18	17.65	9.70	4.67	1.71	86.26
13	$M_w \div 4$	474.43	389.88	308.29	231.73	163.12	106.04	62.91	33.73	16.08	6.38	1.71	0	
14	M_w	1,898	1,560	1,233	927	652	424	252	135	64	26	7	0	
15	$M_w \div \Sigma w$	22.0	18.1	14.3	10.8	7.6	4.9	2.9	1.6	0.7	0.3	0.1	0	
16	$24V_b$	24.0	23.5	22.4	20.5	17.7	14.0	10.0	6.3	3.5	1.5	0.5	0	
17	M_b	−2.0	−5.4	−8.1	−9.7	−10.1	−9.1	−7.1	−4.7	−2.8	−1.2	−0.4	0	
18	$f = M_b \div 48$	−0.042	−0.112	−0.169	−0.202	−0.210	−0.190	−0.148	−0.098	−0.058	−0.025	−0.008	0	

$$V_b = \frac{\Sigma M_p xw}{\Sigma x^2 w} = \frac{M_{xw}}{\Sigma x^2 w}. \qquad M_b = \frac{\Sigma M_p w}{\Sigma w} - \frac{L}{2} V_b = \frac{M_w}{\Sigma w} - \frac{L}{2} V_b.$$

(Note that m times Eq. (13-7) is zero; also all terms that involve Σxw are zero.)

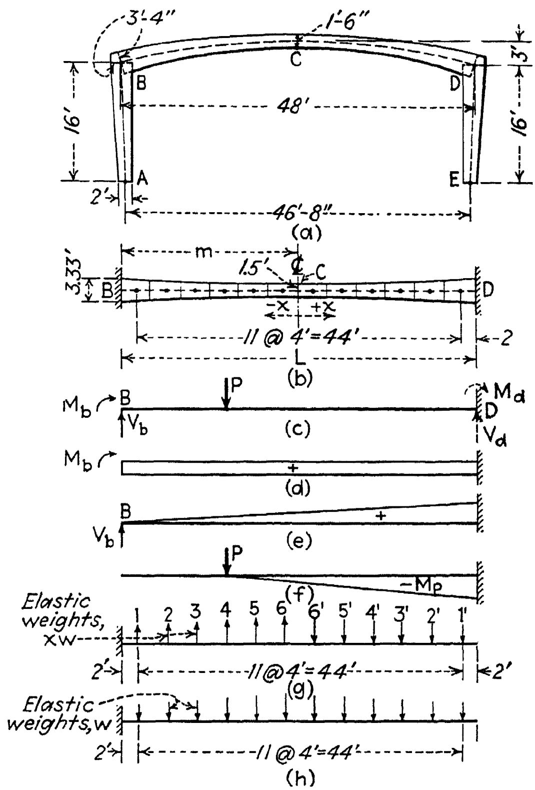


FIG. 13-3.

From Eqs. (13-7) and (13-8),

$$V_a = \frac{\Sigma M_p x w}{\Sigma x^2 w} \tag{13-9}$$

and

$$M_a = \frac{\Sigma M_p w}{\Sigma w} - m V_a. \tag{13-10}$$

Furthermore,

$$M_b = M_a + V_a L - Pb \tag{13-11}$$

s shown by Eq. (13-3).

The details of the application of the preceding equations will be illustrated in the following problem. However, for members with a constant moment of inertia, the fixed-end moments for ordinary conditions of loading are as shown in Fig. 13-2.

Problem 13-1. Find the fixed-end moments for BD of the hinged-end frame of Fig. 13-3(a), assuming it to be 12 in. wide.

This frame is the same structure as that of Fig. 11-22 from which the dimensions are to be copied. It is selected in order to show how to treat slightly curved or inclined members as well as straight ones.

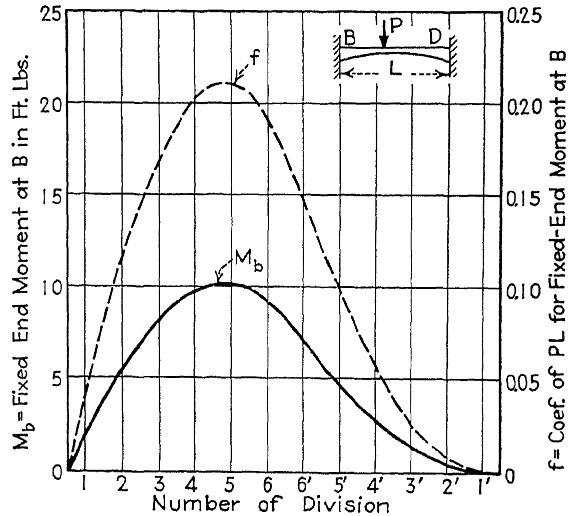


FIG. 13-4.—Influence lines for fixed-end moments.

The deck is shown straightened out in Fig. 13-3(b), the span being taken as the horizontal distance between B and D. Sketches (c), (d), (e), and (f) show the cut member and the moment diagrams. Sketches (g) and (h) picture the imaginary beams with the elastic weights as loads. The symmetry of the deck indicates at once that the elastic center is at mid-span, so that $m = L/2$.

The fundamental data for BD and the solutions of Eqs. (13-9) and (13-10) are given in Table 13-1. The calculations for the bending moments in the imaginary cantilevered beam—and for the influence-line ordinates—have been made by slide rule. The influence lines for the fixed-end moments are plotted in Fig. 13-4. The curve for the coefficient f is the influence-line ordinate for M_b divided by L; i.e., $M_b \div 48$.

When a member is symmetrical, the influence lines for the end moments at the two ends are equal, but they are opposite hand. When a member is unsymmetrical, the influence line for the fixed-end moments at one end must

be found first. It is then easy to use these values and to calculate the ordinates for the influence line for fixed-end moments at the other end.

Naturally, when a member with fixed ends has no loads acting upon it, the fixed-end moments at its ends are zero, provided there is no displacement of joints, as in Art. 13-8.

13-4. Carry-over and Stiffness Factors. When a moment is applied at the simply supported end of a beam, a resisting moment will be induced at the opposite end, if that end is fixed. The "carry-over factor" is the factor by which the applied moment at the simply supported end must be multiplied to obtain the induced moment at the fixed end.

The term "stiffness factor" denotes the rigidity of a member—a measure of its resistance to flexural distortion. The stiffness factor is defined as the moment that must be applied at the simply supported end of a member—whose far end is fixed—in order to produce unit rotation of the simply supported end.

The carry-over and stiffness factors for a beam may be found by the principles of work. Assume the beam of Fig. 13-5(a) which, at first, has hinged ends. Apply a moment M_a at A. This will cause the beam to curve as shown diagrammatically in (b). The rotations of the ends will be called θ'_a and θ'_b. The general work equation for such angular rotation is

$$= \int Mm \frac{dx}{EI}, \tag{13-12}$$

in which M = the bending moment that is caused by the applied forces or moments and in which m = the bending moment that is due to a unit, or "dummy," moment applied at the point where the angular rotation is desired.

Referring to Fig. 13-5 and denoting moments that cause compression in the top fibers as positive, the bending-moment diagram for M_a is as shown in (c); that for $m = 1$ at A, in (d); that for $m = 1$ at B, in (e). Using the values for M and m from Sketches (c) and (d), Eq. (13-12) becomes

$$\theta'_a = M_a \int \frac{(L - x)^2}{L^2} \frac{dx}{EI} = M_a \alpha_1 \tag{13-13}$$

where α_1 = the angular rotation at A [Fig. 13-5(b)] if $M_a = 1$. Using Sketches (c) and (e),

$$\theta_b' = M_a \int \frac{(\quad - x)x \, dx}{} = M_a\alpha \tag{13-14}$$

where α_2 = the angular rotation at B [Fig. 13-5(*b*)], when M_a = 1.

Now, assume another moment M_b to be applied at B, causing the beam to curve as shown in Fig. 13-5(*f*). Sketches (*g*), (*h*),

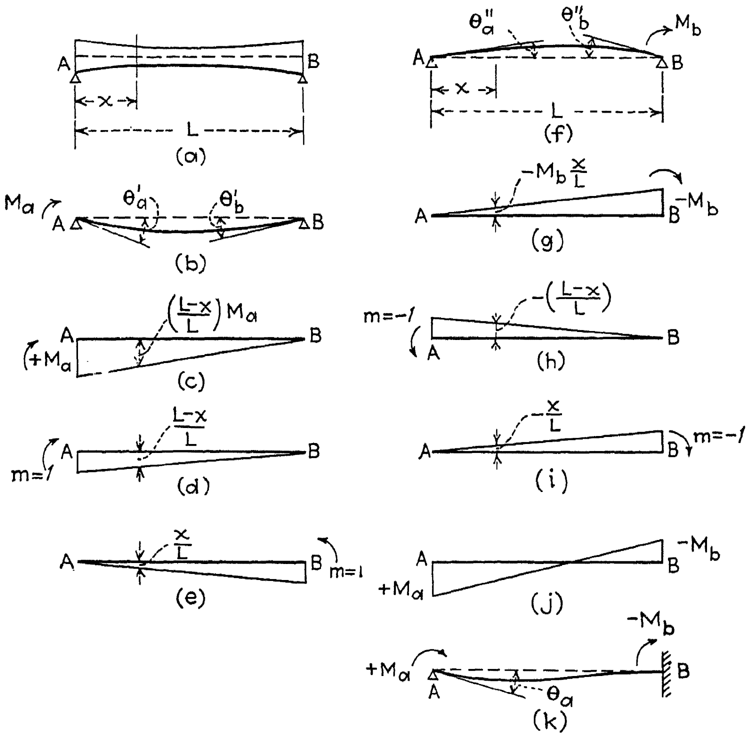


FIG. 13-5.

and (*i*) give the bending-moment diagrams for M_b, $m = 1$ at A, and $m = 1$ at B, respectively. Therefore, using Sketches (*g*) and (*h*), Eq. (13-12) becomes

$$= M_b \int \frac{(}{} \tag{13-15}$$

where α_2 = the same integral as in Eq. (13-14); but in this case, it is the angular rotation at A when a unit moment is applied at

B, showing that Maxwell's theorem of reciprocal deflections applies to rotations as well as to linear deflections. In other words, Eqs. (13-14) and (13-15) show that, when a unit moment is applied at one end of a simply supported beam, the angular deformation at the opposite end is a constant no matter at which end the unit moment is applied. Using values from Sketches (g) and (i), Eq. (13-12) becomes

$$\theta_b'' = M_b \int \frac{x^2}{L^2} \frac{dx}{EI} = M_b \alpha_3 \tag{13-16}$$

where α_3 is the angular rotation at B when a unit moment is applied at B.

It is important to notice that the angles in Eqs. (13-13) to (13-16), inclusive, are expressed in terms of the moments times the angles or coefficients α_1, α_2, and α_3. These coefficients, in turn, are physical constants for the particular member because they depend only upon the material as represented by E and upon the dimensions of the member as determined by L, x, and I. Therefore, up to the elastic limit of the materials, the angles of rotation vary directly as the applied moments.

If M_b in Fig. 13-5(f) is to be sufficient to hold B fixed when M_a is applied as in Sketch (b), then

$$\theta_b' - \theta_b'' = 0 = M_a \alpha_2 - M_b \alpha_3. \tag{13-17}$$

From Eq. (13-17),

$$M_b = M_a \frac{\alpha_2}{\alpha_3}. \tag{13-18}$$

Therefore, the carry-over factor C_A from A to B must be

$$C_A = +\frac{\alpha_2}{\alpha_3}. \tag{13-19}$$

By similar reasoning, the carry-over factor C_B from B to A is

$$C_B = +\frac{\alpha_2}{\alpha_1}. \tag{13-20}$$

Furthermore, when B is fixed—when M_b resists the action of M_a as pictured in Figs. 13-5(j) and (k)—the resultant angular deflection at A, considering Eq. (13-18), is

$$\theta_a = \theta'_a - \theta''_a = M_a\alpha_1 - M_b\alpha_2 = M_a\left(\alpha_1 - \frac{\alpha_2^2}{\alpha_3}\right). \quad (13\text{-}21)$$

By definition, the stiffness factor for the beam at A—called K_A—is the moment at A that will produce unit rotation at A when B is fixed. Equation (13-21) then yields

$$K_A = M_a = 1 \div \left(\alpha_1 - \frac{\alpha_2^2}{\alpha_3}\right) = \frac{\alpha_3}{\alpha_1\alpha_3 - \alpha_2^2}. \quad (13\text{-}22)$$

Again, by similar reasoning for B,

$$K_B = 1 \div \left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1}\right) = \frac{\alpha_1}{\alpha_1\alpha_3 - \alpha_2^2}. \quad (13\text{-}23)$$

Equations (13-19), (13-20), (13-22), and (13-23) are general expressions. In connection with the carry-over factor C and the stiffness factor K, the following should be noted:

1. When one end of a member is hinged, $C = 0$ for a moment that is applied at the other end, because the hinge has zero stiffness compared to that of the member itself.

2. When one end of a member is fixed and a moment is applied at that fixed end, no moment is carried over to the other end, because the fixed end is infinitely stiff compared to the member.

3. When a member is symmetrical, C and K are the same for both ends, since $\alpha_1 = \alpha_3$.

4. When a member is unsymmetrical, C is smaller for the stiffer end than it is for the weaker one, but K is larger because the strong end tends to hold more of the moment to itself.

5. When a member is prismatic, I is constant, and

$$C = \frac{\alpha_2}{\alpha_3} = \left[\frac{Lx^2}{2} - \frac{x^3}{3}\right]_0^L \div \left[\frac{x^3}{3}\right]_0^L = \frac{1}{2} \quad (13\text{-}24)$$

and

$$K = 1 \div \left(\alpha_1 - \frac{\alpha_2^2}{\alpha_3}\right) = 1 \div \frac{1}{EI}\left(\frac{L}{3} - \frac{L^2}{36} \times \frac{3}{L}\right) = \frac{4EI}{L}. \quad (13\text{-}25)$$

This clearly shows that the stiffness factor varies as the I/L of such a member. C and K are the same at both ends of the member.

6. When a moment is applied at one end of a member and the other end is hinged, the stiffness factor is less than that which is

given in Eq. (13-22), because $M_b = 0$ and α_2^2/α_3 in Eq. (13-21) must be zero. Therefore, in such a case, Eq. (13-22) gives

$$K_A' = 1 \div \alpha_1. \tag{13-26}$$

From Eqs. (13-19), (13-20), and (13-22), the expression

$$- C_A C_B) = \frac{\alpha_3}{\cdots - \alpha_2^2}\left(1 - \frac{\alpha_2}{\alpha_3} \times \frac{\alpha_2}{\alpha_1}\right) = \frac{1}{\alpha_1}.$$

Therefore, Eq. (13-26) gives

$$K_A' = K_A(1 - C_A C_B). \tag{13-27}$$

This means that the stiffness factor when the far end of the beam is hinged is equal to the stiffness factor when the far end is fixed times 1 minus the product of the carry-over factors for both ends when those ends are fixed. This is a convenient form to employ when using the tables which will be explained later for members having variable I.

In the case of a prismatic member,

$$K_A' = \frac{1}{\alpha_1} = 3\frac{EI}{L}.$$

This shows that the stiffness factor of such a member with the far end hinged is three-fourths of its value when the far end is fixed.

7. In general, for ends A and B of any member,

$$C_A K_A = \frac{\alpha_2}{\alpha_3}\left(\frac{\alpha_3}{\alpha_1\alpha_3 - \alpha_2^2}\right) = \frac{\alpha_2}{\alpha_1\alpha_3 - \alpha_2^2}$$

and

$$C_B K_B = \frac{\alpha_2}{\alpha_1}\left(\frac{\alpha_1}{\alpha_1\alpha_3 - \alpha_2^2}\right) = \quad \alpha_2$$

This proves that

$$C_A K_A = C_B K_B. \tag{13-28}$$

In other words, the product of the carry-over factor and the stiffness factor at one end of a member equals the same product for the other end. Equation (13-28) gives a valuable check on one's calculations.

Problem 13-2. Find the stiffness and carry-over factors for the deck of the frame of Fig. 13-3(a), using the dimensions from Fig. 11-22 and the data of Table 13-2 which gives Σw, Σxw, and Σx^2w for the frame with the origin at the left end.

Using $w = dx/I = s/I$ in α_2 of Eq. (13-14),

$$\Sigma(L - x)xw$$

Substituting the summations from Table 13-2,

$$\alpha_2 \quad \frac{2{,}070}{48E} \quad \frac{57{,}770}{48^2E} = \frac{18.05}{E}$$

Similarly, from Eq. (13–16),

$$= \quad 1 \quad \Sigma \quad = \quad 57{,}770 \quad = \quad 25.07$$

Equation (13-19) may be rewritten to yield

$$C_B = C_D = \frac{\alpha_2}{\alpha_3} = \frac{18.05}{25.07}$$

The stiffness factor is found by rewriting Eq. (13-22) as follows, remembering that

$$- \alpha_2^2$$

$$\frac{25.07E}{25.07^2 - 18.05^2} = 0.083E.$$

Problem 13-3. Find the stiffness factor at B for the leg AB of the rigid frame of Fig. 13-3(a), A and E being hinged.

The leg is shown in a rotated position in Fig. 13-6. The data in Table 13-3 are worked out with the origin at B. The inclined length of the leg is disregarded.

From Eqs. (13-26) and (13–13),

$$K_B' = 1 \div \alpha_1 = 1 \div \sum \frac{(L - \quad)^2}{EI}$$

$$\frac{L^2E}{\Sigma(L - x)^2w} \quad 636 \qquad 0.403E.$$

13-5. Sign Convention. In applying moment distribution, it is necessary to set up a sign *convention* to use in determining the signs of the end moments in the members. In order to avoid confusion when dealing with multiple frames, the method that makes the sign depend upon the kind of stress in the top fibers will not be used. Instead, a *moment* will be called "positive"[1] when it *tends to rotate the joint in a clockwise direction;* it will be called "negative" when it tends to rotate the joint in a counter-clockwise direction.

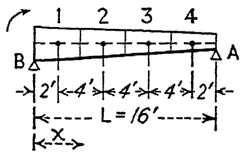


FIG. 13-6.

The important relationships between the signs, rotations, and moments should be noted. The direction of attempted rotation is opposite to the direction of the restraining moment which must act at the end of a member. Figure 13-7 shows that, if a horizontal member has a positive moment at its left end, there will be tension in the top fibers; if there is a positive moment at its right end, there will be compression in the top fibers. The same statements apply to the left-hand fibers of a vertical member when the bottom corresponds to the left end of the horizontal one.

TABLE 13-3.—DATA FOR LEG OF RIGID FRAME TO FIND K

Line	Point	1	2	3	4	Σ
1	t	3.17	2.83	2.5	2.17	
2	I	2.66	1.89	1.30	0.85	
3	$w = s \div I = 4 \div I$	1.50	2.12	3.08	4.70	
4	x	2	6	10	14	
5	$(L - x)$	14	10	6	2	
6	$(L - x)^2$	196	100	36	4	
7	$(L - x)^2w$	294	212	111	19	636

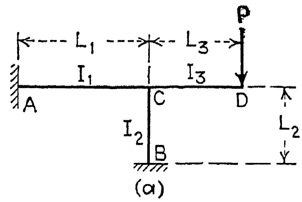
Reference to Fig. 13-7 also shows that when there is no point of contraflexure or when there are two such points, the signs of the end moments are opposite; when there is one point of contraflexure, the signs of the end moments are alike.

The fact that the carry-over factors as given in Eqs. (13-19) and (13-20) are positive shows that the induced moment at one

[1] See "Moment Distribution Applied to Continuous Concrete Structures," Portland Cement Association, No. ST40.

end of a beam has the same sign as the applied moment at the other end.

13-6. Distribution of Moments. When several members meet at a rigid joint, the algebraic sum of all the end moments at the joint must be zero if the point is in equilibrium; otherwise there will be rotation until ΣM becomes zero. For instance, assume the frame of Fig. 13-8(a). Consider C to be fixed. There will then be a bending moment M' at C in the cantilevered beam CD. It is called a fixed-end moment; but since C does not rotate, no bending is carried into AC and CB. Therefore, is not zero, but it is an "unbalanced fixed-end moment," sometimes called U, which must be resisted by some other balancing moment. As soon as the joint C is released, or "unlocked" from its fixed condition,



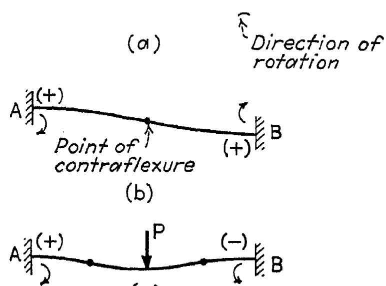


FIG. 13-7.

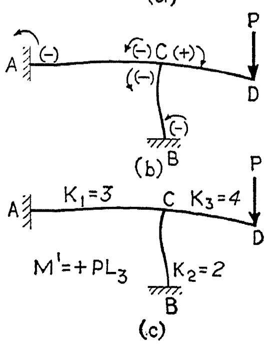


FIG. 13-8.

it rotates until AC and CB stop this rotation, as pictured in Sketch (b), at which time $\Sigma M = 0$ at C.

Obviously, since C rotates as a unit, AC and CB resist the rotation of C in accordance with their ability to do so—in accordance with their stiffnesses—and in so doing, each resists a certain proportion of the unbalanced moment. From Eq. (13-25), the stiffness factor for a prismatic member is $\frac{4EI}{L}$. However, the constants 4 and E may be omitted. Then

$$K_1 \text{ for } AC = \frac{I_1}{L} \qquad \text{and} \qquad K_2 \text{ for } CB = \frac{I_2}{L_2}.$$

Assume these values of K to be as shown in Fig. 13-8(c). The combined stiffness of AC and $CB = 3 + 2 = 5$. (K_3 for CD does not enter the calculation because the stiffness of a cantilevered beam may be considered to be zero.) Therefore, $M_{CA} = -\frac{K_1M'}{\Sigma K} = -\frac{3}{5}M' = -\frac{3}{5}PL_3$, and $M_{CB} = -$

$-\frac{2}{5}M' = -\frac{2}{5}PL_3$. The moments at C are now balanced, $= 0$, and the balancing moment has been "distributed" between AC and CB. Of course, the signs of the distributed moments are opposite to those of the unbalanced moment.

However, the bending in AC and CB at C causes these members to induce bending moments at A and B which are the respective distributed moments M_{CA} and M_{CB} times the carry-over factor, which, according to Eq. (13-24), is $\frac{1}{2}$.

If A of Fig. 13-8 is hinged, it can resist no bending, the member AC is less stiff, and K_1 will be $\frac{3}{4} \times 3 = \frac{9}{4}$. In this case, with B fixed, the total stiffness of AC and CB is $\frac{9}{4} + 2 = \frac{17}{4}$, so that $M_{CA} = -\frac{9}{17}PL_3$, and $M_{CB} = -\frac{8}{17}PL_3$.

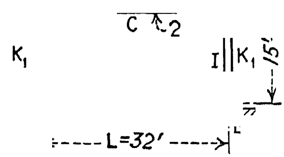
13-7. Procedure in Application of Moment Distribution. An outline of the steps to be followed in the analysis of a structure by moment distribution (distribution of fixed-end moments) is the following:

1. Draw an outline of the structure with all of its loads.
2. Compute the fixed-end moments for each member that supports a load (or is bent by other action).
3. Calculate the stiffness factors for all members, and find the relative stiffnesses of all members that meet at a common joint.
4. Determine the carry-over factors for both ends of all members.
5. "Unlock" one joint, compute the unbalanced moment, distribute the balancing moment among the members that meet at that joint (in proportion to their stiffnesses), then lock the joint again.
6. Carry over the induced moments to the far ends of the members.
7. Unlock the next joint, and proceed as before.
8. Continue from joint to joint until all have been included and the first "cycle" is completed; start over again at the first joint, using the new unbalanced moments, and continue successive cycles until the unbalanced moments are negligible. In many cases, it is advantageous to distribute the moments at *all* joints first, then to perform *all* of the carry-over operations, thus completing the first cycle. Successive cycles are made in the same way. This method will be used in the solutions of problems because it helps to avoid errors.

9. The final end moment in any member will be the algebraic sum of all of the moments that have accumulated at that particular end.

The procedure will be illustrated further by the solution of problems.

Problem 13-4. Find the moments at B and D of Fig. 13-9(a) if A and E are hinged.



	Col. B	Bm.		Bm.	Col. B
$\frac{K}{\Sigma K}$	0.44	0.56	.5	0.56	0.44
fem	0	+100		−100	0
	−44	− 56		+ 56	+44
	0 ↑	+ 28 ←		− 28 →	0 ↑
2-D	−12	− 16		+ 16	+12
2-C.O.	0 ↑	+ 8 ←		− 8 →	0 ↑
3-D	− 4	− 4		+ 4	+ 4
3-C.O.	0 ↑	+ 2 ←		− 2 →	0 ↑
4-D	− 1	− 1		+ 1	+ 1
	−61	+61		−61	+61

A (b)

FIG. 13-9.

The magnitudes of the fixed-end moments (f.e.m.) at B and D, when they are "locked," are

$$M_B = M_D = \frac{PL}{8} = \frac{25 \times 32}{8} = 100 \text{ ft.-kips.}$$

$$K_1 \text{ for } AB \text{ and } ED = \frac{3}{4}\frac{I}{15} = \frac{I}{20}, \qquad K_2 = \frac{2I}{16}.$$

$$\Sigma K = K_1 + K_2 = \frac{9I}{80} \text{ at } B \text{ and } D.$$

Therefore, when a balancing moment is distributed between BA and BD,

$$M_{BA} = \frac{K_1}{\Sigma K}U = 0.44U \quad \text{(approx.), and}$$

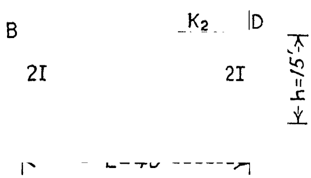


FIG. 13-10.

$M_{BD} = \frac{K_2}{\Sigma K}U = 0.56U$. These coefficients are recorded at the joints as shown in Fig. 13-9(b). The heading "Col. B" denotes the moments in the column below the joint—either B or D. The heading "Bm." denotes the moments in the beam at the side of the joint which corresponds to the position of the tabulated data.

The fixed-end moments are recorded next, and the first distribution is made as shown. A line is drawn under these distributed moments. The

first carry-over moments are then set down under this line, the small arrows at the right of the numbers indicating the directions from which the moments were carried. Notice that no moment can be carried to A and E and that none will be transferred back from these points. The unbalanced moment at B after the first cycle is $+28$. The successive distributing and carrying over of moments are continued until the unbalanced moments are sufficiently small to be neglected, the work ending upon a last distribution of the moments. The symbols "1−D," "1−C.O.", etc., in Fig. 13-9 denote the "first distribution," "first carry over," etc. The tabulated values in each column are summed up to obtain the final moments. (Compare the answer with that for Problem 11-4.) The carry-over factor is .5.

	Col.B	Bm.	Bm.	Col.B
$\overline{\Sigma K}_B$	0.84	0.16	0.16	0.84
fem		+112	−38	
1-D	−94	−18	+6	+32
1-C.O.	0↑	+3←	−9→	0↑
2-D	−3	0	+1	+8
2-C.O.			0→	0↑
3-D			0	0
	−97	+97	−40	+40

	Col.A	Col.A
$\frac{K}{\Sigma K}$		
fem		
1-D		
1-C.O.	−47↓	+16↓
2-D		0
2-C.O.		+4↓
3-D	0	0
ΣM	−48	+20

FIG. 13-11.

Problem 13-5. Find the moments at the joints of the frame of Fig. 13-10 if A and E are fixed. Side sway, which will be discussed in a subsequent article, is assumed to be prevented.

$$K_1 = \frac{2I}{15} = 0.133I, \qquad \frac{\quad}{40} = 0.025I, \qquad \Sigma K \text{ at } B \text{ or } D = 0.158I.$$

$$K_1 \qquad \frac{K_2}{\quad} \; 0.16.$$

$$\text{At } B\text{, f.e.m.} = \frac{+20 \times 10 \times 30^2}{40^2} \quad +112 \text{ ft.-kips.}$$

$$\text{At } D\text{, f.e.m.} \quad \frac{-Pa^2b}{\quad} \quad \frac{-20 \times 10^2 \times 30}{\quad} \quad -38 \text{ ft.-kips.}$$

In this case, because of the fixed ends at A and E, the moments that are carried from B to A or from D to E are immediately locked at A or E, respectively.

This problem is solved in Fig. 13-11. The heading "Col. A" in the tables denotes the moments in the column above the joint—in this case A or E. The final moments at the joints are plotted in Fig. 13-12(a). From these moments, the shears and reactions are found. For instance, the magnitude of H_a is the numerical sum[1] of the end moments in AB (without reference to signs) divided by the height of the leg. Therefore, $H_a = (M_{AB} + M_{BA}) \div h$; $H_e = (M_{ED} + M_{DE}) \div h$; $V_a = 0.75P + (M_{BD} - M_{DB}) \div L$; $V_e = 0.25P - (M_{DB} - M_{BD}) \div L$. The force $S' = H_a - H_e = 5.7$ kips.

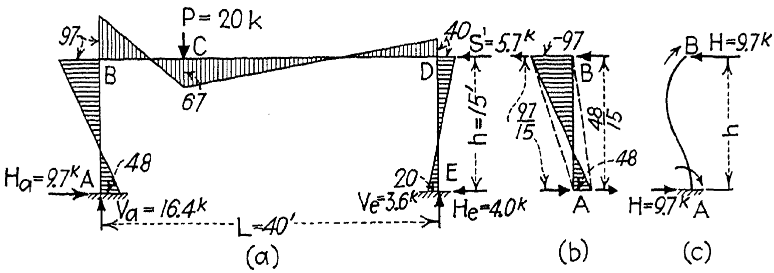


FIG. 13-12.

The direction of H_a can be found by studying Figs. 13-12(b) and (c). From the latter, $H \times h = M_{BA} + M_{AB}$, and H must act toward the left at B and toward the right at A. H is the "apparent" shear in AB.

13-8. Displacement of Joints. Although a member may have no loads acting between its ends, fixed-end moments may be produced by displacement of the joints.

In Fig. 13-12A(a), beam AB is fixed at B. A moment is applied at A, rotating that end through an angle θ_a and causing the tangent to move from B to B'. The angle θ_a is small, and it may be equated to its tangent d/L. The induced moment at B is therefore

$$M_b' = M_a'C_A = K_A\frac{d}{L}C_A.$$

Another moment is then applied at B in Sketch (b), causing the angular deflection $\theta_b = \theta_a$. Therefore,

If the beams in Sketches (a) and (b) are then combined and rotated until the deflected tangents are horizontal, the picture

[1] If M_{BA} and M_{AB} both caused tension on the left side of AB, H_a would be $(M_{BA} - M_{AB}) \div h$.

in (c) will result. This shows that the combined action is equivalent to that which would occur if the end A moved a distance d.

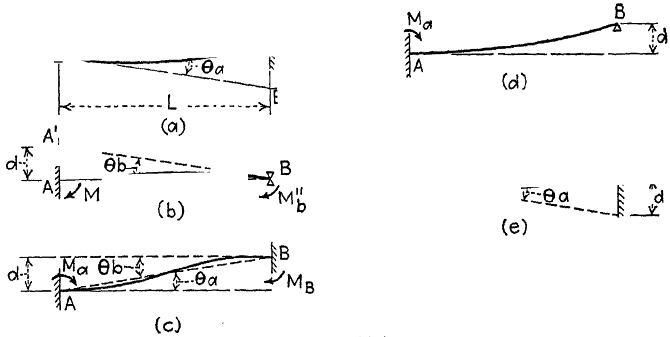


FIG. 13-12A.

From Eq. (13-28), $C_A K_A = C_B K_B$. Therefore, adding the moments at A and B for both of the cases pictured in Figs. 13-12$A(a)$ and (b),

$$
\begin{aligned}
M_a &= M'_a + M''_a = K_A\frac{d}{L} + K_B\frac{d}{L}C_B = K_A\frac{d}{L}(1 + C_A \\
M_b &= M'_b + M''_b = K_A\frac{d}{L}C_A + K_B\frac{d}{L} = K_B\frac{d}{L}(1 + C_B
\end{aligned}
\qquad (13\text{-}29)
$$

These give the fixed-end moments for a displacement equal to d. Therefore, if the beam is prismatic,

$$EId \qquad \cdot\frac{1}{2}\Big) =$$

Incidentally, it must be remembered that the stiffness factor involves E, so that when using tabulated values that omit E, this term must be added. When a member with one hinged end is displaced as shown in Fig. 13-12$A(d)$, $M_b = 0$, and the second term of Eq. (13-29) for M_a which involves K_B and C_B is also zero. Furthermore, K_A in the first term must be changed to K'_A in accordance with Eq. (13-27).

In the case of the rotation of one joint as pictured in Fig. 13-12$A(e)$, the condition is just the same as in Sketch (a). Therefore, the moment at A is the stiffness factor K_A times the ratio $\frac{\theta_a}{1}$, and the moment at B is $M_a C_A$. (θ_a is in radians.)

Equations (13-29) can be used in the solution of problems that involve both vertical settlements and horizontal displacements of joints, such as the movement of foundations, the changes of length caused by shrinkage or variations in temperature, and the lateral displacements resulting from wind or from other forces that cause side sway. These will be illustrated in subsequent problems.

It is often difficult to determine whether or not a structure will be able to adjust itself for side sway. In the case of a rigid-frame bridge, the earth fill may prevent all or part of this motion. However, it is sometimes advisable to analyze a structure with

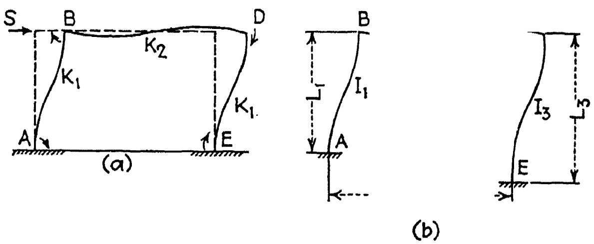


FIG. 13-13.

and without side sway and to design each part for whichever condition requires the greater strength.

Problem 13-6. Find the moments in the frame of Fig. 13-10, considering side sway but using the first part of the solution as already completed in Problem 13-5. $E = 432{,}000{,}000$ lb. per sq. ft.

In examining the results of Problem 13-5, it is apparent that the force S' is needed to hold the frame in equilibrium. If the structure is free to move or to "sway" sidewise with no force S' to stop it, the frame will distort as pictured in Fig. 13-13(a). The effect is equivalent to applying to the frame an equal but opposite force S at B [Fig. 13-13(a)]. Therefore, the moments and forces of Fig. 13-12(a) must be altered before the frame can come to rest.

When the legs are equal in all respects, as in Fig. 13-13(a), they divide the moments and shears equally between them because B and D deflect the same amount. However, when the legs are prismatic but I and L are unequal, as in Fig. 13-13(b), Eq. (13-30) will show that the moments vary as I/L^2; the shears vary as I/L^3.

Since the legs are prismatic and fixed at their supports, they are assumed to have a point of contraflexure at their centers so that the magnitudes of the moments at both ends of each member are equal. When the legs are hinged at the bottom, all of the moments must be resisted by the restrained ends at the top. (These statements refer to f.e.m.)

An examination of Eq. (13-30) shows that the deflection d must be known if M_a is to be evaluated. Therefore, a convenient way to solve this problem is to assume dE, compute the fixed-end moments, analyze the frame, and find the resultant shear, then correct the moments in proportion to the ratio of S to this resultant shear.

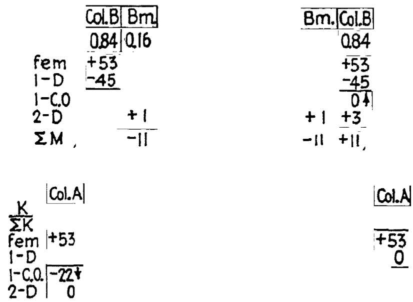


FIG. 13-14.

Therefore, assume $dE = 1{,}000{,}000$, the displacement being to the right, since S in Fig. 13-13(a) acts in this direction. Also, $I = 1$ in this case for the deck and $I = 2$ for the legs. From Eq. (13-30),

$$M_{AB} = \qquad M_{DE} = M_{ED} = \frac{6 \times 2 \times 1{,}000{,}000}{15^2}$$

53 ft.-kips.

The moment distribution is worked out in Fig. 13-14. The procedure is as follows:

1. Distribute the fixed-end moment at B between AB and BD in the ratio of 0.84: 0.16, giving -45 and -8. Do likewise at D.

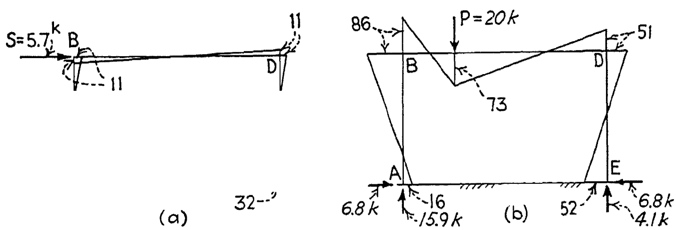


FIG. 13-15.

2. Carry over one-half of -45 to A and one-half of -8 to D, etc. Two cycles are sufficient.

3. The sum of the final moments in AB is $+42$. Therefore,

$$= \frac{42}{h} = \frac{42}{15} = 2.8 \text{ kips}$$

for AB alone, or $2.8 \times 2 = 5.6$ kips for AB and DE combined. Then the factor by which the moments of Fig. 13-14 must be multiplied to correct

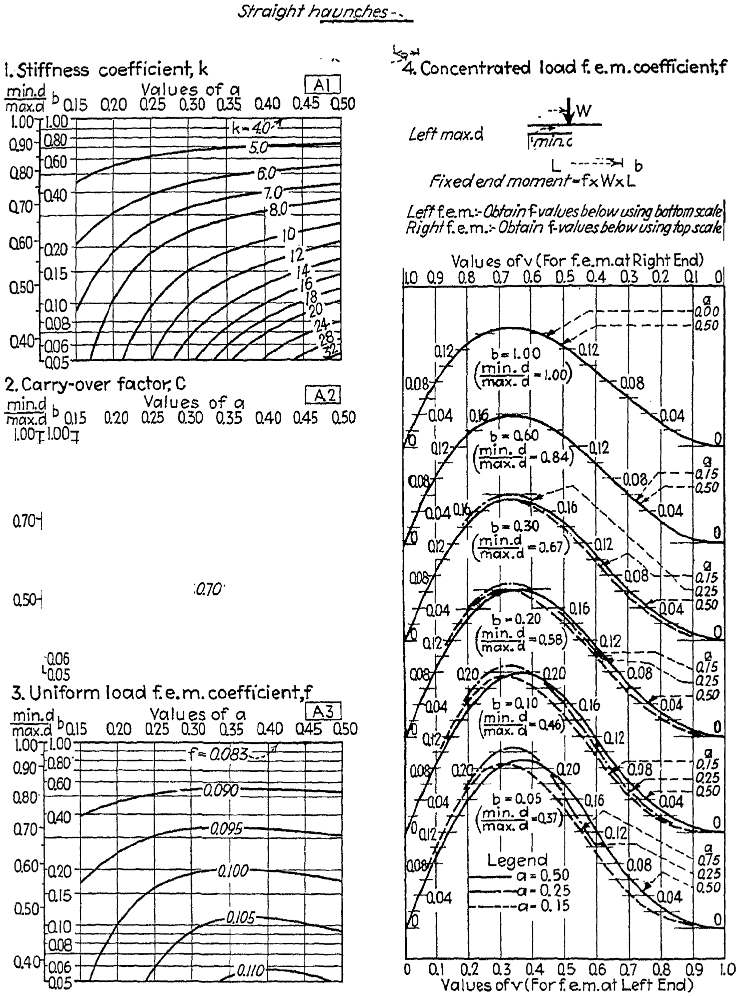


FIG. 13-16.—Symmetrical members with straight haunches.

them is

$$\frac{S}{H} \quad \frac{5.7}{5.6}$$

These corrected moments are shown in Fig. 13-15(a).

4. The final moments are obtained by combining the moments that are given in Fig. 13-12(*a*) with those in Fig. 13-15(*a*). The results are plotted

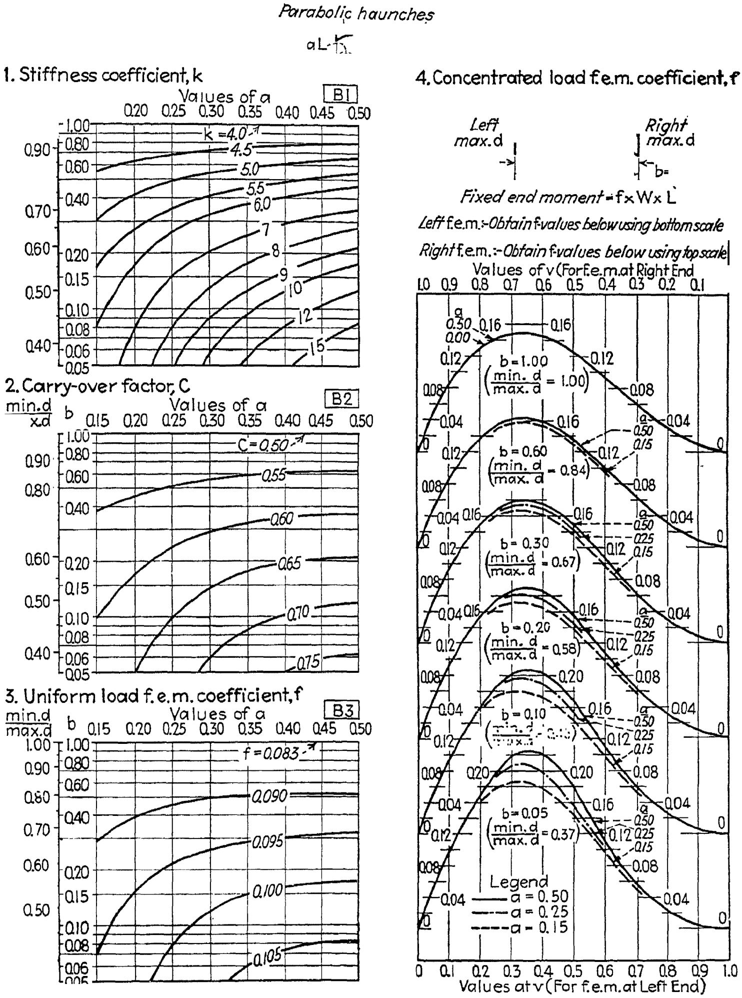


FIG. 13-17.—Symmetrical members with parabolic haunches.

in Fig. 13-15(*b*). They should be compared with the answers for Problem 11-5 of Chap. 11.

13-9. Use of Curves to Determine Fixed-end Moments, Stiffness, and Carry-over Factors. The curves of Figs. 13-16 to

13-21, inclusive,[1] have been prepared to eliminate tedious work in the calculation of fixed-end moments, stiffness factors, and

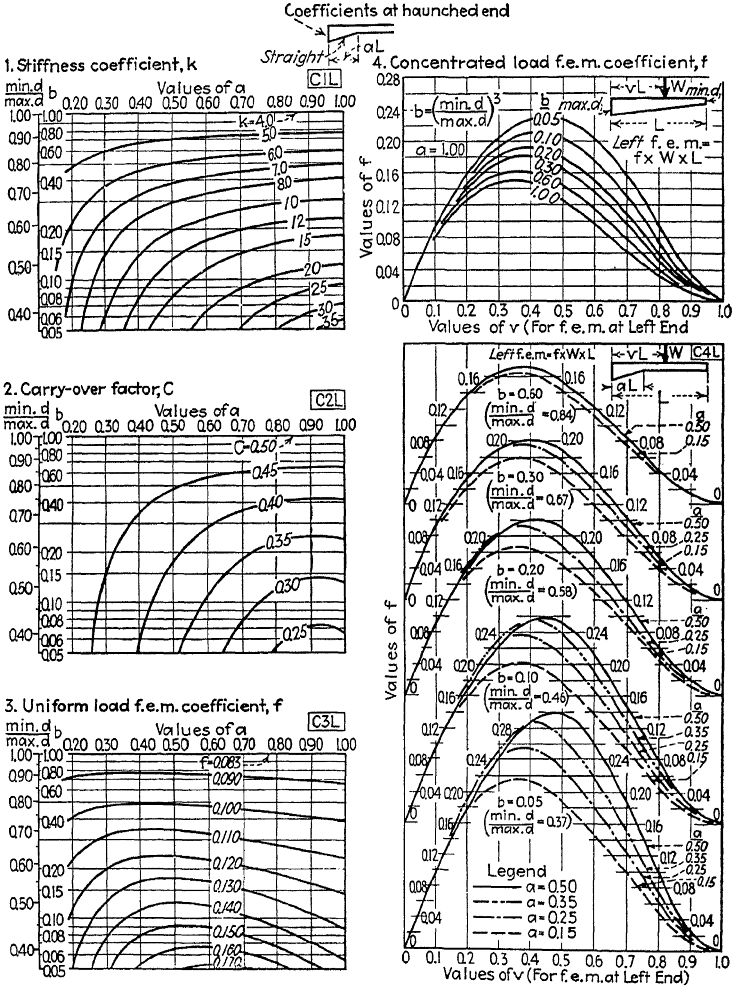


FIG. 13-18.—Unsymmetrical members with straight haunch at one end.

carry-over factors for nonprismatic members. Since E is con-

[1] Figures. 13-16 to 13-21, inclusive, are taken in their entirety from "Concrete Information," Pamphlet ST41, issued by the Structural Bureau of the Portland Cement Association.

stant, it is omitted from the figures, but it must not be forgotten in cases like Problem 13-6.

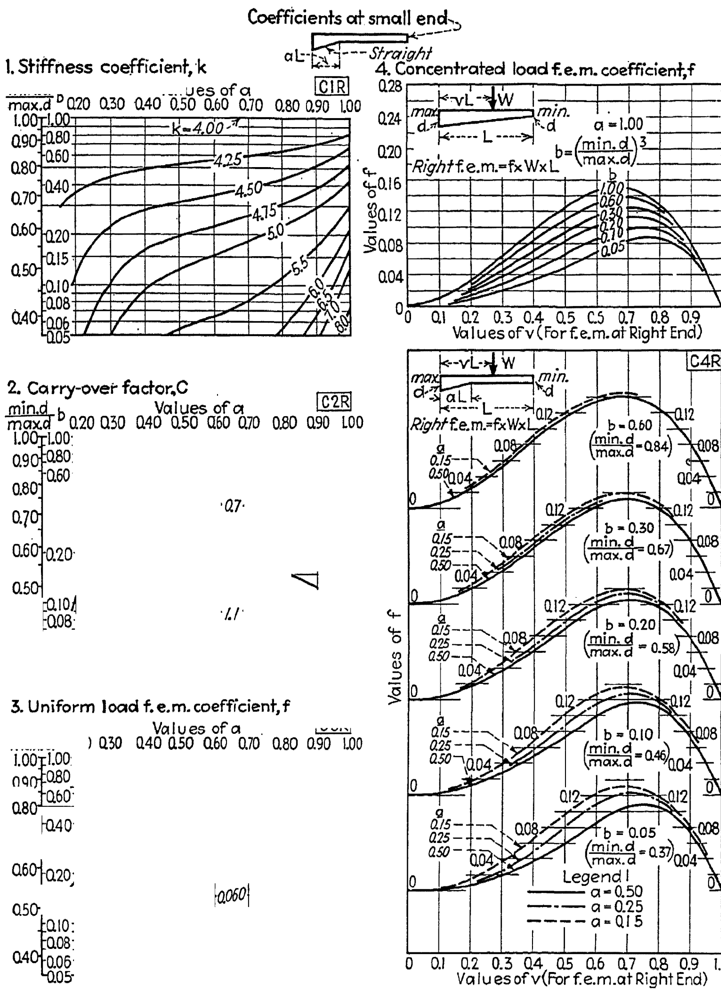


FIG. 13-19.—Unsymmetrical members with straight haunch at one end.

These diagrams will render great service to the designer. For instance, compare the influence line for f (Fig. 13-4) with Diagram $B4$ (Fig. 13-17), using $a = 0.5$ and $b = 1.5^3/3.33^3 = 0.092$. In order to check the stiffness factor K_B for BD of Problem 13-2,

enter Sketch $B1$ (Fig. 13-17), with $a = 0.5$ and $b = 0.092$, reading $k = 14$. When this coefficient is multiplied by (min. I) $1/L$, it

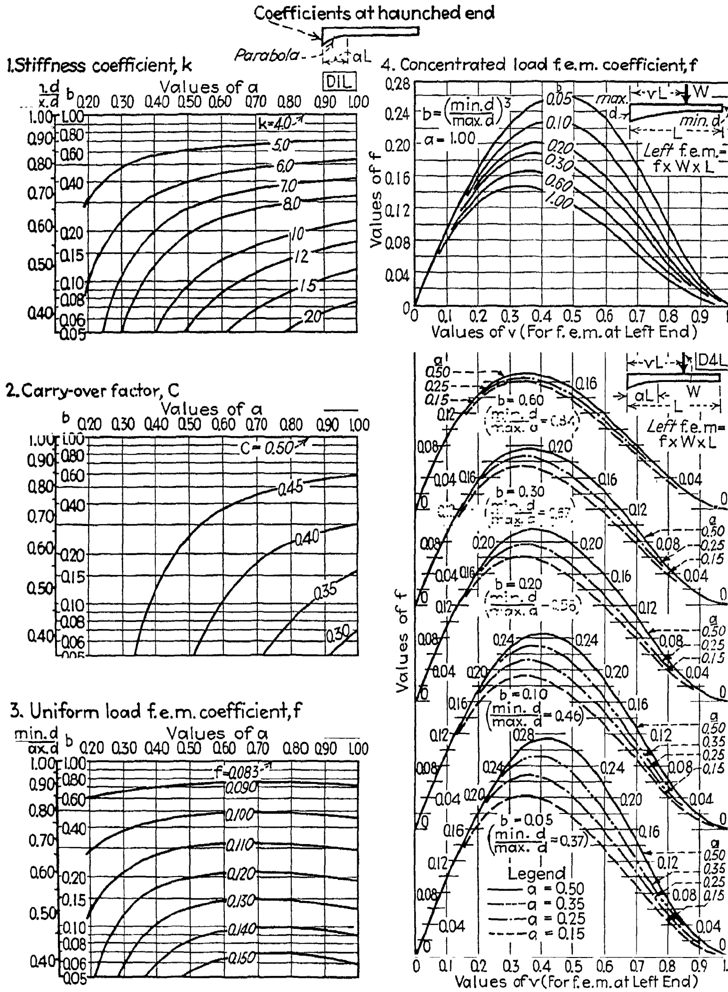


FIG. 13-20.—Unsymmetrical members with parabolic haunch at one end.

gives

$$K_B = \frac{k}{L} \text{ (min. } I) = \frac{14}{48} \times \frac{1.5^3}{12} = 0.082.$$

Again, to find the carry-over factor C, enter Sketch $B2$ with the

same values of a and b, finding $C_B = 0.72$. Now, compare K_B and C_B with their values as computed in Problem 13-2.

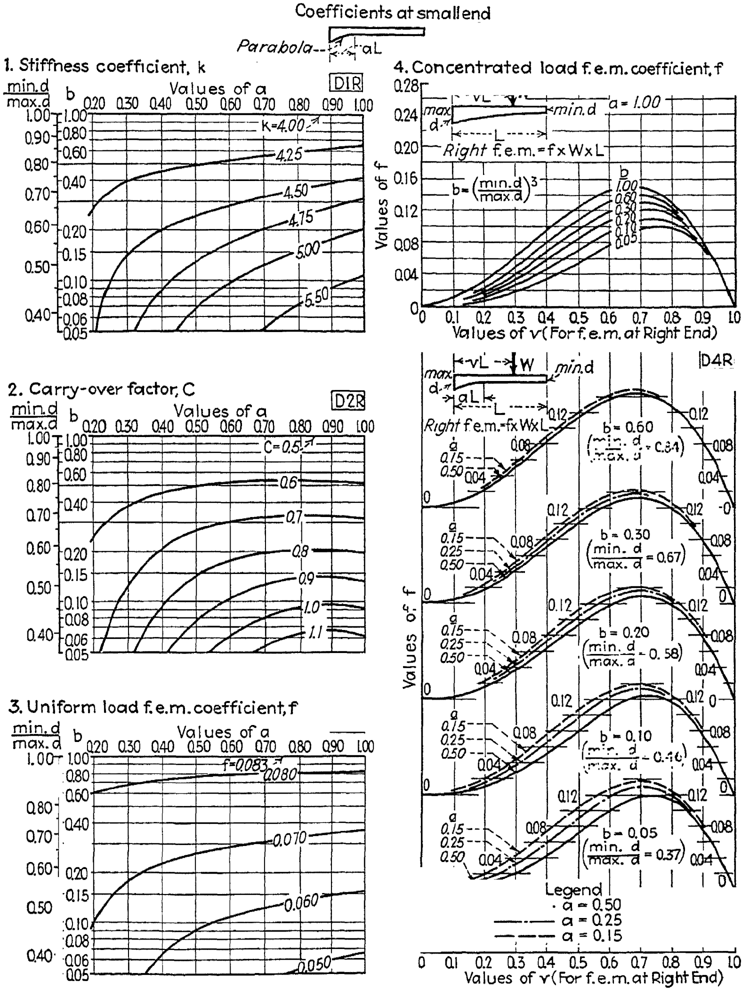


FIG. 13-21.—Unsymmetrical members with parabolic haunch at one end.

As another illustration of the use of these curves, solve Problem 13-3 again for K_B. Equation (13-27) applies to a member with one hinged end. With the alteration of the subscripts, it becomes

$$K'_B = K_B(1 - C_A C_B).$$

Figure 13-18 may be used first. Obviously, $a = 1.0$ and $b = (\text{min. } d/\text{max. } d)^3 = (2/3.33)^3 = 0.216$. Using these values in Diagram $C1L$, $k = 13$. Therefore, $K_B = (\text{min. } I)k/L = \frac{13}{16} \times 2^3/12 = 0.542$. Diagram $C2L$ gives $C_B = 0.34$. The magnitude of C_A must be found from Diagram $C2R$ (Fig. 13-19). Using the same values for a and b, $C_A = 0.74$. From these quantities,

$$K_B' = 0.542(1 - 0.74 \times 0.34) = 0.405.$$

This answer is sufficiently accurate.

13-10. Curved Members. The method of moment distribution is based upon the assumption that the axes of the members of a structure are straight. In many cases, these axes are curved slightly, especially in nonprismatic members, and this curvature

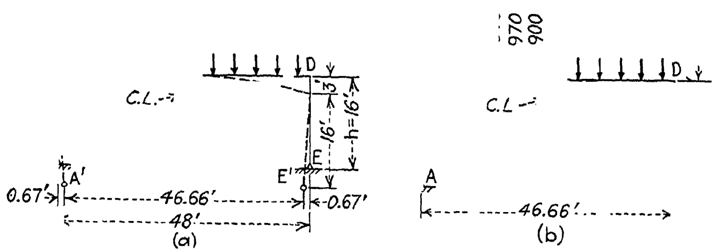


FIG. 13-22.

may be disregarded. However, if it is too great (perhaps a rise of 1 ft. in a span of 25 ft.), the method of work must be used in analysis so as to avoid excessive errors.

The maximum permissible curvature, the positions in which the substituted straight axes should be placed, and the corrections to be made to the results of the calculations—these are matters for which there seems to be no set of rules except such as may seem to be reasonable to the individual. However, some guidance may be secured from the solution of the next problem which is selected in order to illustrate this case.

Problem 13-7. Find the bending moments at B and C in the frame of Fig. 13-3(a) due to the dead load of the deck, using concrete at 150 lb. per cu. ft.

Assume that the real axes of the frame are shown by the dotted lines in Fig. 13-22(a); then, as an experiment, straighten out the deck and the legs, holding point C. This will result in the frame that is shown by solid lines in Sketch (a).

The fixed-end moments can be found from the influence line of Fig. 13-4. Using the ordinates for f as given in Table 13-1,

$$\text{f.e.m.} = L[1{,}870(0.042 + 0) + 1{,}540(0.112 + 0.008) + 1{,}280(0.169 + 0.025) + 1{,}090(0.202 + 0.058) + 970(0.210 + 0.098) + 900(0.190 + 0.148)] = 1{,}399L.$$

Therefore,

$$\text{f.e.m.} = 67{,}100 \text{ ft.-lb.}$$

The carry-over and stiffness factors, as determined in Problems 13-2 and 13-3, omitting E, are as follows:

(1) For the deck: $C_{BD} = C_{DB} = 0.72$; $K_{BD} = K_{DB} = 0.083$.
(2) For the legs: $K_{BA} = 0.403$; $C_{BA} = 0$.
(3) $\Sigma K = K_{BD} + K_{BA} = 0.083 + 0.403 = 0.486$.
(4) For the deck: $K_{BD} \div \Sigma K = 0.083 \div 0.486 = 0.17$.
(5) For the legs: $K_{BA} \div \Sigma K = 0.403 \div 0.486 = 0.83$.

These values of the relative stiffnesses of the members are shown in Fig. 13-23 along with the fixed-end moments. The usual procedure of dis-

B	Col.B	Bm.	Bm.	Col.B
$\frac{K}{\Sigma K}$	0.83	0.17	0.17	0.83
fem		+67.1	−67.1	
1-D	−55.7	−11.4	+11.4	+55.7
1-C.O.			−8.2	
2-D	−6.8	−1.4	+1.4	+6.8
2-C.O.		1.	0	
3-D	−0.8	−0.2	+0.2	+0.8
	−63.3	+63.3	+63.3	−63.3

FIG. 13-23.

B	Col.B	Bm.	Bm.	Col.B
$\frac{K}{\Sigma K}$	0.8	0.2	0.2	0.8
fem		+65.2	−65.2	
1-D	−52.2	−13.0	+13.0	+52.2
1-C.O.		+9.4	−9.4	
2-D	−7.5	−1.9	+1.9	+7.5
2-C.O.		+1.4	−1.4	
3-D	−1.1	−0.3	+0.3	+1.1
	−60.8	+60.8	−60.8	+60.8

A

FIG. 13-24.

tributing the balancing moments and of carrying over these distributed moments is indicated in the sketch. The moment at B is found to be $-63{,}300$ ft.-lb.

The moment at C is merely the moment at C in a simply supported beam with a span of 48 ft. minus the restraining moment at B. Therefore,

$$= +78{,}400 - 63{,}300 = +15{,}100 \text{ ft.-lb.}$$

From the solution of Problem 11-6, M at $B = -52{,}000$ ft.-lb., and M_{CB} at $C = +11{,}700$ ft.-lb. Obviously, these differ too much from the solutions as found herein. It is therefore necessary to review the original assumptions in order to find the reasons for this.

Figure 13-22(a) shows that the tilt of the legs was neglected. When they slope inward, the result is a reduction in the effective span to $A'E' = 46.66$ ft. Furthermore, the 3-ft. rise of the deck is somewhat equivalent to a decrease in the stiffness of the legs. Therefore, assume the frame to be as shown by the heavy lines in Fig. 13-22(b), passing through A, C, and E.

It is now necessary to correct f.e.m., C, and K. For the first, assume that the influence line of Fig. 13-4 still holds, the member being slightly com-

pressed so that

$$\text{f.e.m.} = 1{,}399L = 1{,}399 \times 46.66 = 65{,}200 \text{ ft.-lb.}$$

Next, let $C_{BD} = 0.72$ as before. Finally, following the methods of Art. 13-9, the new magnitude of K_{BD} for the deck will be

$$\frac{14}{46.66} \times \frac{1.5^3}{12} = 0.084.$$

Also, K_{BA} for the leg $= K_{BA}(1 - C_A C_B) = \frac{k}{L}(\text{min. } I)(1 -$

$$K_{BA} = \frac{13}{19} \times \frac{2^3}{12}(1 - 0.71 \times 0.34) = 0.341.$$

$$\Sigma K = K_{BD} + K_{BA} = 0.084 + 0.341 = 0.425.$$

$$K_{BD} \div \Sigma K = 0.084 \div 0.425 = 0.2.$$

$$K_{BA} \div \Sigma K = 0.341 \div 0.425 = 0.8.$$

The frame is now reanalyzed in Fig. 13-24. The moment at B is $-60{,}800$ ft.-lb. The simple beam moment at C is $+73{,}300$ ft.-lb. when the span is 46.66 ft. and when the loads are as shown in Fig. 13-22(b). Therefore,

$$M_{CB} = +73{,}300 - 60{,}800 = +12{,}500 \text{ ft.-lb.}$$

Furthermore, if the moment at B is divided by $h = 19$ ft., the horizontal thrust at A of Fig. 13-22(b) is

$$H_a = \frac{60{,}800}{19} = 3{,}200 \text{ lb.}$$

This value of H_a times 16 ft. will give a partially corrected moment at B which is

$$M_{BA} = -3{,}200 \times 16 = -51{,}200 \text{ ft.-lb.}$$

These adjusted values of the moments now agree closely with those which were calculated in Problem 11-6.

All of this goes to show that members with considerable curvature need special consideration when they are analyzed by moment distribution.

13-11. Two-span Frames.

These structures will be illustrated by a practical problem.

Problem 13-8. Find the bending moments in the symmetrical two-span rigid-frame bridge of Fig. 13-25(a) for a uniform live load of 270 lb. per sq. ft. on span BC, using moment distribution and assuming that side sway can take place.

This problem is the same as Problem 11-14. It is used to illustrate a very general case in which the members are unsymmetrical and somewhat curved.

The first step is the assumption of an equivalent structure with straight axes. The dimensions of the actual structure are shown in Fig. 13-25(a).

The "equivalent" frame with straight axes is determined arbitrarily as follows: The legs are taken as vertical members whose axes pass through the centers of the supports at A and E because this represents the effective span; the axis of the center pier is already vertical, but its length is shortened to 17.5 ft. to allow for the fact that the portion above the springing lines below C is infinitely stiff compared to the pier shaft; the axis of the deck is taken as a horizontal line which passes through the crown of the curve of the deck. The structure of Fig. 13-25(b) results.

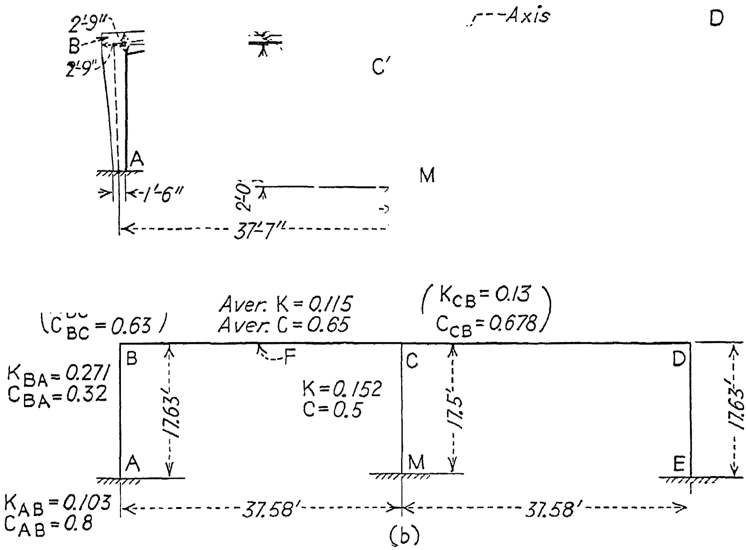


FIG. 13-25.

The next step in the solution of the problem is the determination of the physical constants of the structure—K and C—for a strip 1 ft. wide. These are found from the diagrams of Figs. 13-17, 13-18, and 13-19 as follows:

1. *Leg AB.* Determine

$$I_A = \frac{1.5^3}{12} = 0.281 \text{ ft.}^4$$

$$\frac{\quad}{12} \qquad 1.733 \text{ ft.}^4$$

and

$$b = \left(\frac{\text{min. } d}{\text{max. } d}\right)^3 = \frac{\quad}{2.75} \qquad 0.162.$$

Enter Fig. 13-19($C1R$) with b = 0.162 and a = 1.0, and find k_A at the small end = 6.5. Therefore,

$$= 0.103E.$$

With the same values of b and a, find $C_{AB} = 0.8$ from Fig. 13-19($C2R$).

Similarly, from Figs. 13-18($C1L$) and ($C2L$), obtain $k_B = 17$ and $C_{BA} = 0.32$. Therefore,

$$\frac{17 \times 0.281E}{17.63} =$$

2. *Deck BC.* Determine

$$I_F \text{ (at crown)} = \frac{1.75^3}{12} = 0.477 \text{ ft.}^4$$

$$I_c = \frac{3.25^3}{12} = 2.86 \text{ ft.}^4$$

Since the deck is not symmetrical, it is necessary to approximate its stiffness and carry-over factors. Therefore, assume that both halves are the same as BF so that

$$b = \left(\frac{\text{min. } d}{\text{max. } d}\right)^3 = \left(\frac{1.75}{2.75}\right)^3 = 0.258$$

and, from Figs. 13-17($B1$) and ($B2$), with $a = 0.5$ and $b = 0.258$, $k_B = 8.4$ and $C_{BC} = 0.63$. Therefore,

Also, if both halves are the same as FC,

$$k_C = 10.8 \qquad \text{and} \qquad C_{CB} = 0.678.$$

Therefore,

$$10.8 \times 0.447E$$

By averaging these values, an approximation of the real constants will be obtained; thus,

$$\text{Ave. } K = \frac{0.1E + 0.13E}{2} = 0.115E^*$$

* One should construct the influence line for f.e.m. and should compute the stiffness and carry-over factors as in Arts. 13-3 and 13-4 in order to get more correct values for this case. However, it is assumed here that the data secured from the diagrams will be satisfactory.

$$\text{Ave. } C = \frac{0.63 + 0.678}{2} = 0.65 \text{ (approx.)}.$$

It should be noticed that Fig. 13-17 is for members with parabolic haunches, but its use in this case is sufficiently accurate for practical purposes. Furthermore, E can be omitted from the figures because it is constant.

3. *Pier CM.* Since CM is a prismatic member,

$$K_{CM} = K_{MC} = \frac{4EI}{L} = \frac{4 \times 8E}{17.5 \times 12} = 0.152E$$

and

$$C_{CM} = C_{MC} = 0.5.$$

4. *Relative Stiffnesses.* At B, $\Sigma K = K + K_{BA} = 0.115 + 0.271 = 0.386$.

Therefore, the relative stiffnesses of BC and BA at A are

$$K_1 = \frac{K}{\Sigma K} = \frac{0.115}{0.386} = 0.3 \text{ for } BC$$

and

$$K_2 = \frac{K_{BA}}{\Sigma K} = \frac{0.271}{0.386} = 0.7 \text{ for } BA.$$

At C, there are three members, BC, CD, and CM. Therefore,

$$\Sigma K = 2K + K_{CM} = 2 \times 0.115 + 0.152 = 0.382.$$

Therefore,

$$K_1 \text{ at } C = \frac{0.115}{0.382} = 0.30 \text{ for } CB \text{ and } CD.$$

$$K_2 \text{ at } C = 0.40.$$

It is now necessary to determine the f.e.m. at B and C for a load of 0.27 kip per ft. To do this, refer to Fig. 13-17($B3$). With $b = 0.258$ (as in item 2) and $a = 0.5$,

$$\text{f.e.m.} = 0.097WL.$$

With $b = 0.156$,

$$\text{f.e.m.} = 0.101WL.$$

Therefore,

$$\text{Ave. f.e.m.} = 0.099WL = 0.099 \times 0.27 \times 37.58 \times 37.58 = 37.7 \text{ ft.-kips}.$$

The distribution of the moments when side sway is prevented is completed next, as shown in Fig. 13-26(a). B, C, and D are unlocked in succession, and the balancing moments are distributed; all of the carry-overs are made

as shown by the arrows. In the third cycle at C, the two carry-over moments (-1.4) are combined before the distribution is made. The total moments are plotted in Fig. 13-26(b).

	Col. B	Bm.
$\frac{K}{\Sigma K}$	0.7	0.3
fem	0	+37.7
1-D	-26.4	-11.3
1-C.O.	0 ↑	+7.3 ←
2-D	-5.1	-2.2
2-C.O.	0 ↑	+1.4 ←
3-D	-1.0	-0.4
3-C.O.	0 ↑	+0.5 ←
4-D	-0.4	-0.1
ΣM	-32.9	+32.9

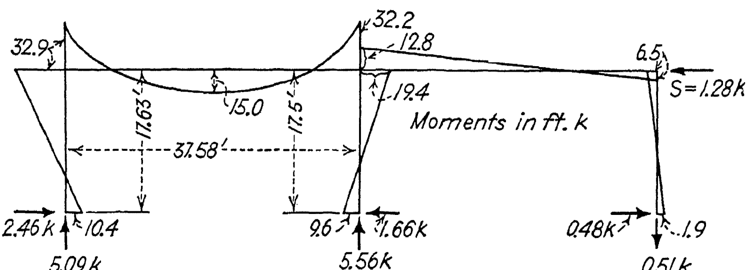
Bm.	Col. B	Bm.
0.3	0.4	0.3
-37.7	C 0	0
+11.3	+15.1	+11.3
-7.3 →	0 ↑	0 ←
+2.2	+2.9	+2.2
-1.4 →	0 ↑	-1.4 ←
+0.8	+1.2	+0.8
-0.3 →	0 ↑	-0.3 ←
+0.2	+0.2	+0.2
-32.2		+12.8

Bm.	Col. B
0.3	0.7
0	D 0
0	0
+7.3 →	0 ↑
-2.2	-5.1
+1.4 →	0 ↑
-0.4	-1.0
+0.5 →	0 ↑
-0.1	-0.4
+6.5	-6.5

(0.32)

	Col. A (A)	Col. A (M)	Col. A (E)
	—	—	—
fem	0	0	0
1-D	0	0	0
1-C.O.	-8.5 ↓	+7.6 ↓	0 ↓
2-D	0	0	0
2-C.O.	-1.6 ↓	+1.4 ↓	-1.6 ↓
3-D	0	0	0
3-C.O.	-0.3 ↓	+0.6 ↓	-0.3 ↓
4-D	0	0	0
ΣM	-10.4	+9.6	-1.9

(a)



(b)

FIG. 13-26.

The horizontal reaction H_a at the bottom of the leg AB is

$$(M_{AB} + M_{BA}) \div AB = (32.9 + 10.4) \div 17.63 = 2.46 \text{ kips.}$$

Similarly,

$$H_m = (19.4 + 9.6) \div 17.5 = 1.66 \text{ kips} \quad \text{and} \quad H_e = (6.5 + 1.9) \div 17.63 = 0.48 \text{ kip.}$$

It is important to find the directions of these reactions. In Fig. 13-27, the legs and pier are isolated, and the end moments are applied. It is easily seen that M_{BA} and M_{AB} tend to rotate AB in a clockwise direction so that, for equilibrium, the forces H_a and H_b must form an opposing couple, acting as shown.

The algebraic sum of H_a, H_m, and H_e = 1.28 kips acting toward the right. Therefore, the unbalanced force at the top is 1.28 kips acting toward the left and preventing side sway.

In order to find the effect of side sway, a force of 1.28 kips is to be applied at B [Fig. 13-25(*b*)]. It will move the top joints toward the right an unknown distance d, all joints being assumed to move equally. However, if Eqs. (13-29) are to be used, it is clear that the end moments cannot be calculated when d is unknown. The procedure is therefore the usual simple expedient of arbitrarily assuming a magnitude for dE (notice that E cannot

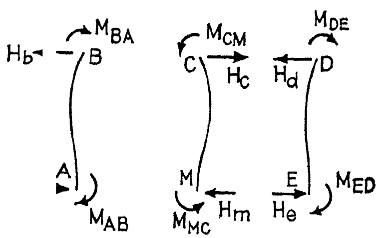


FIG. 13-27.

be omitted), solving for the end moments in the legs and in the pier, distributing these moments, finding the unbalanced horizontal shear which results from these moments, and then adjusting these distributed moments in the ratio of the real force S to the one which resulted from the assumed deflection.

In this case, assume dE = 1,000,000 lb. per ft.
Then Eqs. (13-29) give

$$M_{BA} = M_{DE} = K_{BA}(1 + C_{BA})\frac{dE}{L} = 0.271(1 + 0.32)\frac{1,000,000}{17.63} = 20,300 \text{ ft.-lb.}$$

$$M_{AB} = M_{ED} = K_{AB}(1 + C_{AB})\frac{dE}{L} = 0.103(1 + 0.8)\frac{1,000,000}{17.63} = 10,500 \text{ ft.-lb.}$$

$$M_{CM} = M_{MC} = K_C(1 + C_C)\frac{dE}{L} = 0.152(1 + 0.5)\frac{1,000,000}{17.50} = 13,000 \text{ ft.-lb.}$$

These fixed-end moments (in foot-kips) are applied at the ends of the vertical members; then the moments are distributed as shown in Fig. 13-28(*a*), unlocking points B, D, and C in this order for each cycle. Proceeding as

before, $H_a = H_e = \dfrac{6.9 + 6.2}{17.63} = 0.74$ kip, and $H_m = \dfrac{10.8 + 11.8}{17.5} = 1.29$ kips. $\Sigma H = 2.77$ kips. The factor for adjusting the side-sway moments is

	Col. B	Bm.	Bm.	Col. B	Bm.	Bm.	Col. B
$\frac{K}{\Sigma K}$ B	0.7	0.3	0.3	0.4	0.3	0.3	0.7
fem	+20.3	0	0	C+13.0	0	0	D+20.3
1-D	−14.2	−6.1	−3.9	−5.2	−3.9	−6.1	−14.2
1-C.O.	0	−2.5	−4.0	0	−4.0	−2.5	0
2-D	+1.8	+0.7	+2.4	+3.2	+2.4	+0.7	+1.8
2-C.O.	0	+1.6	+0.5	0	+0.5	+1.6	0
3-D	−1.1	−0.5	−0.3	−0.4	−0.3	−0.5	−1.1
3-C.O.	0	−0.2	−0.3	0	−0.3	−0.2	0
4-D	+0.1	+0.1	+0.2	+0.2	+0.2	+0.1	+0.1
ΣM	+6.9	−6.9	−5.4	+10.8	−5.4	−6.9	+6.9

(0.32)

	Col. A (A)	Col. A (M)	Col. A (E)
	−	−	−
fem	+10.5	+13.0	+10.5
1-D	0	0	0
1-C.O.	−4.5	−2.6	−4.5
2-D	0	0	0
2-C.O.	+0.6	+1.6	+0.6
3-D	0	0	0
3-C.O.	−0.4	−0.2	−0.4
4-D	0	0	0
Σ M	+6.2	+11.8	+6.2

(a)

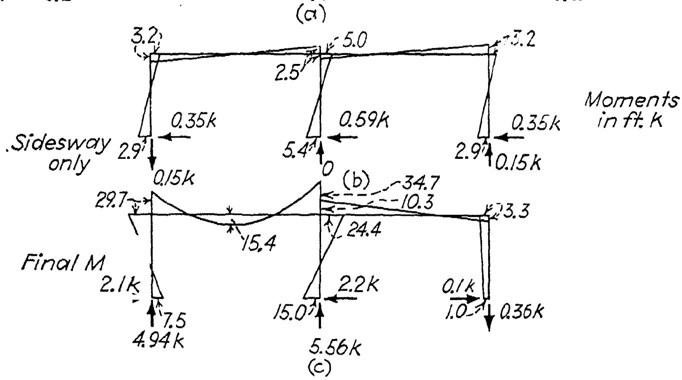


FIG. 13-28.

1.28/2.77 = 0.46. This is applied to the results of Fig. 13-28(*a*) to get the moments that are plotted in Sketch (*b*).

The combined moments from Figs. 13-26(*b*) and 13-28(*b*) are shown in Fig. 13-28(*c*). These results should be compared with those in Table 11-11 of Chap. 11. The critical moments check with reasonable accuracy. How-

ever, there is one important point to notice; viz., the method of work automatically includes side sway, but moment distribution gives the results with or without side sway. This is another distinct advantage in favor of analysis by moment distribution.

Problem 13-9. Find the moments in the frame of Fig. 13-25(a) for a fall in temperature of 40°F. if E = 432,000,000 lb. per sq. ft. and ω = 0.000006.

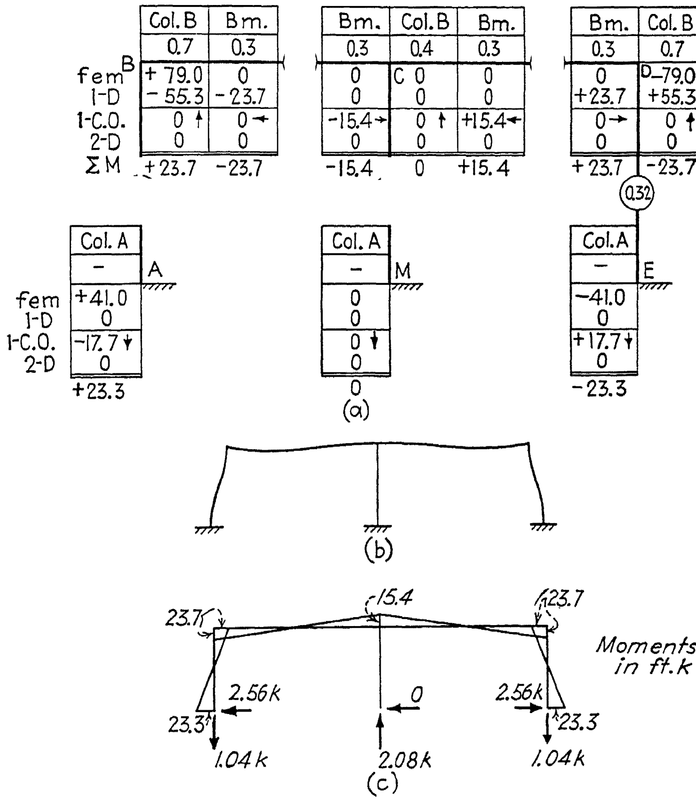


FIG. 13-29.

The shortening of each half of the bridge is $d = \Delta L = \omega t L = 0.000006 \times 40 \times 37.58 = 0.009$ ft. Therefore,

$$dE = 0.009 \times 432{,}000{,}000 = 3{,}890{,}000 \text{ lb. per ft.}$$

The structure will act symmetrically about the center pier so that the change in temperature will not affect the pier. However, Eqs. (13-29), adding E, will give the bending moments in the legs:

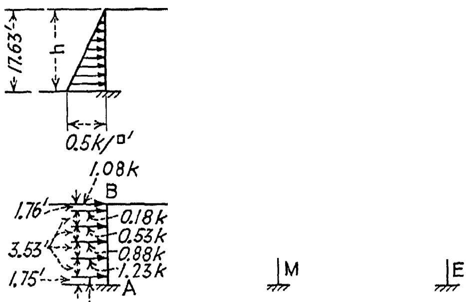
$$M_{BA} = M_{DE} = K_{BA}(1 + C_{BA})\frac{dE}{L} = 0.271(1 + 0.32)\frac{3{,}890{,}000}{17.63}$$

79,000 ft.-lb.

$$M_A \qquad + C_{AB})\frac{dE}{L} \qquad \cdot 0.8)\,3{,}890{,}000$$

41,000 ft.-lb.

The moments will have positive signs for AB and negative ones for DE. The analysis is shown in Fig. 13-29. One cycle is sufficient to complete the work.



(b)

(c)

FIG. 13-30.

Problem 13-10. Find the moments in the frame of Fig. 13-25(a) for the unsymmetrical earth pressures and the braking force which are shown in Fig. 13-30(a) as acting upon the substitute frame of Fig. 13-25(b).

The uniformly varying earth pressures are changed to five approximately equivalent concentrated forces at $0.1h$, $0.3h$, etc., as pictured in Fig. 13-30(b). The braking force is also changed by the principle of the lever to an equivalent force at B, 1.08 kips.

Interpolation in Figs. 13-18($C4L$) and 13-19($C4R$), with $b = 0.162$ and $a = 1.0$, gives the ordinate by which each intermediate lateral load of Fig. 13-30(b) must be multiplied to obtain the f.e.m. at A and B. When these products are summed up, f.e.m. at A = 5.7 ft.-kips, and f.e.m. at B = 8.0 ft.-kips. The distribution of these moments is given in Fig. 13-31(a).

Notice that the carry-overs to the fixed bases at A, M, and E are arbitrarily extended to one more cycle than was done for the deck.

	B: Col. B	B: Bm.	C: Bm.	C: Col. B	C: Bm.	D: Bm.	D: Col. B
K/ΣK	0.7	0.3	0.3	0.4	0.3	0.3	0.7
fem	−8.0	0	0	0	0	0	0
1-D	+5.6	+2.4	0	0	0	0	0
1-C.O.	0 ↑	0 ←	+1.6 →	0 ↑	0 ←	0 →	0 ↑
2-D	0	0	−0.5	−0.6	−0.5	0	0
2-C.O.	0 ↑	−0.3 ←	0 →	0 ↑	0 ←	−0.3	0 ↑
3-D	+0.2	+0.1	0	0	0	+0.1	+0.2
ΣM	−2.2	+2.2	+1.1	−0.6	−0.5	−0.2	+0.2

32

	A: Col. A	M: Col. A	E: Col. A
K/ΣK	−	−	−
fem	+5.7	0	0
1-D	0	0	0
1-C.O.	+1.8 ↓	0 ↓	0 ↓
2-D	0	0	0
2-C.O.	0 ↓	−0.3 ↓	0 ↓
3-D	0	0	0
3-C.O.	+0.1 ↓	0 ↓	+0.1 ↓
4-D	0	0	0
ΣM	+7.6	−0.3	+0.1

(a)

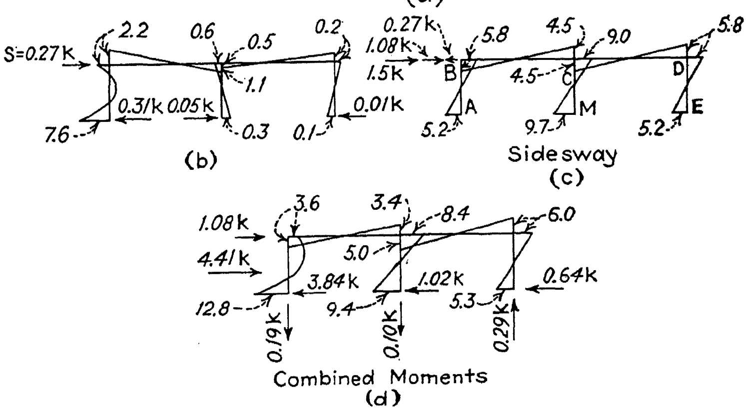


FIG. 13-31.

The moments from Fig. 13-31(a) are shown in Sketch (b). From this diagram it is clear that

0.31 kip acting toward the left.

17.63

$$H_m \quad \frac{0.6 + 0.3}{17.5} \quad 0.05 \text{ kip acting toward the right.}$$

$$H_e \quad \frac{0.2 + 0.1}{17.63} \quad 0.01 \text{ kip acting toward the left.}$$

Therefore,

0.27 kip acting toward the left.

Therefore, the force $S = 0.27$ kip (acting toward the right) must be added to produce equilibrium.

The moments that are due to side sway are found as follows, referring to Fig. 13-31 (c):

1. Apply 0.27 kip at B, acting toward the left to take into account the elimination of S in Sketch (b).
2. Find the forces at A and B for the loads on AB of Fig. 13-30(b) by taking moments about A. $\Sigma W = 4.41$ kips, $R_a = 2.91$ kips, and $R_b = 1.5$ kips. R_a acts toward the left.
3. The braking force is 1.08 kips acting toward the right in Fig. 13-30(b). Therefore, the resultant force which deflects the frame toward the right is $1.5 + 1.08 - 0.27 = 2.31$ kips as shown in Fig. 13-31(c).
4. The moments in Fig. 13-28(b) were computed for a force of 1.28 kips toward the right. If these values are multiplied by $2.31/1.28 = 1.8$, the diagram of Fig. 13-31(c) results.
5. The final moments are found by combining Sketches (b) and (c). The results are shown in (d). They can be checked by noting that the sum of the end reactions at A, M, and E is 5.50 kips (with $R_a = 2.91$ kips from item 2 included). This compares with $4.41 + 1.08 = 5.49$ kips, the total of the applied loads.

Problem 13-11. Find the moments in the frame of Fig. 13-25(a) if point A is assumed to be displaced 0.02 ft. toward the left and 0.04 ft. downward. Also, assume A to be rotated 10 min. in a counterclockwise direction. Use the substitute frame of Fig. 13-25(b), with $E = 432{,}000{,}000$ lb. per sq. ft.

A diagram to picture the distorted structure is given in Fig. 13-32(a).

By means of Eqs. (13-29), adding E, find the following:

$$dE = 0.02 \times 432{,}000{,}000 \quad 8{,}640{,}000 \text{ lb. per ft. (horiz.)}$$
$$dE = 0.04 \times 432{,}000{,}000 \quad 17{,}280{,}000 \text{ lb. per ft. (vert.)}$$

$$M_{BA} = K_{BA}(1 + C_{BA})\frac{dE}{L} = 0.271(1 + 0.32)\frac{8{,}640{,}000}{17.63} \quad 175{,}000 \text{ ft.-lb.}$$

$$M_{AB} = K_{AB}(1 + C_{AB})\frac{dE}{L} \quad 0.103(1 + 0.8)\frac{8{,}640{,}000}{17.63} = 91{,}000 \text{ ft.-lb.}$$

$$M_{BC} = M_{CB} = K_{BC}(1 + C_{BC})\frac{dE}{} \quad 0.115(1 \quad 37.58 \quad \text{ft.-lb.}$$

The moments in AB that result from the rotation of A are found from the stiffness and carry-over factors at A, as given in Fig. 13-25(b). The value

of K_{AB} as given in the diagram should include E. By definition K_{AB} is the moment that is required to rotate A through one radian. Therefore, by proportion,

$$M'_{AB} = K_{AB}\frac{E\theta_a}{1} = 0.103 \times 432{,}000{,}000 \times \frac{10}{57.3 \times 60} = 129{,}000 \text{ ft.-lb.}$$

The induced moment at B must be

$$M'_{BA} = M_{AB}C_{AB} = 129{,}000 \times 0.8 = 103{,}000 \text{ ft.-lb.}$$

It is necessary to find the directions of these various end moments before they can be combined. The individual members are shown in Fig. 13-32(b) in their distorted positions for each of the three separate displacements. The arrows show the rotations that the members try to produce at the joints —opposite to the directions of the restraining moments. Therefore, M_{BC} and M_{CB} are negative, but the others are positive. The total f.e.m. at the bottom of AB is 91 + 129 = 220 ft.-kips; that at the top of AB is 175 + 103

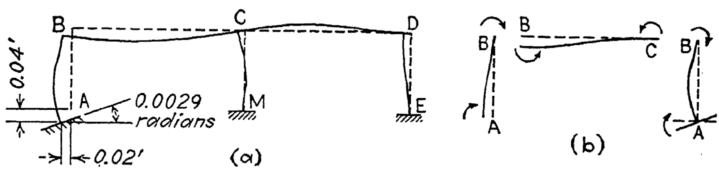


FIG. 13-32.

= 278 ft.-kips; those at the ends of BC are −87 ft.-kips. Figure 13-33(a) shows the distribution of these moments, neglecting side sway.

Dividing the sums of the moments in the legs by the lengths of the respective members gives an unbalanced force of 20.3 kips (17 acting left at A, 4.5 acting left at M, and 1.2 acting right at E). On the other hand, moments have already been worked out in Fig. 13-28(b) for an unbalanced force of 1.28 kips acting toward the right on this frame. If these moments are multiplied by $-20.3/1.28 = -16$, the effects of side sway are automatically found to be as shown in Fig. 13-33(b). The sign is negative because the unbalanced force in this problem acts toward the left. When these moments are combined with those of Fig. 13-33(a), the final diagram of Fig. 13-33(c) will result. The reactions practically balance, showing that the frame is now in equilibrium.

When the final moments as given in Fig. 13-33(c) are compared with those in Figs. 13-28(c), 13-29(c), and 13-31(d), one gets a clear picture of the seriousness of the failure or yielding of foundations that support continuous structures.

13-12. Construction of Influence Lines by Moment Distribution. In the case of a rigid-frame structure like that of Fig. 13-25(a) which must support moving live loads, it is advantageous to construct influence lines for the critical moments.

This can be done easily by using the frame of Fig. 13-25(*b*); computing the f.e.m. for 1 kip successively at 0.2*L*, 0.4*L*, 0.6*L*, and 0.8*L* of the left span *BC*; and analyzing the frame by moment distribution for each individual case, with or without side sway. The moments thus found, together with the other moments

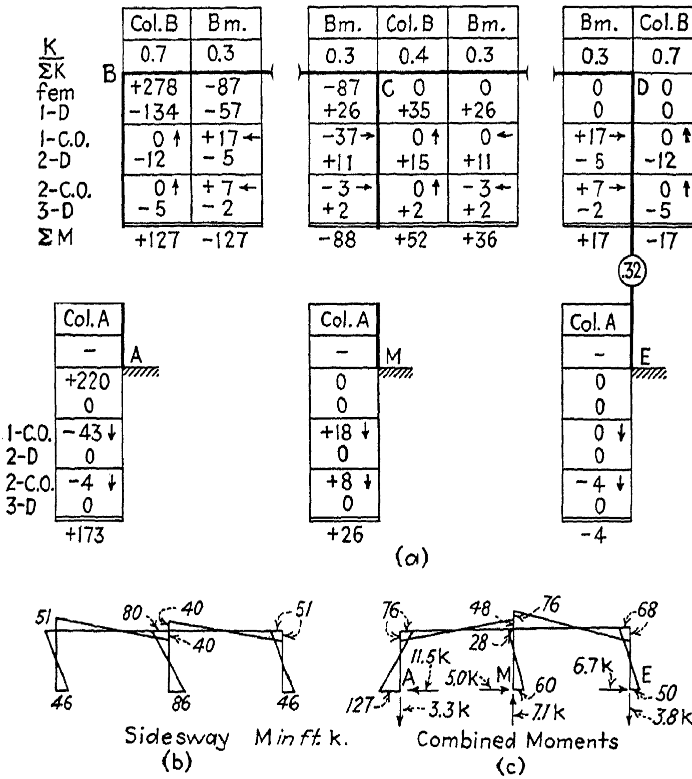


FIG. 13-33.

which can be computed easily at each two-tenths of the span, will give sufficient data for the construction of practically all of the influence lines that are needed.

A similar procedure will apply equally well to almost any structure for the analysis of which moment distribution is adaptable.

CHAPTER 14

BUILDING FRAMES

14-1. Introduction. One common type of building has slab and beam floors, with girders along the lines of the columns. In a reinforced-concrete structure of this type, there is almost inevitably a large degree of continuity at the junctions between beams, girders, and columns. This interaction of the members must be considered in the design, and the reinforcement must be placed so as to avoid cracks near the columns in the region of negative bending moments. It is therefore the purpose of this text to show some of the theoretical and practical considerations that are encountered in the design of the frame of such a building.

14-2. Preliminary Design of Members. One must make so many assumptions regarding loads, sections, degrees of continuity, and moments of inertia in designing a building that the refinement of the calculations need not be greater than the accuracy of these assumptions. If Fig. 14-4 represents one bent (line of columns and beams) in a small building, it is obvious that analysis of this frame by the so-called "exact" method of work is too laborious to be considered because there will be such a large number of unknowns when the structure is "cut" so as to make it statically determinate. However, moment distribution can be used to great advantage in analyzing such a frame, but it is necessary to find the approximate sizes of the members before the calculations can be made.

The preliminary design requires considerable labor and the exercise of the good judgment that comes through experience. It is impossible to set up any infallible rules for such work, but the following information may be used as a guide in the design of a simple building which has the floor panels subdivided to form T-beams and which has T-beams or T-beam girders (called "headers" when they support other beams) along the column lines:

1. Design the floor slab for the specified uniformly distributed live load and the dead load, using for the maximum moment $M = wL^2/8$ for two spans or $M = wL^2/10$ for more than two spans, where w = the intensity of the total live load + dead load on the slab.

2. Design a typical interior T-beam or header as a fixed-end, rectangular beam. The maximum moment will therefore be the f.e.m. which will be used later in the distribution of moments. Compute the trial reactions as for a simply supported beam.

3. Design a typical T-beam in an outer bay (or a beam that is discontinuous) as a rectangular beam with one end fixed (the inner end for outside panels). One or two relatively large concentrated loads should be considered separately, but a series of small concentrations may be treated as uniformly distributed loads ($M = wL^2/8$). In general, assume that the reaction at the fixed end is $\frac{5}{8}wL$ and that the one at the hinged end is $\frac{3}{8}wL$.

4. Design interior columns of moderate size for direct compression over the entire area, considering concrete alone and using the allowable f_c for compression in the type of column that will be used (tied or spirally reinforced). The sizes of the beams should also be considered in proportioning small columns, because the latter should generally be at least as wide as the stems of the beams that they support.

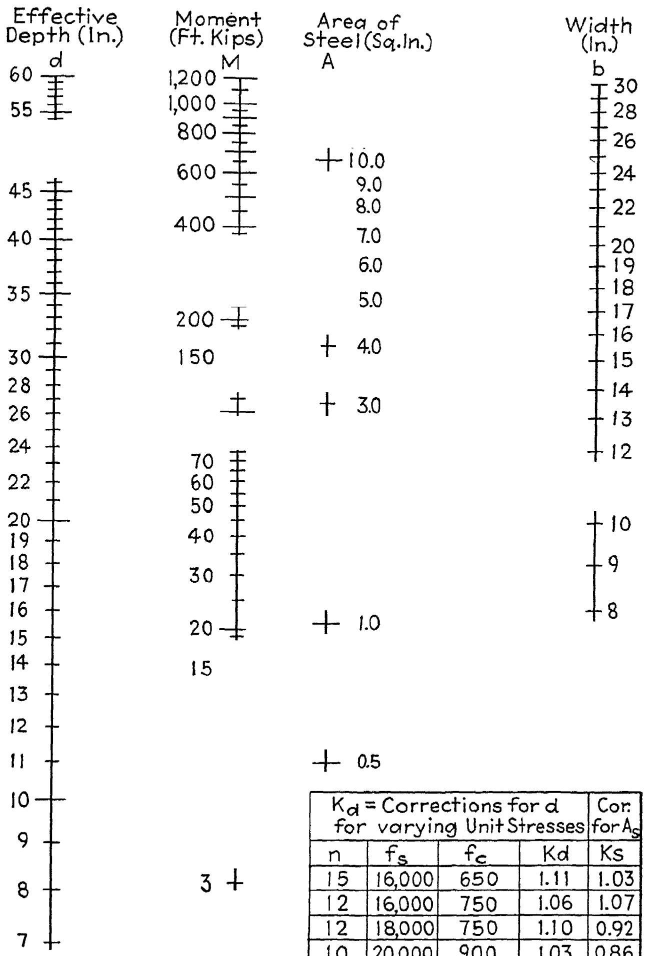
5. Design outer columns (or others at which the beams are not continuous) as plain concrete columns, using about $0.9f_c$ to allow for relatively large bending moments.

Design all beams as rectangular sections with a width equal to that of the stem b' of the T-beam and with a depth D equal to that of the T-beam. This is permissible because the critical bending moments are usually at the columns, where the flanges cannot be relied upon for resisting tension.

The diagram in Fig. 14-1 is useful for the determination of these preliminary sections. Use a triangle, and place its edge at the required value of M in foot-kips; rotate the triangle about this point until the simultaneous readings from the scales of depths and widths seem to be reasonable ($d = 2b\pm$); then add 2 or 3 in. to the depth for the cover over the steel. Shearing stresses must be considered also in choosing b' and D.*

One must remember that, when a long, heavy beam abuts a short, light one, the end moments of the former will be less than

* The table of corrections that is given in Fig. 14-1 is made to apply for several combinations of n, f_s, and f_c. Use the diagram as described above, but multiply d as read from the chart by K_d to get the corrected effective depth. Similarly, multiply the magnitude of A_s as given in the chart by K_s to find the corrected area of the tensile reinforcement. These data are for rectangular beams of balanced design.



K_d = Corrections for d for varying Unit Stresses				Cor. for A_s
n	f_s	f_c	Kd	Ks
15	16,000	650	1.11	1.03
12	16,000	750	1.06	1.07
12	18,000	750	1.10	0.92
10	20,000	900	1.03	0.86
8	20,000	1,000	1.01	0.87

FIG. 14-1.—Diagram for beam design. (Based upon f_c = 900 and f_s 18,000 lb. per sq. in., and n = 10.) (*Courtesy of S. Potashnick.*)

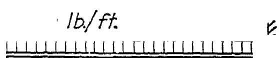


FIG. 14-2.

the f.e.m. and that of the latter will be greater than the f.e.m. This principle is illustrated in Fig. 14-2 where the slope of the tangent to the beam's axis at B shows the relief of the curvature at the right of B but an increase of curvature at the left. This will aid in making some preliminary adjustments when selecting b' and D.

The *nominal* dimensions of columns and of the stems of beams should be in multiples of 2 in. so as to facilitate the use of commercial sizes of lumber in the building of forms. The variations of sizes should also be minimized in order to reduce the cost of forms because of greater duplication in their use.

It is also important to consider the relationship of beams in any one row because, when a large beam like A of Fig. 14-3 is opposite a smaller one like B, it is often desirable to use a fillet under B in order to facilitate the transmission of compressive stresses between the bottoms of A and B. However, this fillet is not considered in computing the strength of B unless it is a real, sizable haunch.

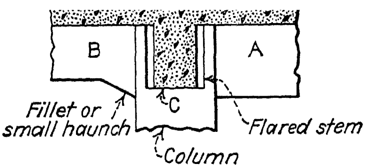


FIG. 14-3.

Sometimes, stems of beams are flared as shown for C in Fig. 14-3. The Code permits these thickened portions to be counted upon in computing shearing stresses, but generally they are disregarded in calculating the strength of a member in bending.

14-3. Calculation of Stiffnesses of Members. After the tentative sections are established, it is necessary to determine their relative stiffnesses. The members may be assumed to be prismatic sections unless large haunches are used at the columns, in which case the stiffness factors are to be calculated as described in Chap. 13.

Inasmuch as the members are T-beams, there are wide differences of opinion among engineers regarding their real stiffnesses. However, studies of the moments of inertia of some typical beams, using $f_c = 900$ lb. per sq. in., $f_s = 18{,}000$ lb. per sq. in., and $n = 10$, seem to indicate the following approximate relationships, where $I_r =$ the moment of inertia of the "uncracked" concrete rectangular beam with a section equal to $b'D$, neglecting

the steel, and where I_c for the "cracked" members = that of the transformed section:

I for uncracked T-beam = 1.5 to 2 times I_r.
I_c for cracked T-beam = 0.5 to 1 times I_r.
I_c for cracked rectangular beam = 0.4 to 0.6 times I_r.

It is obvious that the portions of any given beam that are in the regions of large bending moments must be cracked, otherwise the steel cannot be stressed as assumed in the design; other regions near the points of contraflexure are uncracked. Therefore, the real I is less than I_r for part of the member, and it exceeds I_r for the balance so that it is reasonable to use $I = I_r$ to $1.5I_r$. In any event, the relative stiffnesses are desired rather than the absolute ones.

If I of a beam is assumed to be too large in comparison with that of the columns to which it is rigidly connected, the effect is a relief of the bending in the columns, especially the outside ones, because the angular rotation of the assumed stiffer beam at its junction with the column may be less than the actual rotation of the real beam, thereby causing a smaller theoretical bending moment in the column.

The relative stiffnesses of the columns and the beams are important. The direct compression in the former will prevent the formation of any appreciable tensile cracks in these members. Therefore, the moments of inertia of the columns may be computed generally upon the basis of I_r for the uncracked concrete, or, if they are to be heavily reinforced, 10 to 20 per cent may be added as an allowance for the effect of the steel. If one wishes to use the I of the uncracked T-beams, Fig. 5 of the Appendix will be a help in computing these values. However, since the beams must crack somewhat, it seems to be reasonable to assume that $I = I_r$.

It is necessary to make sure that the assumed restraints are actually developed at the ends of the beams, because, if they are not, the positive bending moments in the members may increase to the danger point.

The stiffness of each member is computed as $K = I/L$ for fixed ends, or $K = 0.75I/L$ when the far end is hinged, unless there are haunched or tapered sections.

14-4. Analysis of Frame by Moment Distribution. Using the tentative design as a basis, compute the relative stiffnesses of the members at each joint as explained in Chap. 13, calculate the fixed-end moments for all members, and then proceed with the detailed analysis.

The positions of live loads that cause maximum stresses will vary. In general, the critical moments in the columns and the greatest positive moments in the beams occur when alternate

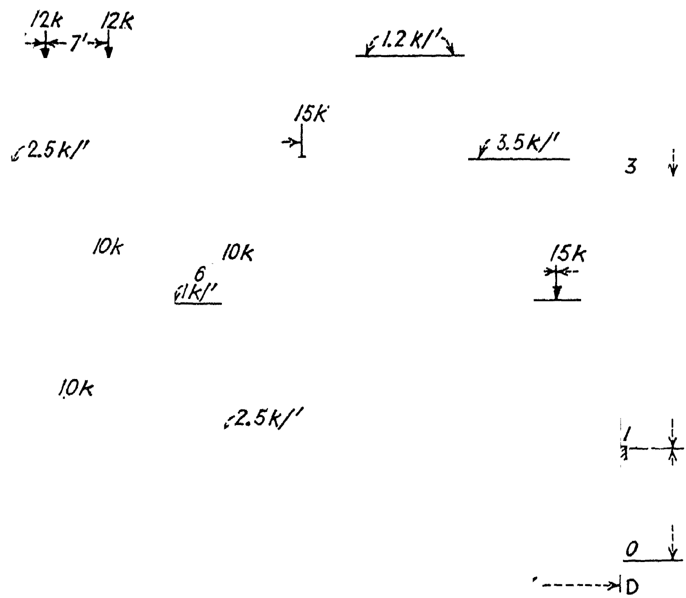


Fig. 14-4.

bays are loaded; the maximum negative moments in the beams, when two adjacent spans are loaded.

It is not necessary to analyze a large structure completely for localized live loads because a few adjacent spans can be "removed" with the columns above and below them. The far ends of these columns and the extreme ends of the outer beams can be assumed to be fixed. This portion of the structure can then be analyzed by moment distribution,[1] and the sizes of the

[1] See "Continuity in Concrete Building Frames," 2d ed., Portland Cement Association, for the "two-joint method."

members can be checked. Later on, the entire frame may be considered as a whole if greater accuracy is desired.

In ordinary buildings of moderate size, side sway need not be considered because the shearing strength of the outer walls and of the interior partitions is so great that lateral movement is prevented. Even in the case of wind loads, these walls and partitions are so stiff that they constitute the major line of resistance. They should be designed accordingly.

Problem 14-1. Find the bending moments at the joints of the building frame of Fig. 14-4 for the loading conditions that are shown. For convenience, these loads are assumed to include live load and dead load. Neglect side sway. All members are to be treated as prismatic sections with restrained ends. Assume f_c for the columns = 660 lb. per sq. in.

This frame is not used as a model of building construction, but it is for the purpose of illustrating several different conditions in one problem.

Table 14-1 gives the data for the preliminary design of the beams; Table 14-2, for that of the columns. Figure 14-1 is used in obtaining the sizes of the beams.

The following points should be noticed in the preparation of this preliminary material:

1. Points 1*A* and 1*C* are fixed to heavy foundation walls so that the beams and columns at these points are assumed to have no effect upon each other.

TABLE 14-1.—PRELIMINARY DESIGN OF BEAMS

Member	M, ft.-kips	b', in.	D, in.	R_L, kips	R_R, kips	I/L, ft.[3]
RA-RB	118*	14	30	17	28	0.072
RB-RC	78	14	26	17	17	0.035
RC-RD	66*	12	24	16	9	0.032
3*A*-3*B*	138*	16	30	20	32	0.083
3*B*-3*C*	183	18	34	36	36	0.102
3*C*-3*D*	193*	18	34	46	28	0.135
2*A*-2*B*	170†	16	30	32†	57†	0.083
2*B*-2*C*	78	14	26			
2*C*-2*D*	188‡	18	32	31	31	0.113
1*A*-1*B*	132	16	30	. .	36	0.083
1*B*-1*C*	163	18	32	35	. .	0.085

* Fixed one end.

† Computed as positive moment in overhanging beam.

‡ Simply supported, positive moment.

TABLE 14-2.—PRELIMINARY DESIGN OF COLUMNS

Member	R_N, kips	ΣW, kips	f_c, kips per sq. in.	Required area, sq. in.	Width b, in.	Depth D, in.	$1.1I/L$, ft.^{3}
RA-$3A$	22	41	0.59	70	14	14	0.014
$3A$-$2A$	60	124	0.59	210	16	16	0.021
$2A$-$1A$	50	210	0.59	356	20	20	0.047
RB-$3B$	53	100	0.66	152	14	14	0.014
$3B$-$2B$	100	271	0.66	410	20	20	0.051
$2B$-$1B$	75	407	0.66	615	24	24	0.098
$1B$-$0B$	130	613	0.66	925	30	30	0.300
RC-$3C$	61	96	0.66	142	14	14	0.014
$3C$-$2C$	125	306	0.59	515	24	24	0.105
$2C$-$1C$	40	381	0.59	642	24	24	0.098
RD-$3D$	38	49	0.59	83	14	14	0.014
$3D$-$2D$	75	155	0.59	263	18	18	0.033
$2D$-$1D$	40	230	0.59	390	20	20	0.047

2. Beam $2A$-$2B$ is assumed to be a simply supported member with an overhanging end. However, the size of the trial beam is made smaller than the bending moment seems to require so as to allow for the restraining effect of the columns.

3. Beam $2C$-$2D$ is assumed to be simply supported, but its trial section is also reduced because of the action of the columns.

4. The slab thickness is assumed to be 6 in.

5. R_N in Table 14-2 represents the reactions of beams that are normal to the plane of the frame.

6. ΣW gives the total load upon each column, allowing 2 kips for the columns of the third story, 3 kips for the second story, etc. In practice, the live load is often reduced for the proportioning of columns when a series of floors is loaded, but such reductions are not included here for the preliminary calculations.

7. The widths of the columns are purposely made at least equal to the stems of the beams that they support. Also, the height is to be not over ten times the least lateral dimension.

8. I of the columns $= bD^3/12$; L is the height of the story; and the stiffness is assumed to be $1.1\frac{I}{L}$.

Next, the fixed-end moments are calculated for all of the beams, using the coefficients from Fig. 13-2 of Chap. 13. These moments are recorded in Figs. 14-5 and 14-6. The relative stiffnesses of the members at all joints are calculated and recorded in these same figures.

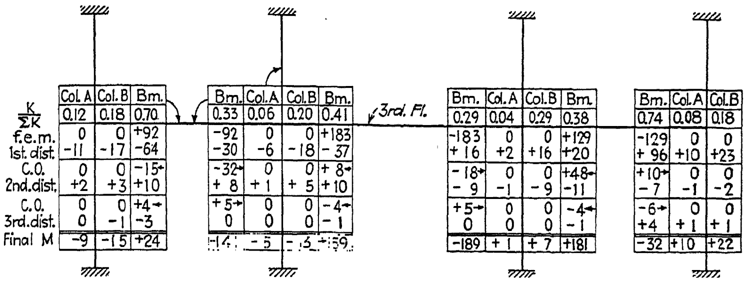


FIG. 14-5

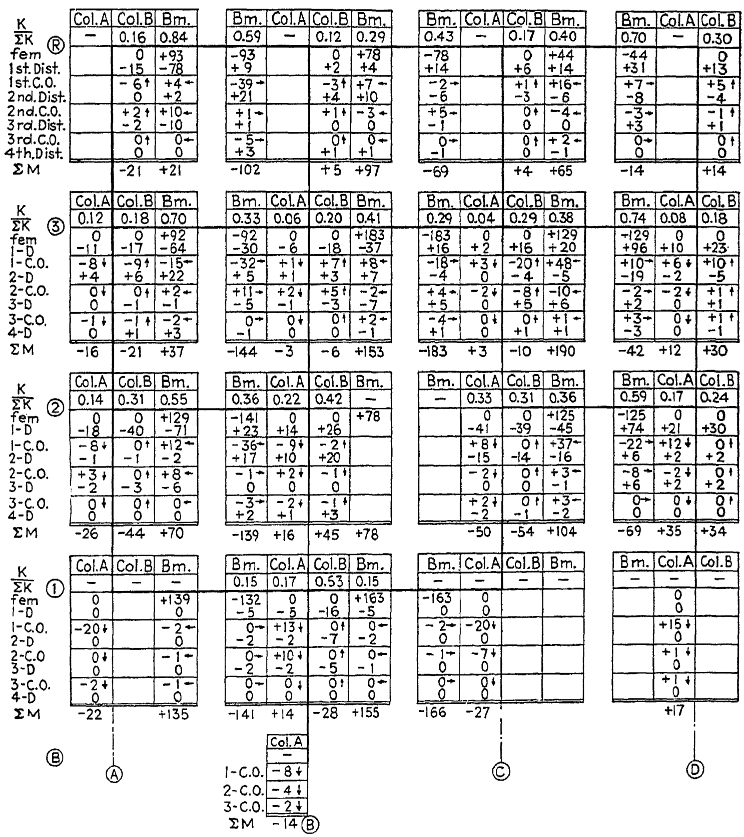


FIG. 14-6.

The third floor is analyzed alone in Fig. 14-5 in order to show the results when a small portion of the frame is isolated. The moments carried over to the far ends of the columns are not considered.

The entire frame is analyzed as shown in Fig. 14-6. To do this without making errors, the following procedures are suggested:

1. Distribute the balancing moments at all joints, and draw a line under them.
2. Carry over all beam moments. (Carry-over factor = ½.)
3. Carry over the moments from the upper ends of the columns to the lower ends (from the right side of the column in the block above to the left side in the block below).
4. Carry over the moments from the lower ends of the columns to the upper ends (from the left side of the column in the lower block to the right side in the upper one).

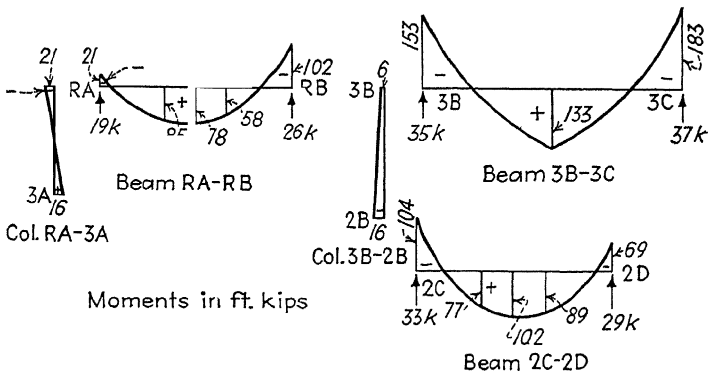


FIG. 14-7.

5. Distribute the balancing moments again, and repeat the process until the unbalanced moments are negligible. Finish the computations by making the last distributions; then algebraically sum up the moments in each column of figures. In this case, there is little advantage in performing the fourth cycle in Fig. 14-6.

A comparison of the results given in Figs. 14-5 and 14-6 shows that the maximum negative moments in the beams agree reasonably well but that those in the columns vary considerably. On the other hand, referring to Fig. 14-6, a checkup of the sizes of the beams by the use of Fig. 14-1 shows that beam *RB-RC* is somewhat weak and that *2C-2D* is too large.

The positive bending moments in all beams must be computed after the end moments are known. Figure 14-7 shows the moment diagrams for a few typical beams and columns as determined from Fig. 14-6. Of course, each column must be designed for the combined loads and bending moments that are due to the action of the two frames which are normal to each other and of which the column forms a part.

The checking of column loads and sizes, the design of the reinforcement, the calculation of shearing stresses, etc., should be completed by the student.

CHAPTER 15

ARCHITECTURAL CONSIDERATIONS

15-1. Basic Principles Underlying Architectural Treatment. The design of the architectural features of any reinforced-concrete structure must be based upon the nature of concrete itself and upon the processes that are to be used in the building of that structure. In other words, concrete is a substance that,

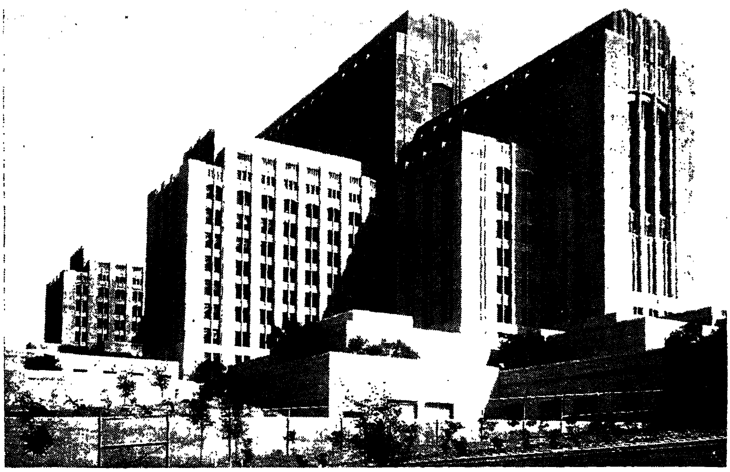


FIG. 15-1.—Los Angeles County Hospital, Los Angeles, Calif. (*Courtesy of Portland Cement Association.*)

in a plastic state, is placed against forms that give to it whatever shape the forms possess so that, after the concrete has set, these outlines and shapes are permanently maintained by the concrete. Therefore, the details of the whole subject must be thought about in terms of forms, simplicity, pouring schedules, and other practical matters of construction so that the finished structure will become one harmonious entity whose parts automatically blend to produce the desired architectural ensemble. Figure 15-1 is an example of this.

Furthermore, the general architectural conception of the structure should be based upon these considerations, along with those of beauty, proportion, surface texture, permanence, color, economy, and functional effect. The designer should abandon many of the ideas that have been developed through the past in using stone and brick masonry which emphasize moldings, cornices, decorative carvings, and other features for which concrete is not suited. Concrete is a special material, and the architecture of a concrete structure should be adapted to it.

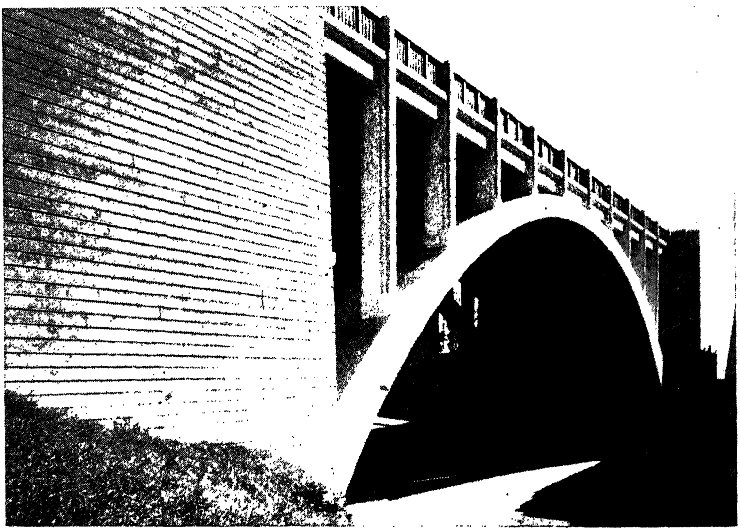


FIG. 15-2.—Dyckman St. Arch, Henry Hudson Parkway, New York City. (*Courtesy of Portland Cement Association.*)

For instance, Fig. 15-2 shows a four-ribbed reinforced-concrete arch bridge which has been designed upon this basis.

Much of the architectural detail of a concrete structure should also be based upon a consideration of light and shadows. Excessively large, flat surfaces should be avoided; they should be broken up by horizontal or vertical markings or moldings, by moldings in both directions, or by offsets. Lines should be straight and distinct, avoiding intersections at large obtuse angles. The special detail features should be coordinated with the necessities of the building operations so that the construction joints can be placed at markings or offsets, the joints themselves thus being indistinguishable.

An interesting treatment of large surfaces is shown in Fig. 15-3. With the base and panel treatments, one hardly realizes that this might have been an unattractive retaining wall.

Another manner of breaking up large surfaces is shown in the treatment of the abutments of Fig. 15-2. The surface gives a sort of clapboard effect with a multitude of small horizontal shadows.

15-2. Forms. Besides possessing adequate strength, forms should be true to shape, well braced, and tight.

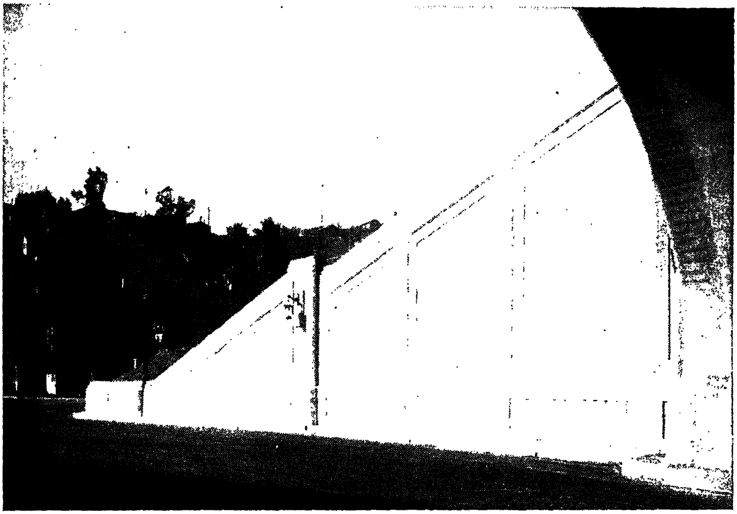


FIG. 15-3.—Retaining wall at portal of Figueroa Tunnel, Los Angeles, Calif. (*Courtesy of Portland Cement Association.*)

The need for the first of these is obvious, but it emphasizes the advantage of simplicity because forms are made generally of wood, by carpenters, or of metal, by ironworkers. The designer must try to visualize just how the workmen are going to make these forms; he must be sure that his design is practical.

The second feature is primarily a problem for the men in the field, but the designer can do some things to facilitate their work. One of these is the location of construction joints so as to provide shoulders against which the forms for subsequent pours may bear, thus making it easy to hold the forms to the proper line.

The third point is very important. If the forms are not tight, the mortar will leak out, leaving the aggregate behind and causing

honeycombed places in the surface of the concrete. Sometimes, ragged fins of mortar will be made at the loose joints. When these fins are knocked off, there will be unsightly streaks in the surface of the concrete. Figure 15-4* illustrates one satisfactory way of making tight corners.

In planning details to accommodate formwork, one must consider the duplication of forms, also the use of commercial sizes of materials. Plywood is a common lining for the forms for architectural concrete. However, the greatest reasonable length of

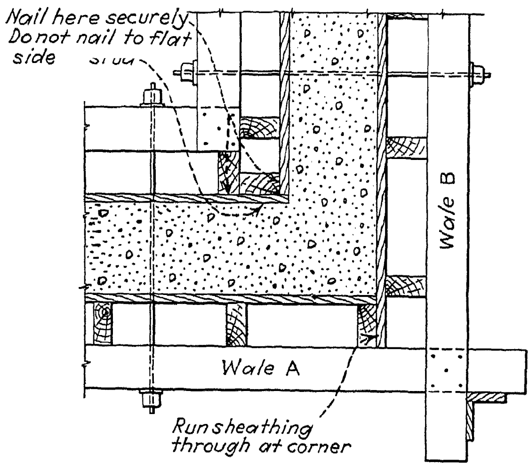


FIG. 15-4.

plywood is about 8 ft. (maximum 10 ft.). This should be the limit between moldings, offsets, or other details; it will avoid the marks that might otherwise be visible in the concrete at the junctions of the ends of abutting pieces. Also, with the panels of plywood laid with their long dimension horizontally, joints must often occur between the upper and lower pieces, but these joints can be braced thoroughly. If slight marks occur in the concrete at such points, they are not usually objectionable because of their horizontal position. However, care should be used to line up these joints across or around the structure. A patchy effect should be avoided.

* Taken from "Forms for Architectural Concrete," Portland Cement Association.

It is generally necessary to use form ties as shown in Fig. 15-4. There are various types of these, but the builder should be sure to use one that can be completely removed after the forms are stripped, or at least a type that can be broken off or disconnected back of the surface of the concrete so that the hole can be pointed up, to avoid staining. These ties should also be arranged so that the marks, if any, will not form an objectionable pattern. Another desirable feature is to have form ties so that they can be tightened up shortly before the pouring of the concrete.

Warping of forms can usually be prevented if they are wetted down before the concrete is poured. Drain holes which can be plugged up later are therefore desirable at the bottoms of the forms. When the backs of the forms are exposed to the hot sun

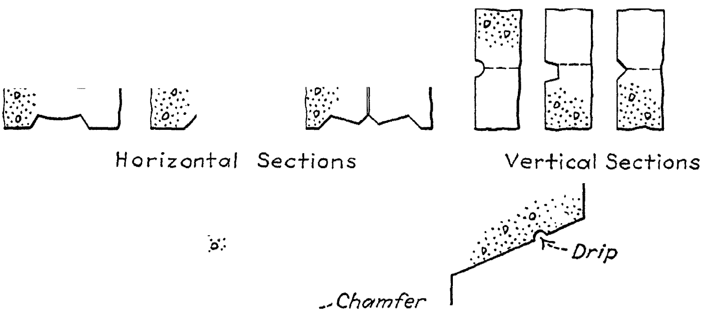


FIG. 15-5.

while the concrete is still plastic, the wetting of the backs will often avoid warping.

The details of forms should be made with the consideration of their removal and further use. Much damage can be done by spalling corners, marring the surface, etc., if the stripping of forms is difficult or if it is done carelessly.

15-3. Moldings and Minor Details of Forms. The moldings or markings that are used to break up flat surfaces should be detailed properly. It is very easy to nail strips on the insides of the forms, thereby causing recessed moldings or markings in the finished concrete, but it is rather difficult to make recesses in the forms themselves. It is also better to recess these cuts in the concrete, because small projections are likely to be damaged

The following points, some of which are illustrated in Fig. 15-5, should also be considered:

1. Wide, shallow recesses do not cast sharp shadows.

2. Horizontal, recessed moldings or markings should be V-shaped, beveled outwardly at the top and bottom, or horizontal at the top and beveled at the bottom. Horizontal ledges are likely to collect and hold dirt; they are also more difficult to fill so as to produce sharp edges, unless the hydrostatic head of the concrete is considerable.

3. Sharp, acute angles in moldings are difficult to fill properly and may spall off easily.

4. Chamfering of projecting corners is desirable, but chamfering is difficult at reentrant angles.

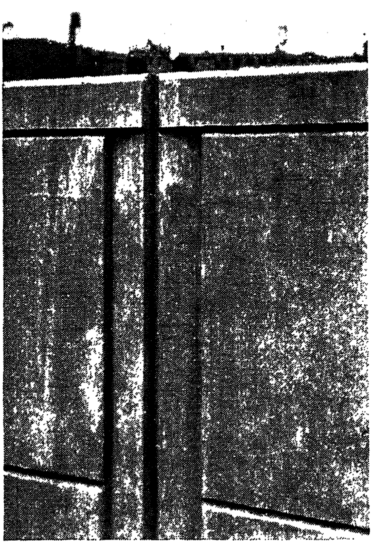


FIG. 15-6.—Details at expansion joint.

5. Small, recessed V-strips or half-rounds (1 to $1\frac{1}{2}$ in. deep) should be used at construction joints. If this is not permissible, a straight finish strip should be attached to the forms at the line of the joint so as to avoid a ragged line in the concrete. Sometimes the top of the forms can be a straight edge along which the concrete can be finished with a trowel for a depth of about an inch.

6. Lining up of moldings in successive pours is very important; so is proper mitering of the form strips at intersections.

7. Vertical keyways should be made, if possible, by fastening the key strip on the inside of the forms of the first pour.

8. Moldings must be sloped along their edges sufficiently to facilitate the removal of the forms.

9. The depths of moldings and V-cuts automatically affect the location of the reinforcement, because the rods should have $1\frac{1}{2}$ to 2 in. of cover at the deepest part of the recess. When the cuts are too deep, the reinforcement that crosses behind them has to be bent in order to keep the rods near enough to the main surface.

A very simple but effective treatment at a contraction joint in a retaining wall is shown in Fig. 15-6. The grooves are only $1\frac{1}{2}$ in. deep, but they are sufficient to cast sharp shadows.

15-4. Surface Finish and Appearance. It is ordinarily desirable to have a smooth finish on the surface of the concrete. Plywood or similar materials generally serve this purpose very well. However, in the case of buildings or other structures that will be seen at close range, it is often desirable to rub the concrete

with a carborundum stone and water so as to remove all objectionable irregularities and variations in the pattern or the texture of the surface.

Bushhammering or other treatment that destroys the mortar coating over the aggregate is sometimes used on massive structures, but the effect upon durability is open to question. However, this type of finish has the advantage of being very difficult to deface with chalk and similar materials; it also can be used in securing paneled effects, but the aggregate should be suitable in color, about $\frac{3}{4}$ to 1 in. in size, a good gravel or crushed stone. The general layout must be simple, and the bushhammering must be done thoroughly if the effect is to be satisfactory.

When any painting of exterior surfaces is desirable, a wash or paint of cement is recommended by the Portland Cement Association.

The finishing of horizontal surfaces should be done by screeding, using a wood float, or troweling moderately. Excessive troweling or other working that flushes too much of the mortar to the surface will be likely to cause "crazing"—very fine cracking of the surface.

The builder should be very careful to use aggregates that are properly graded and that are uniform in color. Otherwise, variations between pours will be distinguishable. Also, he should be careful to avoid segregation of the aggregates during the placing of the concrete, or else the bottom of a pour will appear coarse while there is an accumulation of finer material at the top of the previous one. This requires careful placing, the use of a concrete that is not too dry or too wet, and careful spading of the surface. Generally, the use of vibrators which are applied to the forms or in the body of the wet concrete is not sufficient to guarantee a good surface regardless of all other conditions. Great care must be exercised when using vibrators to make sure that the forms do not become bulged, displaced, or leaky at joints.

Another thing to be avoided in the surface of concrete is cracking due to shrinkage or variations in temperature. Adequate reinforcement will prevent this usually if the pours are not too large and if the details are such that the necessary deformation can occur at vertical and horizontal construction joints along the lines of the moldings. Doors, windows, and

other openings in walls are likely to cause such cracking of the concrete because they are points of relative weakness. In these cases, "dummy joints" such as that of Fig. 15-7* may be used together with a decrease in the reinforcement across the joint, thus inviting the crack to occur at a predetermined, concealed point. In very long buildings, it is even desirable to divide the structure into units about 200 ft. or less in length by using expansion joints which completely isolate adjacent units.

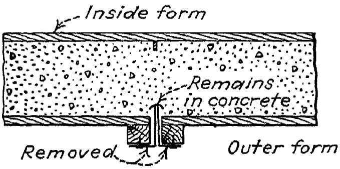


FIG. 15-7.

Other features that are desirable in order to maintain a satisfactory appearance of concrete surfaces are the following:

1. Waterproofing of the back of the concrete when it is subjected to the penetration of moisture—as in retaining walls.
2. Avoidance of metallic accessories which are almost certain to cause streaking.
3. Provision of a wash (or slope) on the tops of parapets so that drainage will carry dirt back away from the surface which one wishes to maintain in a satisfactory condition.
4. Provision of a "drip," or groove, under cantilevered construction as shown in Fig. 15-5. This is very important to prevent streaking of the surface due to rains.

These are some of the things that one should bear in mind when he is designing a concrete structure. Continued study by architects and engineers, more extensive experience by building contractors, and greater skill and care on the part of the men in the field—all of these things together will bring still further advances in this great field of construction.

* A. M. Young, Crack Control in Concrete Walls, *Eng. News-Record*, Aug. 11, 1938.

CHAPTER 16

PRACTICAL DETAILS AND MISCELLANEOUS DATA

16-1. Introduction. There are many practical details which are important in the design of satisfactory reinforced-concrete structures. Some of these are treated herein, but the chief objective of this chapter is that of showing various fundamental principles which affect these details so that the young engineer may use them in the solution of other problems that will confront him.

16-2. Temperature Reinforcement. If a piece of concrete could be rigidly connected to immovable supports, and if this piece could be subjected to a fall in temperature of 1°F., the resultant tension in the concrete due to the latter's attempt to contract would be

$$f_c = E_c\frac{\Delta L}{L} = E_c\omega$$

where ω = the coefficient of thermal expansion or contraction. Then, if $E_c = 3{,}000{,}000$ lb. per sq. in. and $\omega = 0.000006$, $f_c = 18$ lb. per sq. in. However, since the range of temperature variations is frequently ± 60F., the corresponding maximum stresses in this concrete would be $f_c = \pm 18 \times 60 = \pm 1{,}080$ lb. per sq. in. From this, it is clear that the compressive stresses alone would not be critical, but the concrete would surely crack if it were subjected to such a tensile stress. The amount of steel that would be needed to carry this tension is clearly excessive.

In practice, structures are not often fully restrained this way unless they are keyed to rock foundations. However, if they are very long, there may be sufficient restraint or frictional resistance to cause them to crack in tension. Therefore, the engineer should build his structures in short units so that these cracks cannot develop.

When one considers that ω for steel is practically the same as for concrete, he realizes that the use of rods to prevent cracks

due to temperature variations is not for the purpose of stopping the expansion and contraction but merely to knit the structure together and to avoid localized cracks. The amount of steel that is required to do this is not definite, but it may be assumed to be about 0.0025 times the cross-sectional area of the concrete. However, parapets and similar relatively thin parts which are attached to more massive structures should be reinforced more heavily.

The temperature reinforcement should be placed near the surfaces of concrete structures; when only one side is exposed, about 60 to 70 per cent of the steel should be near the exposed side, unless the walls are thin. Even when the foundations are on rock, the parts of the structure that are above the footings should be provided with adequate joints and reinforcement.

Structures that are narrow and high—such as piers—do not need steel to resist cracking due to temperature to the same extent as do long ones. They are free to expand or contract vertically. Their dead load will not let cracks open up.

When concrete is used in places that are subjected to very high temperatures,[1] there is a far different problem. Ordinary concrete may be weakened by dehydration above 500 to 600°F.; at 1200°F. it may be almost worthless. No amount of reinforcement will stop this action. On the other hand, concrete made with Lumnite and low-silica aggregate may withstand about 1000°F. If ladles of molten metal are to be placed on or alongside concrete surfaces, these should be protected by replaceable materials such as bricks. Unless ventilation can remove the heat, even these are inadequate; e.g., an uncooled concrete foundation slab for a reverberatory furnace, if placed on earth, will have its temperature gradually increased to somewhere near that of the furnace in spite of the latter's lining because the heat has no way of being dissipated. Surface spalling may occur when one side of a section is subjected to sudden high temperature locally applied.

[1] Alfred L. Miller and Herbert F. Faulkner: A Comparison of the Effect of High Temperatures on Concretes of High Alumina and Ordinary Portland Cements, University of Washington, *Bull.*, University Experiment Station, Series 43, Sept. 15, 1927; also Concrete Subjected to High Temperatures, A.C.I. *Proc.* Vol. 35, p. 417, 1938–1939.

16-3. Shrinkage. Shrinkage of the concrete in setting is somewhat like a drop in temperature of 30 to 80°F. (depending upon the "richness" of the concrete), except that the reinforcement does not shrink simultaneously. The shrinkage actually sets up compressive stresses in the rods so that, if the area of the steel is too great, the rods will be stronger in compression than the concrete is in tension, thereby producing the cracking that the reinforcement was supposed to eliminate.

The best way to handle shrinkage is to provide joints so that the concrete can shrink without causing trouble. With proper planning, the work may be arranged so as to build long structures in alternate sections, pouring rather long portions first, then filling in the shorter, intermediate sections later—preferably allowing the first ones to set for one or two weeks. In multistory structures with heavy, rather solid walls and intermediate floors or in massive ones poured in complete horizontal lifts, the lower portions necessarily shrink first. When a higher lift or a floor is poured, its shrinkage is restrained so that it may develop cracks or, if the previous work is weak in shear, the former's shrinkage may crack the latter. Proper jointing is the best remedy, but full-height wall reinforcement inclined upward toward the center of shrinkage may minimize cracking. Buildings with reinforced-concrete framework and filled-in walls do not have this difficulty because the columns can safely deform if full-height expansion joints are used about 300 ft. apart.

When shrinkage causes bending moments in such structures as arches and rigid frames, the resulting stresses should be considered in the design.

16-4. Construction Joints. Construction joints must be located so as to cause no serious weakness in the structure. It is therefore desirable to place them in regions where the shearing stresses and the bending moments are small or where the joints will be supported by other members. However, they must be located so as to facilitate construction.

Such joints must be adequately keyed in order to transfer the necessary shearing forces. Figure 16-1 shows various arrangements. The numbers in the circles denote the sequence of the pours.

The following comments should be noted, the letters referring to the various sketches in Fig. 16-1:

(*a*) This key is easy to form, but it holds water when it is horizontal. The water should be removed. When a keyway is vertical, the form for it should be attached on the inside of the forms for the first pour.

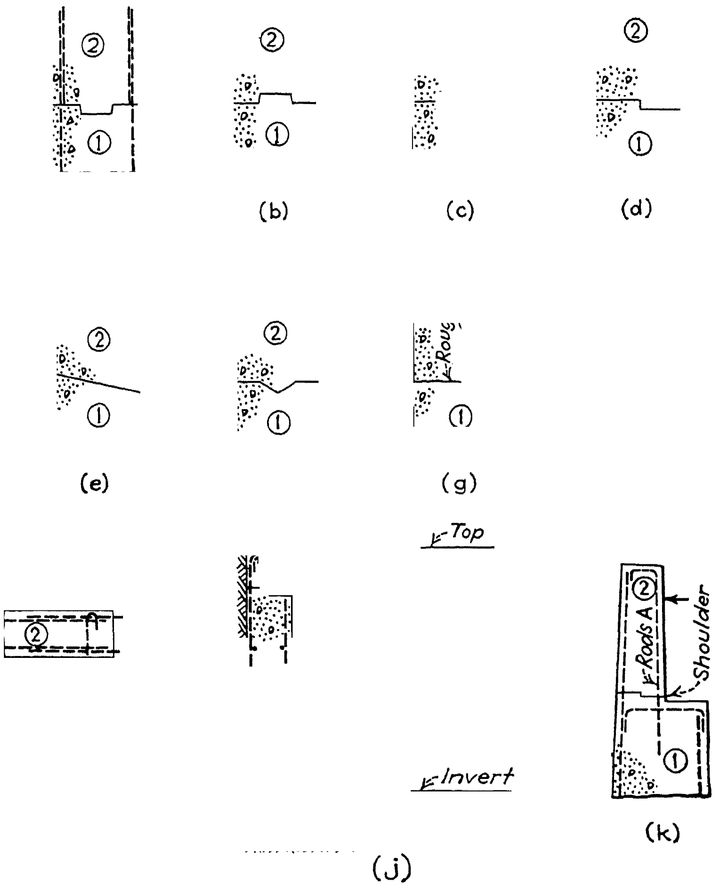


FIG. 16-1.

(*b*) The raised key causes objectionable formwork.

(*c*) These intermittent precast blocks set by hand in the wet concrete are not very strong. They are easily forgotten, and they require extra hand operations. This intermittent key idea can be used to advantage in vertical joints that must withstand vertical and horizontal shears by placing pieces of 2 by 6 planks about 8 in. long and 18 in. c.c. on the inside of the end forms of the first pour.

(*d*) This type of key can be used only when the shearing forces are as shown, but it is good for such cases.

(*e*) This is theoretically better than (*d*); but when the rods are close together, it is difficult to finish properly. It is also easily forgotten by the workmen.

(*f*) This V can be made by hand after the concrete is poured, thus eliminating form strips which would interfere with the pouring of the concrete. It is good for thin walls and often for others (if it is not forgotten).

(*g*) If the shears are small or the direct compressive loads are large, this hand-roughened surface is often sufficient. Trowel the edge to get a straight line for appearance.

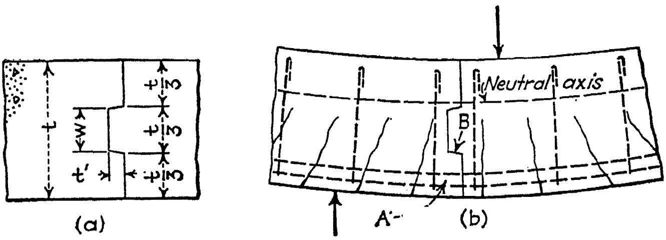


FIG. 16-2.

(*h*) This type of key is good for arches or other structures that it is desirable to pour in sections so as to minimize shrinkage stresses.

(*i*) This is a possible arrangement. There is no need for a true key when the joint is supported. This may weaken the T-beam.

(*j*) This shows suggested keyways at corners—such as for box culverts where the invert must be troweled or screeded. The wall forms can be braced on the inside. It is assumed that the shears in the walls are small compared to those which are in the top and bottom slabs. The vertical pressure on the walls causes friction that helps the keying action, whereas, if the full width of the wall were recessed into the horizontal slabs, the effective thickness of the latter for resisting shears would be seriously decreased.

(*k*) This illustrates the provision of a shoulder for use in setting forms.

A key should be designed so that its width w [Fig. 16-2(*a*)] is sufficient to transfer the shearing force, using ordinarily a shearing unit stress of about $0.1f'_c$. When the shearing force is reversible, the width of the key should be $t/3$ or slightly smaller, so that all three parts of the joint will have approximately equal strength. The thickness t' should provide sufficient area in bearing along the edge at a unit stress of not over about $0.2f'_c$. A thick but narrow key (large t') is likely to break off.

Figure 16-2(*b*) shows why a key at a point of large bending stresses may have very little value. The tensile forces may

cause cracks which destroy the ability of a piece like *A* in the figure to resist a pressure applied at *B*. In such cases, it is advantageous to use vertical or inclined stirrups both sides of the joint, as shown in the figure.

It is usually necessary to have reinforcement which passes through construction joints. If so, the rods from the first pour should project through the joint enough to secure the desired bond, as in Figs. 16-1(*a*), (*h*), (*i*), (*j*), and (*k*). In this way, the rods in the later pour can rest upon or bear against the concrete of the previous pour. In some special cases, separate dowels like rods *A* of Sketch (*k*) can be set by hand in the wet concrete of the first pour. However, rods or dowels should not be relied upon for resisting the shear at the joint, because they will crush the concrete locally before they will withstand much shearing force, unless special provision is made to avoid this action.

In all cases, construction joints must be cleaned thoroughly before the next pour is made. All laitance must be removed, using wire brushes, water under high pressure, or other means. It is often desirable to coat the joint with a little mortar just prior to the placing of the concrete upon it. The exposed edges of the joint should be finished straight, or they should have a small V-strip on the forms ($\frac{1}{2}$ to 1 in. deep).

The volume of concrete that can be deposited in one continuous pour will influence the locations of construction joints in massive structures. These should be planned far in advance. The other extreme occurs in the case of very thin walls—4 to 6 in. thick. These cause difficult pouring if they have any great height. They must be built in short lifts, by the use of pumping or by depositing through a small spout and hopper called an "elephant's trunk."

Needless construction joints should be avoided, especially in retaining walls and other structures in which such joints may be a cause of the seepage of water.

16-5. Expansion Joints. One of the first things to consider about expansion joints (or contraction joints) is that of the best locations for them from the standpoint of their proper functioning. They should be at points of change in thickness, at offsets, and at other points where the concrete will tend to crack if shrinkage and temperature deformations are restrained or prevented. The engineer must study a structure carefully in order

to discover these points. Ordinarily, joints should be about 30 ft. c.c. in exposed structures.

The second consideration should be that of coordination with the pouring schedule and avoidance of extra construction joints.

The third matter is that of satisfactory details. For these, the following points are mentioned:

1. The keyways should be of the types shown in Figs. 16-1(*a*), (*b*), and (*d*) if there is a considerable shearing force at the joint. However, there should be a space between the abutting ends of the concrete. This can be provided readily by the use of cork board, premolded mastic fillers, or other compressible materials. Figure 16-3(*a*) shows a vertical expansion joint which has been used in some of the retaining walls of the approaches to the Lincoln Tunnel. The compressible material can be fastened to the first pour by tacking it to the forms and by having nails protruding so as to bond into the concrete. However, one must be careful to use materials that will not squeeze out, slump when heated by the sun, or stain the surface of the concrete. In the space outside the key itself, beveled strips may be used (instead of fillers) and later withdrawn, but this is difficult when the walls are thick. In some cases it is satisfactory to paint the end of the first pour with asphalt. Then, when the shrinkage of the next section occurs, the latter will cause a slight clearance at the joint which may be sufficient, inasmuch as tensile rather than compressive forces are generally critical.

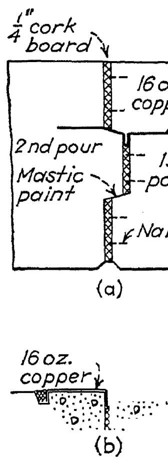


FIG. 16-3.

2. The edges of the keyways should be beveled slightly; they should be coated with mastic paint or with some material that will break the bond but that is not thick enough to destroy the bearing value of the key.

3. When the joints are likely to leak, they should be sealed in some way. Copper flashing is sometimes used as in Fig. 16-3(*a*). This copper should be folded into the joint so as to permit it to open slightly without rupturing the flashing; it must also be strong enough to hold its position during the placing of the concrete—an operation that must be done very carefully.

A second method of flashing, when the back of the joint is accessible, is pictured in Fig. 16-3(*b*). This is expensive because it is a sort of roofing job, but it is sometimes more reliable.

4. When it is possible to do so, expansion joints should be entirely open with an air space of 1 to 2 in. between the concrete sections. This is especially desirable in bridges where considerable motion occurs. However, to avoid visibility through parapets and similar parts, the joints may be offset in plan instead of being straight. In any case, one should be able to clean out such joints.

5. V-cuts should be used at the joints, or the expansion joints should be placed at moldings, reentrant corners, and other suitable points. The V-cuts should be 1 to 2 in. deep or large enough (and the joint fillers thick enough) to guard against spalling of the edges because of the compressive forces which are caused at the joint by expansion.

"Dummy" joints like that which is shown in Fig. 15-7 are useful in some cases as contraction joints that will avoid unsightly shrinkage cracks. The reinforcement should be weaker at these joints than it is elsewhere.

16-6. Waterproofing. Retaining walls, basement walls, spandrels of earth-filled arches, subways, tunnels, and similar structures must be watertight if they are to present a pleasing appearance. The denser the concrete itself is the more impervious it becomes, so that too lean a mix is likely to facilitate leakage. However, it is exceedingly difficult—or almost impossible—to keep construction joints from leaking when there is an appreciable pressure of water behind them.

There are three ways in which this problem of seepage may be attacked, viz., integral waterproofing, inside surface coatings, and outside surface coatings.

Integral waterproofing denotes materials added to the concrete when it is mixed in order to make the concrete itself impervious. The admixture supposedly fills all of the voids through which the water might pass. In any event, complete reliance upon integral waterproofing is dangerous, especially at the construction joints.

Coatings on the insides of the walls are used sometimes as a means of stopping leaks, but they are very likely to be unsatisfactory. Mortars with impervious mixtures in them, paints with

waterglass or other chemicals which evaporate or congeal and leave crystals or chemicals in the pores of the surface of the concrete, and asphaltic paints—these are some of the coatings used. However, these are expensive, and it is unreasonable to expect them to stop the water at the last line of defense—the inside surface—especially when the structures are subject to temperatures that cause the water to freeze behind the surfacing.

The best place to stop the leakage is at the outside surface—the point of entrance of the water. There are two customary ways of doing this. The first, and the most effective, is the use of membrane waterproofing which forms a continuous, water-tight sheet outside the structure; the second is the use of asphaltic emulsions or similar bituminous coatings forming waterproofing without membrane.

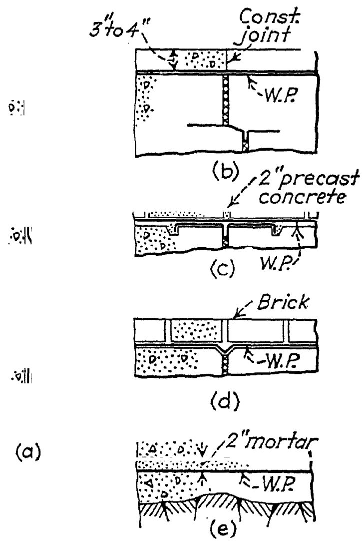


FIG. 16-4.

Membrane waterproofing is usually built up by coating the surface that is to be waterproofed with hot asphalt or coal-tar pitch, laying thereon successive layers of special fabric—placed shingle fashion and each layer coated with the mastic—until the desired number of layers or "plies" is obtained (two-ply, three-ply, etc.). Figure 16-4(*a*) shows this principle.

It is necessary to protect the membrane waterproofing against damage during backfilling; against penetration of oils, gasoline, or other solvents; and against the cutting tendency of sharp stones in the backfill. The first of these may be accomplished by the use of plywood which is laid against the membrane, but this is only a temporary material. Better protective coatings are poured concrete 3 or 4 in. thick and precast-concrete blocks or bricks set in mortar as shown in Figs. 16-4(*b*), (*c*), and (*d*), which picture details at expansion joints—always troublesome points.

It is best to place membrane waterproofing directly upon the outside of the wall, but this is not always economical, desirable, or possible. When one is waterproofing subways, tunnels, and other structures which are not accessible from the outside, it is necessary to provide a surface on which the membrane may be placed before the concrete of the main structure is poured. A sample of this work is shown in Fig. 16-5, which pictures part

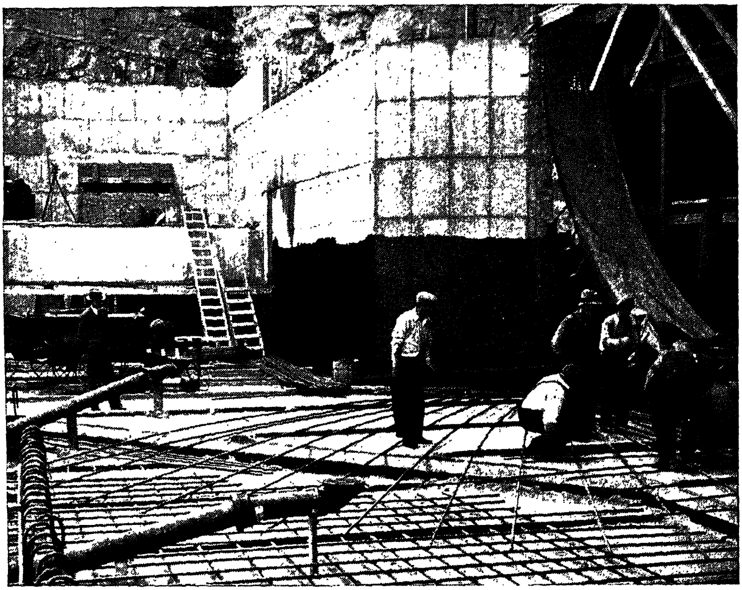


Fig. 16-5.—Construction of invert of New Jersey shaft of the Lincoln Tunnel at New York City.

of the bottom of the New Jersey shaft of the Lincoln Tunnel. A concrete lining, or "sand wall," is poured against the rock as shown above the line of black waterproofing. The membrane is then applied as shown, using a 2-in. coating of mortar on the invert or bottom to protect the waterproofing when the reinforcement and concrete are placed, as shown in Fig. 16-4(*e*). Of course, all roughnesses, projecting form ties, and other sources of damage are removed before the membrane is applied. It is necessary to keep the steel far enough away so that the subsequent spading of the concrete will not cause damage to the membrane. It is also obvious that such waterproofing work should not be attempted when any reinforcement is in the way.

The concrete must be dry when membrane waterproofing is applied. It is also necessary to keep water pressure from building up behind the sand walls, which are shown in Fig. 16-5, in order to avoid bulges in the membrane or displacement of the concrete. There is a shallow, gutter-like drainage system behind the walls in the picture. It leads to a temporary sump.

Waterproofing without membrane—sometimes called "dampproofing"—is merely the application of bituminous materials to the concrete as a paint coat or as multiple coats. It is beneficial, but it can be applied only from the outside; it is easily damaged; it has no particular elastic properties; and it is likely to be ineffective after a number of years. However, it is far better than no waterproofing at all.

It is less detrimental if waterproofing is omitted from stone-faced walls than from plain concrete-surfaced ones if the stones and the concrete are placed monolithically, because slight staining will not show very much, and the leakage is not likely to be serious. However, it is hardly worth while to take such chances. At least, one should apply waterproofing without membrane.

16-7. Drainage. When the conditions are such that natural drainage can be provided readily, it is advisable to construct a drainage system whose function is to remove the ground water behind or outside the structure. This is desirable even with waterproofed structures, especially when they rest upon rock. Such installations vary from simple weep holes through the walls to elaborate, interconnected systems.

Any such drainage system is subject to clogging by silt and to freezing. It should therefore be laid out so that it can be cleaned by rodding or flushing and so that it is below or behind the frost line, if possible. This can be done by a system of manholes, Y-connections at intervals, or even galleries such as that of Fig. 16-6(*c*) which can be inspected.

A few means of drainage are shown in Fig. 16-6. Sketch (*a*) shows a simple "blind," or stone, drain behind a wall. The water passes through the ground to this drain, thence to outlets through the wall, and finally through another system to a sewer or outlet. The addition of a vitrified-clay pipe line which has open joints wrapped with burlap or tar paper, as shown in Sketch (*b*), facilitates the removal of the water. The gallery drain in Sketch (*c*), which can be entered at manholes, has been used behind some

important walls along the approaches of the Lincoln Tunnel. The adaptation of the same principles for use under pavements—called a "French drain"—is pictured in Sketch (*d*).

16-8. Encasement of Structural Steel. Concrete is often used as an encasement around steel beams, girders, and columns in order to fireproof them or to protect them from corrosion. Gunite is used for the same purpose. However, it is necessary to do this work correctly so as to avoid unsightly cracking.

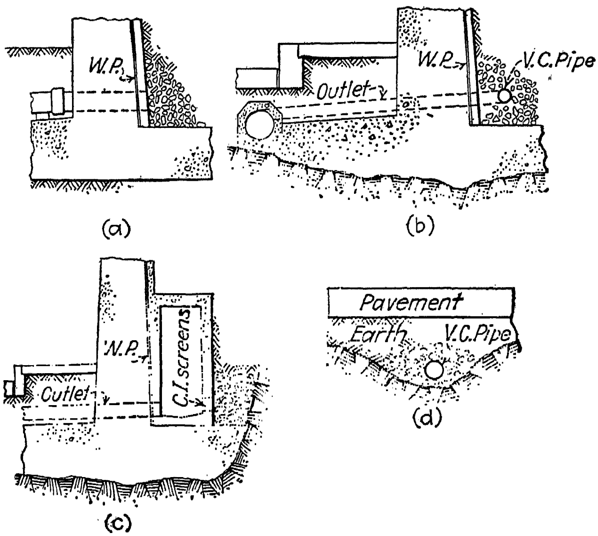


FIG. 16-6.

In the first place, if this encasement is placed on beams and girders before the major part of the dead load is applied, there is a probability that cracking of the encasement will occur, especially if the member has smooth flanges. For this reason, and in order to avoid spalling, the bottom flanges should be covered with wire mesh or beam wrapper, using 4- by 4- or 6- by 6-in. mesh and No. 8 or 9 gauge wires—preferably welded. The minimum encasement should be about 2 in. for smooth flanges or $2\frac{1}{2}$ in. when there are rivet heads to be covered. The concrete on the sides of the webs should be fastened together by rods or wires which pass through the webs. These details are pictured in Fig. 16-7.

Stiffeners on the webs of encased girders are likely to cause cracks. Thin encasement is relatively light, but it must be carried out around these stiffeners, and that causes expensive formwork. With heavy encasement, the longitudinal rods along the web should be carried through holes in the stiffeners. However, unnecessary stiffeners and other details that complicate the work of encasement should be eliminated.

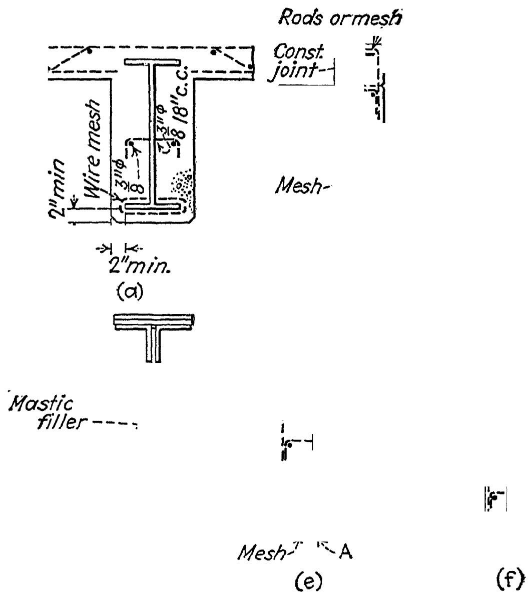


Fig. 16-7.

When a bridge deck is poured against exposed steel, as in Fig. 16-7(*d*), the load of the slab should be carried on a shelf angle; welded ties should bond it to the metal; the surface should be pitched slightly away from the steel for drainage; and some good expansion joint cement or mastic should be placed in a small, tooled groove at the junction with the metal because the joint must be sealed to prevent rusting of the steel. Similar treatment is generally advisable in any case where bare steel enters concrete if it is exposed to the weather. In important cases, it is often advisable to keep the concrete entirely free from the steel so that the latter can be painted. It is obvious that channels with the flanges turned down, beams with horizontal

webs, and similar steelwork causing air pockets are undesirable. They should have at least 3- or 4-in. holes about 18 in. c.c. in the webs to permit concreting or grouting the hollow space.

Figure 16-7(*e*) shows another problem which arises sometimes when an offset in levels occurs at an encased beam. The lower slab must be thick enough at *A* to deliver its reaction on to the top surface of the bottom flange of the I-beam, because the part

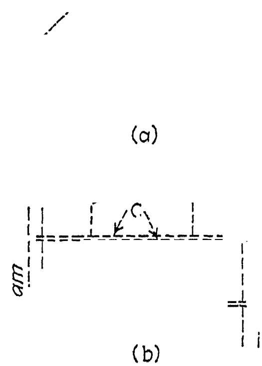


FIG. 16-8.

of the slab that is below this flange is almost valueless for transmitting shear.

When a parapet or projection is supported as shown in Fig. 16-7(*f*), the tendency of the steel beam to act by itself is likely to cause cracks at *B* due to longitudinal shear. The parapet should have joints at close intervals, or else the member should be designed so as to act as a composite beam. Furthermore, special care must be exercised to tie the parapet down to the encasement of the beam itself.

16-9. Reinforcement around Openings. When a rectangular opening must be made in a wall or some other member that is not subjected to bending, it is advisable to add special reinforcement, such as rods *A* in Fig. 16-8(*a*), in order to avoid the formation of cracks due to shrinkage, settlement, and changes in temperature. Such corners are points of weakness. However, the number of rods and the size to use are matters of judgment. In general, rods *A* should be at least sufficient to replace the temperature

steel that the opening has eliminated; in other words, the area of two sets of rods should be 0.0025 times the area of the cross section of the cut—or more.

In cases where there are large shearing forces acting in the plane of the opening that is shown in Fig. 16-8(*a*), it is advisable to use additional diagonal rods like those marked *B*. If there is a series of openings in a long structure, it may be advisable to reinforce the solid portions above and below the openings as girders with special full-height rods designed as stirrups. The area of the steel must be determined by one's judgment of the seriousness of the particular situation.

When the opening occurs in a slab that carries bending, as in Fig. 16-8(*b*), rods *C* should be added to make up for the bars that have been eliminated by the opening—or these rods should be sufficient to carry all the loads. Rods *D* are used to secure lateral spreading. The area of metal in them should equal that of the missing lateral rods or 0.0025 times the cross section of the opening. When the opening is small, the slab may not need to be thickened; but if it is large, small reinforced-concrete beams should be used instead of rods *C*, extending the beams across to the nearest main supports. Of course, when the conditions are very severe, rods *D* should also be replaced by beams.

16-10. Torsion in Concrete. There are two fundamentally different cases to consider regarding torsion in concrete members, i.e., columns or other members that have compression over the entire section, and beams that have tension over part of the cross section. The former will be considered first.

When a rectangular reinforced-concrete member fails in torsion, it generally starts to crack near the middle of the long side, the crack tending to follow a helix at approximately 45° with respect to the longitudinal axis of the member. This failure is primarily due to tension in the concrete. The longitudinal reinforcement should not be relied upon, because the rods are slender, and they depend upon the concrete for their torsional resistance. The most effective reinforcement for such a case is a spiral with its rods at an inclination of 45° to the member's axis so as to cross the possible cracks.

The maximum shearing stresses[1] in sections of isotropic material may be assumed to be the following:

[1] See Paul Andersen, Rectangular Concrete Sections under Torsion,

1. *Edge of Circular Section:*

$$v_e = 0.637\frac{M}{r_e^3} \tag{16-1}$$

2. *Middle of Long Side of Rectangular Section:*

$$\left(3 + \frac{2.6}{0.45 + \dfrac{B}{a}}\right)\frac{M}{8Ba^2} \tag{16-2}$$

3. *Middle of Short Side of Rectangular Section.*

$$\left(3 + \frac{2.6}{0.45 + \dfrac{B}{a}}\right)\frac{M}{8aB^2} \tag{16-3}$$

where M = the twisting moment, r_e = radius of concrete section, B = one-half of the long side, and a = one-half of the short side. In general, B should not exceed $1.5a$. The diagonal tensions and compressions that result from these shearing stresses may be considered to be equal to the intensities of the shearing stresses themselves.

The designer should endeavor to avoid torsional stresses that, when computed by the preceding formulas, will cause in the concrete a net tension that exceeds the allowable tensile stress in the concrete (after overcoming the compressive stresses due to longitudinal loads, if any).

When the torsional moments cause excessive tensile stresses in a circular member with a 45° spiral, Prof. Andersen states that the following relation exists:

$$= (v_e - t_c)^2(3v_e^2 + 2v_et_c + t_c^2) \tag{16-4}$$

where

$$K = \sqrt{2} \times N \times t_s \times A_s \times r_s/\pi r_e^3, \tag{16-5}$$

t_c = allowable tension in concrete, N = number of spiral bars 45° to the axis that are cut by a horizontal plane, t_s = allowable tensile stress in the steel, A_s = cross-sectional area of one spiral bar, and r_s = radius to the line of the spiral. Even when the member is square, it may be considered to be approximately

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A.C.I. *Jour.*, September–October, 1937. Equations (16-1) to (16-5), inclusive, are taken from this publication with minor changes in symbols.

the same as one whose cross section is the circle that may be inscribed in the square.

Professor Andersen's experiments also indicate that the ultimate torsional shearing unit stress in specimens that have no 45° spirals is approximately $v_m = 0.1f'_c$; spirals generally increase the magnitude of this ultimate stress; and the modulus of elasticity in torsion is about $0.45E_c$. The safe working stress in torsional shear should be about $0.03f'_c$ when no spirals are used.

When a beam that carries a severe bending moment is subjected to torsional action, the cracked concrete below the neutral axis should not be relied upon. The resisting moment should be computed from Eqs. (16-2) and (16-3), using only the uncracked part of the section bkd. A careful study of Fig. 2-2 will show the reasons for assuming it to be so. When the member is twisted, the cracks below the neutral axis prevent the transfer of shearing stresses across themselves, leaving only the uncracked portion of the beam to offer resistance. This seems to neglect the action of stirrups, but the stirrups themselves depend upon the uncracked portion for their holding power. In such a case as this, the torsional shearing stress on one side of the member adds directly to the transverse shearing stress which is due to beam action. The combined shearing stresses in the uncracked portion should not exceed about $0.2f'_c$ (even less if the torsion is relatively large).

When a beam like that at B (Fig. 10-9) supports a slab or another beam on one side only, the angular rotation of the end of the latter causes torsion in the edge support. In such a case, for a rectangular beam with a depth h which is not over 1.5 times the width b, the total angular rotation θ_T of the edge support may be approximated by assuming[1]

$$\theta_T = \frac{3.33ML}{E} \times$$

where M = the twisting moment, and E = the modulus of elasticity in torsion. Of course, h is used in Eq. (16-6) in order to obtain a measure of the maximum torsional stiffness of the beam; h should be replaced by kd when one wishes to find the torsional resisting moment that can be developed safely by the

[1] Taken from Mauer and Withey, "Strength of Materials," John Wiley & Sons, Inc., with changes in symbols.

member for any particular angle of rotation that is impressed upon it. However, the *exact* conditions in any given case are almost impossible to ascertain.

By finding θ_T for the edge support when $M = 1$, by finding the angular rotation of the end of the intersecting member as a simply supported beam that carries the given loads, and by finding the angular deflection of the end of this intersecting member when a unit moment is applied at its end, one can approximately balance the angular deflections and therefrom determine the torsional moment. However, this problem is generally

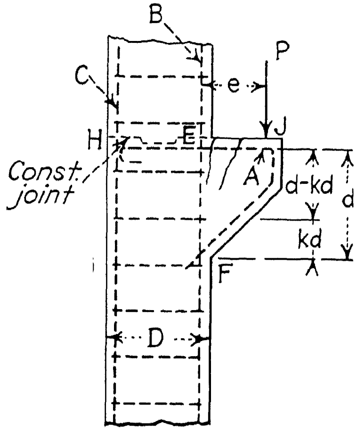


FIG. 16-9.

neglected, and the effect of the twisting is supposed to be covered by the safety factor (not always a wise procedure).

In Eq. (16-6), L is the length of the part of the member through which the torsion acts—the distance from the point of application of the torque to the reaction point. In such a case as Fig. 10-9 with a continuous slab, the magnitude of L is very uncertain; so also is the rotation of the end of the slab.

16-11. Reinforced-concrete Brackets. When a load is applied upon a bracket of reinforced concrete as pictured in Fig. 16-9, the tendency is to cause tension in rods A, also to open up the column at E and G so as to produce tensile stresses (or reduced compressive ones) in rods B and C at these two points, with compressive stresses at F and H.

The eccentricity of the load, as far as the bracket is concerned, may be assumed to be e, the distance from the nearer row of steel

to the load, because the point of compressive resistance is near F at the bottom. When e is less than d, it is sufficient to assume that the cracks will be about as shown in the figure so that the compressive stresses will be small and the rods A can be designed by the simple formula

$$M = Pe = A_s f_s jd = 0.87 A_s f_s d.$$

When e exceeds d, the bracket should be analyzed, as a cantilever beam, for compressive, tensile, and shearing stresses. The applied moment may still be called Pe.

For the column itself, the effect of the load P should be determined upon the basis of combined bending and direct stress (Chap. 7).

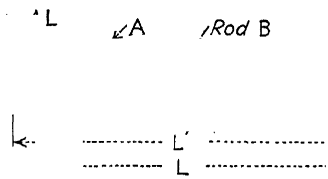


FIG. 16-10.

16-12. Prestressed Concrete. Prestressed concrete—the "preload system"—is used in special cases to prevent cracks due to shrinkage, loads, or other things when these cracks might be very objectionable—such as in a water tank.[1]

The principle of prestressed concrete is shown in Fig. 16-10. Assume that A is a block of concrete of length L and that B is a rod which is covered with a special mastic compound so as to prevent any bond between the concrete and the steel. If tensile forces P are to be applied by the loads so as to stretch the member an amount equal to ΔL (as shown by the dotted lines), the concrete must crack when its ultimate tensile strength is exceeded. However, if rod B is stressed by tightening up the nuts, it will stretch inside the concrete while the latter itself is compressed. If this tightening is continued until the concrete has been compressed an amount $\Delta L'$, and the rod B exerts a tensile force P, then the length of the member will be L'. In actual cases $\Delta L'$ and the stress in the concrete are very small. When the external loads are applied now, the rod will not stretch any more (assum-

[1] See Lt. Comdr. John L. Mason, C.E.C., U.S.N.R., Prestressed Circular Concrete Tanks, Portland Cement Association.

ing $\Delta L' = 0$) if the force P in the rod is not exceeded. Therefore, there will be no tensile cracks in the concrete.

In a case of avoiding shrinkage cracks, the prestressing of the rods is made such that their initial stretching exceeds the shrinkage deformation. Therefore, the rods will continually compress the concrete. The use of rods for such a purpose is shown in Fig. 11-42(*a*).

Prestressed concrete is not reinforced concrete in the ordinary sense, because there is no bond between the concrete and the steel. It is more nearly like highly stressed bands inside a concrete filler. Special details are required to permit the tightening of rods.

16-13. Miscellaneous Details. There are many practical and theoretical points which the engineer learns by experience. A few miscellaneous things are pointed out herein, because they sometimes cause trouble. Referring to the sketches in Fig. 16-11, note the following:

(*a*) When a beam parallel to a wall is poured against it, when it is keyed or bonded to it, or when it rests upon the edge of the wall, it will try to shift its load to the wall because its deflection is prevented. The result is a breaking of the junction or an eccentric load on the wall. The beam and the wall should be separated by a deflection joint (tar paper or similar isolating material), or the construction should be made wall bearing.

(*b*) When an expansion bearing is provided as shown in the sketch, the frictional resistance to motion will set up tensile stresses (at both fixed and expansion ends) which require special hooks in the rods.

(*c*) and (*d*). When rods are curved or offset, tensile stresses in them will tend to straighten them out and to spall the concrete unless they are tied back or otherwise detailed properly.

(*e*) When horizontal shelves, offsets, or brackets occur, like AB in the sketch, it is not advisable to try to pour the top surface against forms because of the uplift and the difficulty of filling the forms properly. When AB is sloped appreciably, forms can be used, but uplift must be guarded against.

(*f*) Corner reinforcement should be made so that one set of outside rods is bent around the corner. When there is a tendency to open up the inside corner, the inner rods A should be hooked near the outside of the wall so as to reinforce this corner. Without these hooks, rods A have very little effective anchorage. If the walls are thin, the continuity will be poor. There is insufficient room for the hooks so that it may be best to use a single rod for A, to bend it around 270° to the right so as to form an inside loop, the other end extending to the left.

(*g*) Rods at corners and in projecting parts—like A in the sketch—may cause trouble unless they are tied in properly. They should be placed so as not to interfere with the placing of the concrete.

(*h*) In cases of shelves where rods *A* are bent back, one must be careful to space them far enough apart to avoid forming a screen effect that will interfere with the placing of the concrete. Small U-shaped rods which are placed by hand are often more advantageous than bending of the main rods.

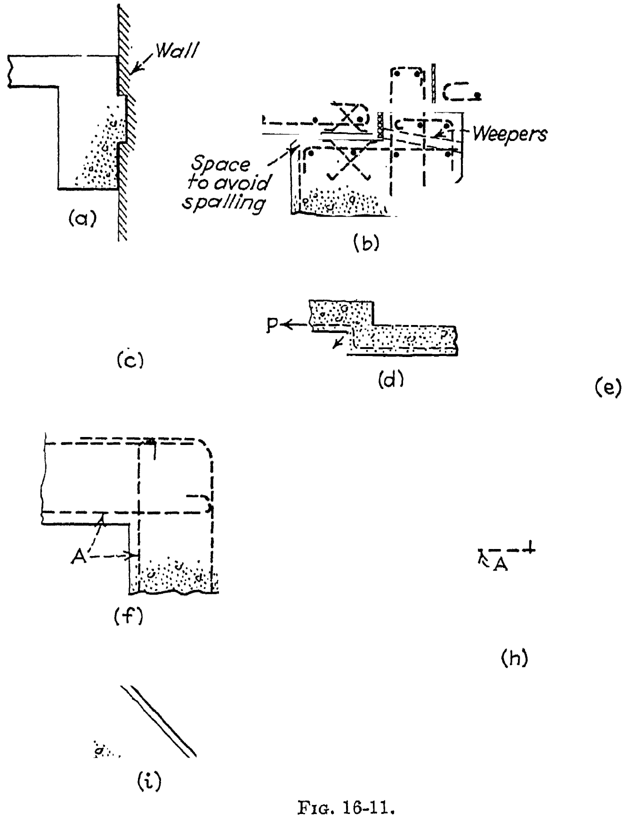


FIG. 16-11.

The vertical rods *B* should rest on the construction joint, with dowels to tie them into the lower pour. Any horizontal ties between the slab and the wall should be above the construction joint if the floor is poured last.

(*i*) Sharp corners such as at this expansion joint should be avoided because they may break off.

In planning all layouts of reinforcement, one must consider that wire chairs or mortar pads must be used to support rods that are above horizontal forms; vertical rods cannot hang in the air but must rest upon a support, such as the previous pour, or they must be wired to other rods that are so supported;

horizontal rods must be wired to the vertical ones which act as small columns in supporting them; intersecting rods must be tied together thoroughly; tie rods or spacer rods must be used to hold the main reinforcement in line; and multiple layers of rods must be held by ties and separators so as to make sure that their proper relative positions are maintained.

When surfaces have to be screeded, it is important to avoid projecting reinforcement that will impede the screeding. Small dowels or other reinforcement may be inserted after the finishing is complete. This often applies to bridge decks.

Excessively large groups of closely spaced rods may look well on a drawing, but they are very difficult to place accurately and to hold in position in the field. If they get out of line, if the upper layers sag too close to the lower ones, or if one of the rods is bent sidewise so as to get too close to the forms or to an adjacent bar, they tend to act as a screen which makes the placing of the concrete very difficult—and the development of adequate bond very questionable. Honeycombing, segregation of the aggregate, and air pockets are likely to be the result. When such a group of rods is at the top of a large member, it is desirable to arrange the bars so that there will be one or two strips of clear space (about 5 or 6 in. wide) through which the concrete can be deposited and compacted.

When planning structures, remember that beams will deflect. If a long beam passes close to a column, or a wall, but is not supported thereby, the beam's deflection may crack the floor slab if the latter passes over both the beam and the column.

These matters that have been discussed here may seem to be minor details, but careful attention to such practical things often makes the difference between a good job and an unsatisfactory one. Furthermore, good judgment, common sense, and the ability to supervise work carefully will always be among the greatest assets of the designer and the builder of reinforced-concrete structures.

APPENDIX

EXPLANATION OF DATA

A few tables and diagrams are given herein. They are to be of special assistance to the designer of reinforced-concrete structures. They are not intended to take the place of extensive information which is given in handbooks. The following instructions are given for each one:

Table 1. This contains the ordinary sizes of reinforcing rods, with the corresponding cross-sectional areas, perimeters, and weights. They are sufficiently exact for practical purposes.

Tables 2 *and* 3. These multiplication tables have been prepared in accordance with the data in Table 1.

TABLE 1.—TABLE OF AREAS, PERIMETERS, AND WEIGHTS OF DEFORMED RODS

Size, in.	Net area, sq. in.	Perimeter, in.	Weight, lb. per lin. ft.
$\frac{1}{4}$ round	0.05	0.78	0.17
$\frac{3}{8}$ round	0.11	1.18	0.38
$\frac{1}{2}$ round	0.20	1.57	0.68
$\frac{1}{2}$ square	0.25	2.00	0.86
$\frac{5}{8}$ round	0.31	1.96	1.06
$\frac{3}{4}$ round	0.44	2.36	1.52
$\frac{7}{8}$ round	0.60	2.75	2.07
1 round	0.79	3.14	2.70
1 square	1.00	4.00	3.44
$1\frac{1}{8}$ square	1.27	4.50	4.35
$1\frac{1}{4}$ square	1.56	5.00	5.37

Table 4. This gives the most common forms for use in the method of integration by volumes. The numbers in parentheses under the headings y and z refer to the sketches, which are numbered similarly. The y figure may be taken as the base of the solid, and the z figure as its altitude. The distance x is in the direction of the length L.

Figures 1 *and* 2. These figures give approximate formulas for use in preliminary or check computations of rigid-frame bridges. The supports are hinged.

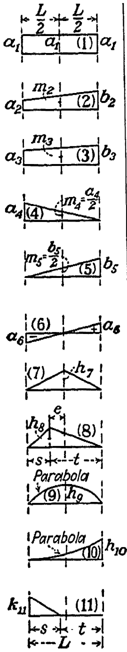
These data have been included by the permission of Walter H. Weiskopf and John W. Pickworth, together with that of the A.I.S.C. Further references are given below the figures.

Spacing, in.	Size of rod																					
	$\frac{1}{4}''\phi$		$\frac{3}{8}''\phi$		$\frac{1}{2}''\phi$		$\frac{1}{2}''\square$		$\frac{5}{8}''\phi$		$\frac{3}{4}''\phi$		$\frac{7}{8}''\phi$		$1''\phi$		$1''\square$		$1\frac{1}{8}''\square$		$1\frac{1}{4}''\square$	
	A_s	Σo	A_s	Σo	A_s	Σo	A_s	Σo	A_s	Σo	A_s	Σo	A_s	Σo	A_s	Σo	A_s	Σo	A_s	Σo	A_s	Σo
2	**0.30**	*4.68*	**0.66**	*7.08*	**1.20**	*9.42*	**1.50**	*12.00*	**1.86**	*11.76*												
2½	**0.24**	*3.74*	**0.53**	*5.66*	**0.96**	*7.54*	**1.20**	*9.60*	**1.49**	*9.41*	**2.11**	*11.33*										
3	**0.20**	*3.12*	**0.44**	*4.72*	**0.80**	*6.28*	**1.00**	*8.00*	**1.24**	*7.84*	**1.76**	*9.44*	**2.40**	*11.00*	**3.16**	*12.56*	**4.00**	*16.00*				
3½	**0.17**	*2.67*	**0.38**	*4.05*	**0.69**	*5.39*	**0.86**	*6.86*	**1.06**	*6.72*	**1.51**	*8.09*	**2.06**	*9.43*	**2.71**	*10.77*	**3.43**	*13.72*	**4.35**	*15.43*		
4	**0.15**	*2.34*	**0.33**	*3.54*	**0.60**	*4.71*	**0.75**	*6.00*	**0.93**	*5.88*	**1.32**	*7.08*	**1.80**	*8.25*	**2.37**	*9.43*	**3.00**	*12.00*	**3.81**	*13.50*	**4.68**	*15.00*
4½	**0.13**	*2.08*	**0.29**	*3.15*	**0.53**	*4.19*	**0.67**	*5.33*	**0.83**	*5.23*	**1.17**	*6.29*	**1.60**	*7.33*	**2.11**	*8.37*	**2.67**	*10.67*	**3.39**	*12.00*	**4.16**	*13.33*
5	**0.12**	*1.87*	**0.26**	*2.83*	**0.48**	*3.77*	**0.60**	*4.80*	**0.74**	*4.70*	**1.06**	*5.66*	**1.44**	*6.60*	**1.90**	*7.54*	**2.40**	*9.60*	**3.05**	*10.80*	**3.74**	*12.00*
5½	**0.11**	*1.70*	**0.24**	*2.57*	**0.44**	*3.43*	**0.55**	*4.36*	**0.68**	*4.28*	**0.96**	*5.15*	**1.31**	*6.00*	**1.72**	*6.85*	**2.18**	*8.73*	**2.77**	*9.82*	**3.40**	*10.91*
6	**0.10**	*1.56*	**0.22**	*2.36*	**0.40**	*3.14*	**0.50**	*4.00*	**0.62**	*3.92*	**0.88**	*4.72*	**1.20**	*5.50*	**1.58**	*6.28*	**2.00**	*8.00*	**2.54**	*9.00*	**3.12**	*10.00*
6½	**0.09**	*1.44*	**0.20**	*2.18*	**0.37**	*2.90*	**0.46**	*3.69*	**0.57**	*3.62*	**0.81**	*4.36*	**1.11**	*5.08*	**1.46**	*5.80*	**1.85**	*7.38*	**2.34**	*8.31*	**2.88**	*9.23*
7	**0.09**	*1.34*	**0.19**	*2.02*	**0.34**	*2.69*	**0.43**	*3.43*	**0.53**	*3.36*	**0.75**	*4.05*	**1.03**	*4.71*	**1.35**	*5.38*	**1.71**	*6.86*	**2.18**	*7.71*	**2.67**	*8.57*
7½	**0.08**	*1.25*	**0.18**	*1.89*	**0.32**	*2.51*	**0.40**	*3.20*	**0.50**	*3.14*	**0.70**	*3.78*	**0.96**	*4.40*	**1.26**	*5.02*	**1.60**	*6.40*	**2.03**	*7.20*	**2.50**	*8.00*
8	**0.08**	*1.17*	**0.16**	*1.77*	**0.30**	*2.36*	**0.38**	*3.00*	**0.46**	*2.94*	**0.66**	*3.54*	**0.90**	*4.12*	**1.18**	*4.71*	**1.50**	*6.00*	**1.90**	*6.75*	**2.34**	*7.50*
9	**0.07**	*1.04*	**0.15**	*1.57*	**0.27**	*2.09*	**0.33**	*2.67*	**0.41**	*2.61*	**0.59**	*3.15*	**0.80**	*3.67*	**1.05**	*4.19*	**1.33**	*5.33*	**1.69**	*6.00*	**2.08**	*6.66*
10	**0.06**	*0.94*	**0.13**	*1.42*	**0.24**	*1.88*	**0.30**	*2.40*	**0.37**	*2.35*	**0.53**	*2.83*	**0.72**	*3.30*	**0.95**	*3.77*	**1.20**	*4.80*	**1.52**	*5.40*	**1.87**	*6.00*
11	**0.05**	*0.85*	**0.12**	*1.29*	**0.22**	*1.71*	**0.27**	*2.18*	**0.34**	*2.14*	**0.48**	*2.57*	**0.65**	*3.00*	**0.86**	*3.43*	**1.09**	*4.36*	**1.39**	*4.91*	**1.70**	*5.46*
12	**0.05**	*0.78*	**0.11**	*1.18*	**0.20**	*1.57*	**0.25**	*2.00*	**0.31**	*1.96*	**0.44**	*2.36*	**0.60**	*2.75*	**0.79**	*3.14*	**1.00**	*4.00*	**1.27**	*4.50*	**1.56**	*5.00*

No. of rods	Size of rod																					
	$\frac{1}{4}''\phi$		$\frac{3}{8}''\phi$		$\frac{1}{2}''\phi$		$\frac{1}{2}''\square$		$\frac{5}{8}''\phi$		$\frac{3}{4}''\phi$		$\frac{7}{8}''\phi$		$1''\phi$		$1''\square$		$1\frac{1}{8}''\square$		$1\frac{1}{4}''\square$	
	A_s	Σo	A_s	Σo	A_s	Σo	A_s	Σo	A_s	Σo	A_s	Σo	A_s	Σo	A_s	Σo	A_s	Σo	A_s	Σo	A_s	Σo
2	**0.10**	*1.56*	**0.22**	*2.36*	**0.40**	*3.14*	**0.50**	*4.00*	**0.62**	*3.92*	**0.88**	*4.72*	**1.20**	*5.50*	**1.58**	*6.28*	**2.00**	*8.00*	**2.54**	*9.00*	**3.12**	*10.0*
3	**0.15**	*2.34*	**0.33**	*3.54*	**0.60**	*4.71*	**0.75**	*6.00*	**0.93**	*5.88*	**1.32**	*7.08*	**1.80**	*8.25*	**2.37**	*9.42*	**3.00**	*12.00*	**3.81**	*13.50*	**4.68**	*15.0*
4	**0.20**	*3.12*	**0.44**	*4.72*	**0.80**	*6.28*	**1.00**	*8.00*	**1.24**	*7.84*	**1.76**	*9.44*	**2.40**	*11.0*	**3.16**	*12.6*	**4.00**	*16.0*	**5.03**	*18.0*	**6.24**	*20.0*
5	**0.25**	*3.90*	**0.55**	*5.90*	**1.00**	*7.85*	**1.25**	*10.0*	**1.55**	*9.80*	**2.20**	*11.8*	**3.00**	*13.8*	**3.95**	*15.7*	**5.00**	*20.0*	**6.35**	*22.5*	**7.80**	*25.0*
6	**0.30**	*4.68*	**0.66**	*7.08*	**1.20**	*9.42*	**1.50**	*12.0*	**1.86**	*11.8*	**2.64**	*14.2*	**3.60**	*16.5*	**4.74**	*18.8*	**6.00**	*24.0*	**7.62**	*27.0*	**9.36**	*30.0*
7	**0.35**	*5.46*	**0.77**	*8.26*	**1.40**	*11.0*	**1.75**	*14.0*	**2.17**	*13.7*	**3.08**	*16.5*	**4.20**	*19.2*	**5.53**	*22.0*	**7.00**	*28.0*	**8.89**	*31.5*	**10.9**	*35.0*
8	**0.40**	*6.24*	**0.88**	*9.44*	**1.60**	*12.6*	**2.00**	*16.0*	**2.48**	*15.7*	**3.52**	*18.9*	**4.80**	*22.0*	**6.32**	*25.1*	**8.00**	*32.0*	**10.2**	*36.0*	**12.5**	*40.0*
9	**0.45**	*7.02*	**0.99**	*10.6*	**1.80**	*14.1*	**2.25**	*18.0*	**2.79**	*17.6*	**3.96**	*21.2*	**5.40**	*24.8*	**7.11**	*28.3*	**9.00**	*36.0*	**11.4**	*40.5*	**14.0**	*45.0*
10	**0.50**	*7.80*	**1.10**	*11.8*	**2.00**	*15.7*	**2.50**	*20.0*	**3.10**	*19.6*	**4.40**	*23.6*	**6.00**	*27.5*	**7.90**	*31.4*	**10.0**	*40.0*	**12.7**	*45.0*	**15.6**	*50.0*
11	**0.55**	*8.58*	**1.21**	*13.0*	**2.20**	*17.3*	**2.75**	*22.0*	**3.41**	*21.6*	**4.84**	*26.0*	**6.60**	*30.2*	**8.69**	*34.5*	**11.0**	*44.0*	**14.0**	*49.5*	**17.2**	*55.0*
12	**0.60**	*9.36*	**1.32**	*14.2*	**2.40**	*18.8*	**3.00**	*24.0*	**3.72**	*23.5*	**5.28**	*28.3*	**7.20**	*33.0*	**9.48**	*37.7*	**12.0**	*48.0*	**15.2**	*54.0*	**18.7**	*60.0*
13	**0.65**	*10.1*	**1.43**	*15.3*	**2.60**	*20.4*	**3.25**	*26.0*	**4.03**	*25.5*	**5.72**	*30.7*	**7.80**	*35.8*	**10.3**	*40.8*	**13.0**	*52.0*	**16.5**	*58.5*	**20.3**	*65.0*
14	**0.70**	*10.9*	**1.54**	*16.5*	**2.80**	*22.0*	**3.50**	*28.0*	**4.34**	*27.4*	**6.16**	*33.0*	**8.40**	*38.5*	**11.1**	*44.0*	**14.0**	*56.0*	**17.8**	*63.0*	**21.8**	*70.0*
15	**0.75**	*11.7*	**1.65**	*17.7*	**3.00**	*23.6*	**3.75**	*30.0*	**4.65**	*29.4*	**6.60**	*35.4*	**9.00**	*41.2*	**11.8**	*47.1*	**15.0**	*60.0*	**19.0**	*67.5*	**23.4**	*75.0*
16	**0.80**	*12.5*	**1.76**	*18.9*	**3.20**	*25.1*	**4.00**	*32.0*	**4.96**	*31.4*	**7.04**	*37.8*	**9.60**	*44.0*	**12.6**	*50.2*	**16.0**	*64.0*	**20.3**	*72.0*	**25.0**	*80.0*
17	**0.85**	*13.3*	**1.87**	*20.1*	**3.40**	*26.7*	**4.25**	*34.0*	**5.27**	*33.3*	**7.48**	*40.1*	**10.2**	*46.8*	**13.4**	*53.4*	**17.0**	*68.0*	**21.6**	*76.5*	**26.5**	*85.0*
18	**0.90**	*14.0*	**1.98**	*21.2*	**3.60**	*28.3*	**4.50**	*36.0*	**5.58**	*35.3*	**7.92**	*42.5*	**10.8**	*49.5*	**14.2**	*56.5*	**18.0**	*72.0*	**22.9**	*81.0*	**28.1**	*90.0*
19	**0.95**	*14.8*	**2.09**	*22.4*	**3.80**	*29.8*	**4.75**	*38.0*	**5.89**	*37.2*	**8.36**	*44.8*	**11.4**	*52.2*	**15.0**	*59.7*	**19.0**	*76.0*	**24.1**	*85.5*	**29.6**	*95.0*
20	**1.00**	*15.6*	**2.20**	*23.6*	**4.00**	*31.4*	**5.00**	*40.0*	**6.20**	*39.2*	**8.80**	*47.2*	**12.0**	*55.0*	**15.8**	*62.8*	**20.0**	*80.0*	**25.4**	*90.0*	**31.2**	*100.0*
21	**1.05**	*16.4*	**2.31**	*24.8*	**4.20**	*33.0*	**5.25**	*42.0*	**6.51**	*41.2*	**9.24**	*49.6*	**12.6**	*57.8*	**16.6**	*65.9*	**21.0**	*84.0*	**26.7**	*94.5*	**32.8**	*105.0*
22	**1.10**	*17.2*	**2.42**	*26.0*	**4.40**	*34.5*	**5.50**	*44.0*	**6.82**	*43.1*	**9.68**	*51.9*	**13.2**	*60.5*	**17.4**	*69.1*	**22.0**	*88.0*	**27.9**	*99.0*	**34.3**	*110.0*
23	**1.15**	*17.9*	**2.53**	*27.1*	**4.60**	*36.1*	**5.75**	*46.0*	**7.13**	*45.1*	**10.1**	*54.3*	**13.8**	*63.2*	**18.2**	*72.2*	**23.0**	*92.0*	**29.2**	*103.5*	**35.9**	*115.0*
24	**1.20**	*18.7*	**2.64**	*28.3*	**4.80**	*37.7*	**6.00**	*48.0*	**7.44**	*47.0*	**10.6**	*56.6*	**14.4**	*66.0*	**19.0**	*75.4*	**24.0**	*96.0*	**30.5**	*108.0*	**37.4**	*120.0*
25	**1.25**	*19.5*	**2.75**	*29.5*	**5.00**	*39.2*	**6.25**	*50.0*	**7.75**	*49.0*	**11.0**	*59.0*	**15.0**	*68.8*	**19.8**	*78.5*	**25.0**	*100.0*	**31.8**	*112.5*	**39.0**	*125.0*
26	**1.30**	*20.3*	**2.86**	*30.7*	**5.20**	*40.8*	**6.50**	*52.0*	**8.06**	*51.0*	**11.4**	*61.4*	**15.6**	*71.5*	**20.5**	*81.6*	**26.0**	*104.0*	**33.0**	*117.0*	**40.6**	*130.0*
27	**1.35**	*21.1*	**2.97**	*31.9*	**5.40**	*42.4*	**6.75**	*54.0*	**8.37**	*52.9*	**11.9**	*63.7*	**16.2**	*74.2*	**21.3**	*84.8*	**27.0**	*108.0*	**34.3**	*121.5*	**42.1**	*135.0*
28	**1.40**	*21.8*	**3.08**	*33.0*	**5.60**	*44.0*	**7.00**	*56.0*	**8.68**	*54.9*	**12.3**	*66.1*	**16.8**	*77.0*	**22.1**	*87.9*	**28.0**	*112.0*	**35.6**	*126.0*	**43.7**	*140.0*
29	**1.45**	*22.6*	**3.19**	*34.2*	**5.80**	*45.5*	**7.25**	*58.0*	**8.99**	*56.8*	**12.8**	*68.4*	**17.4**	*79.8*	**22.9**	*91.1*	**29.0**	*116.0*	**36.8**	*130.5*	**45.2**	*145.0*
30	**1.50**	*23.4*	**3.30**	*35.4*	**6.00**	*47.1*	**7.50**	*60.0*	**9.30**	*58.8*	**13.2**	*70.8*	**18.0**	*82.5*	**23.7**	*94.2*	**30.0**	*120.0*	**38.1**	*135.0*	**46.8**	*150.0*

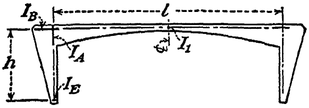
TABLE 4.—INTEGRATION BY VOLUME

(Table of values of $\Sigma M_y M_z\, dx$)



y	z	
(1)	(1)	La_1^2
(1)	(2)	$\frac{L}{6}(a_1a_2 + 4a_1m_2 + a_1b_2)$
(1)	(4)	$\frac{L}{2}a_1a_4$
(1)	(6)	0
(1)	(7)	$\frac{L}{2}a_1h_7$
(1)	(8)	$\frac{s}{2}a_1h_8 + \frac{t}{2}a_1h_8$
(1)	(9)	$\frac{2}{3}La_1h_9$
(1)	(10)	$\frac{1}{3}La_1h_{10}$
(1)	(11)	$\frac{s}{2}a_1k_{11}$
(2)	(2)	$\frac{L}{6}(a_2^2 + 4m_2^2 + b_2^2)$
(2)	(3)	$\frac{L}{6}(a_2a_3 + 4m_2m_3$
(2)	(4)	$\frac{La_4}{6}(a_2$
(2)	(5)	$\frac{Lb_5}{6}$
(2)	(7)	
(2)	(8)	$\frac{m_2}{6}[a_2(L + t) + b_2(I$
(2)	(9)	$\frac{2}{3}Lm_2h_9$
(2)	(10)	$\frac{h_{10}L}{12}(3b_2 + a_2)$
(2)	(11)	$\frac{k_{11}s}{2}\Big[\quad \frac{2s(b_2 -}{3L}$
	(4)	$\frac{L}{3}a_4^2$
(4)	(5)	$\frac{L}{6}a_4l$
(4)	(6)	$-\frac{L}{6}$
	(7)	
	(11)	$\frac{}{6}a$
	(6)	$\frac{L}{6}b$
(5)	(11)	$6L$
	(6)	$\frac{L}{3}a_6$
	(7)	0
	(8)	$-\frac{}{3}a_6$
(6)	(11)	
(7)	(7)	$\frac{L}{3}h_7^2$ or $\frac{L}{3}h$
(7)	(8)	$\frac{L}{24}h \quad 2s^2$
(7)	(11)	$\frac{}{3L}h_7k_{11}$
(8)	(8)	$\frac{L}{3}h_8^2$ or $\frac{L}{3}h_8h_8'$

APPROXIMATE FORMULAS.—SYMMETRICAL RIGID FRAME



$$\frac{I_A - I_1}{I_1}$$

[Approximations]

$n, m = 2$

$$B = \frac{I_B - I_E}{I_E}$$

R

Concentrated load at any point

$$= \frac{2Pl}{hR}[3k - 3k^2 + A(k$$

Concentrated load at center of span

Uniform load w per unit length

Tractive force P_1 at deck

Horizontal force

$$\frac{2hI_A}{5lRI_B}$$

$$- 5k_1^4 + 3k_1^5)]$$

$= P_1 -$

Earth pressure

$$\frac{e}{2} + \frac{f}{3}) -$$

$$[e(105 + 21B) + f(49 + 11B)$$

Earth pressure one side only.

$$h(\frac{e}{4} + \frac{5f}{12}$$

$$\times$$

$$[e(105 \qquad + f(49 + 11B)$$

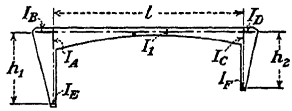
Temperature

$= \pm$

In temperature formula, if E is expressed in pounds per square inch, h must be expressed in inches. H then will be in pounds.

FIG. 1.—(*Weiskopf and Pickworth*, "*Symmetrical Rigid Frames*," *A.I.S.C.*) For the mathematical development of these formulas, see A.S.C.E. *Trans.*, Vol. 102, p. 1, 1937. These data are for the case of hinged supports.

APPROXIMATE FORMULAS.—UNSYMMETRICAL RIGID FRAME



$B =$ $- I_E$ D $I_D - I_F$

Approximations
$m, n, p, q =$
$r, s = 0.5$

$30I_D [10 + D]$

$A[8h_1^2$

$120I_C$ $3h_1^2]\}$

Concentrated load

48 $- 24k + A(5 - 16k^3)] + \frac{1}{I_C}[6 + 3C]\}$

$- h_1)$ $\frac{1}{r}[20 - 40k^2 + A(9 - 48k^4)] +$
240

$\frac{}{I_C}[20 + 9C]\}$

$U_3 + U_4$

Uniform load
w per unit length

$25h_2] + A[22h_1$

U_8 $\frac{l^2}{1920I_C}$ $25h_1]$

wl $U_7 + U_8$

FIG. 2.—(*Weiskopf and Pickworth, "Unsymmetrical Rigid Frames,"* A.I.S.C.) For the mathematical development of these formulas, see A.S.C.E. *Trans.*, Vol. 102, p. 1, 1937. These data are for the case of hinged supports.

APPROXIMATE FORMULAS.—UNSYMMETRICAL RIGID FRAME.—(*Continued*)

Tractive force

$$\frac{h_2 - h_1)}{120}\left\{\frac{1}{I_A}[5 + 3A] + \frac{1}{I_C}[35 + 8C]\right.$$

$$H_1 = \quad + U_{10} \quad + \quad P_1 - H_1$$

Horizontal force $\quad h_1^3 k_1 \quad k_1^2 + B[2 -$

$$P_1 - H_1$$

Earth pressure $\quad H_{12} = \frac{n_1}{2520 I_B}\{e_1[105 + 21B] + f_1[49 + 11B]\}$

$$\frac{}{2520 I_D}\{e_2[105 + 21D] + f_2[49$$

$$+ U_2 + U_3 + U_4$$

Temperature

$$+$$

In temperature formula, if E is expressed in pounds per square inch, U and l must be expressed in inches. H then will be in pounds.

FIG. 2.—(*Continued*)

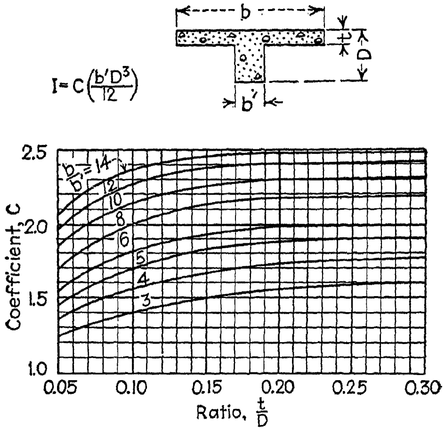


FIG. 3.—Approximate moments of inertia of T-shaped sections.

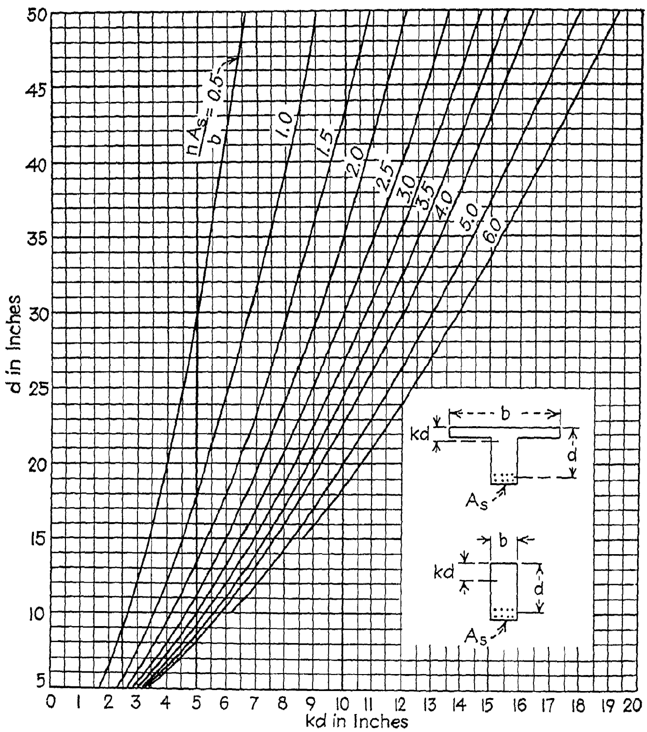


FIG. 4.—Location of neutral axis of beam with tensile steel only.

Figure 3. This diagram is for use in approximating the moments of inertia of uncracked T-shapes. It is for special use in the analysis of building frames. The values of C modify I for the "stem portion" alone.

Figures 4 *to* 9, *inclusive*. The analysis of rectangular beams can be greatly expedited by the use of these diagrams; they are also applicable for the analysis of T-beams if the neutral axis lies within the flange or very close to it. In any case, the values of nA_s/b and $\frac{(n-1)}{b}A'_s$ per inch of width of the beam can be found, using b as the divisor for T-beams as well as for rectangular ones when the bending moment is positive, or using b' for T-beams when the bending moment is negative.

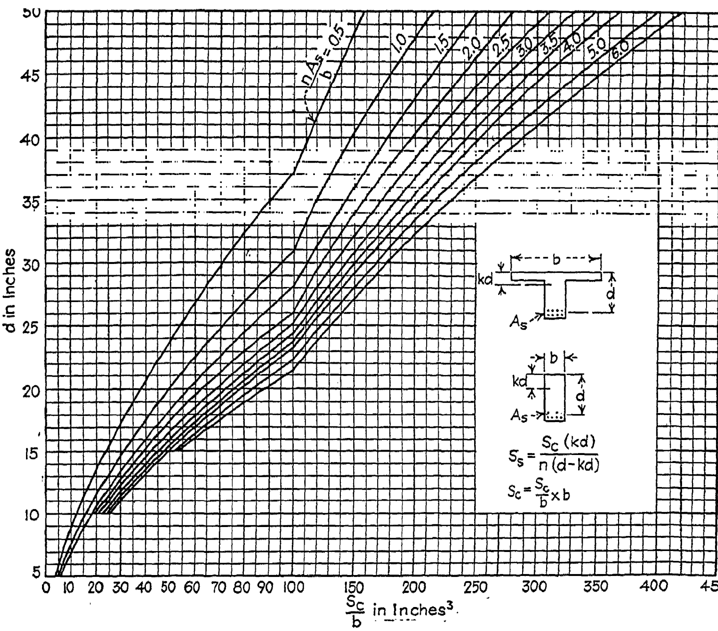


FIG. 5.—Section modulus of a 1-in. width of a beam with tensile steel only.

The procedure in using the diagrams is as follows: When there is little or no steel in compression, use Figs. 4 and 5; otherwise use the curves that are prepared for the nearest value of $\frac{(n-1)}{b}A'_s$; with d as the ordinate, cross horizontally to the proper (or interpolated) value of nA_s/b, then read the corresponding magnitudes of kd and S_c/b; multiply S_c/b by b (or b') to find S_c, then compute $S_s = \frac{S_c(kd)}{n(d-kd)}$ from the quantities already found. Results from two diagrams may be used for interpolation if greater accuracy is desired.

The advantage of these diagrams is the fact that they enable one to find the section moduli, and thereby compute f_c and f_s. Of course, important

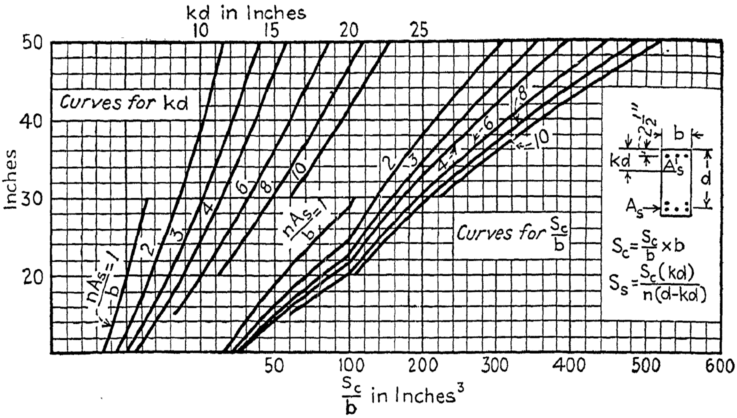


FIG. 6.—Location of neutral axis, and magnitude of section modulus of a 1-in. width of a beam, when $\frac{(n-1)}{b}A'_s = 1.0$.

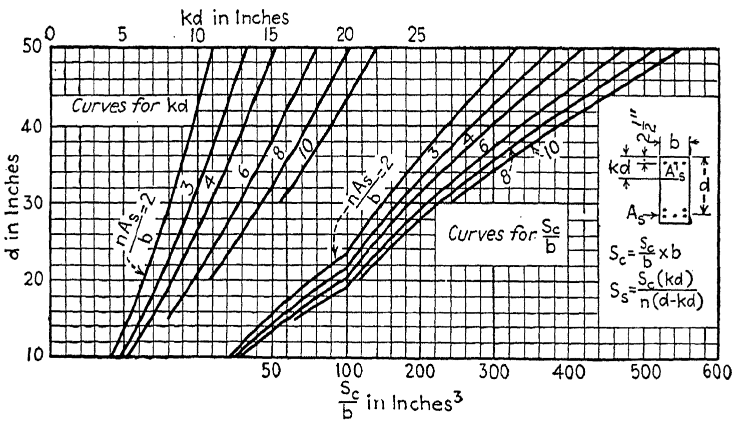


FIG. 7.—Location ot neutral axis, and magnitude of section modulus of a 1-in.
$\frac{(n-1)}{b}A'$

members should be checked analytically after the diagrams have been used for approximate analysis.

In using these diagrams, it will generally be sufficient to assume that

1 *kd*

1

Tables 5 *and* 6. These tables are prepared for use in preliminary design to obtain the theoretically most efficient beam, the results then being modified by practical considerations if necessary.

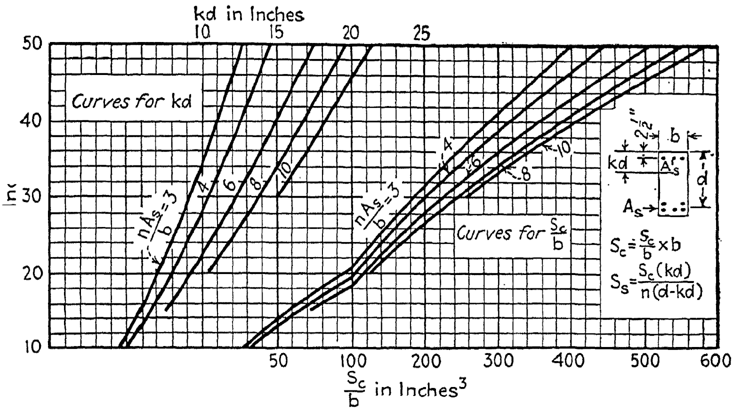


FIG. 8.—Location of neutral axis, and magnitude of section modulus of a 1-in. width of a beam, when $\frac{(n-1)}{b}A'_s = 3.0$.

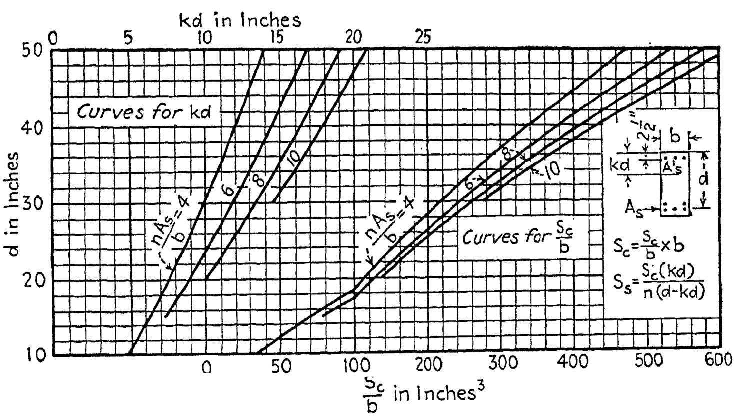
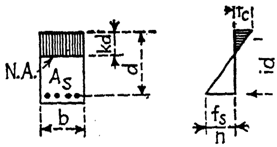


FIG. 9.—Location of neutral axis, and magnitude of section modulus of a 1-in. width of a beam, when $\frac{(n-1)}{b}A'_s = 4.0$.

Figures 10 *and* 11. These diagrams are for the analysis of members. If k for any T-beam lies below and to the right of the straight line in Fig. 11(*a*), it means that the neutral axis is within the flange and that it can be located by means of Fig. 10.

TABLE 5.—COEFFICIENTS p, k, j, AND K FOR RECTANGULAR SECTIONS FOR BALANCED DESIGNS



A_s

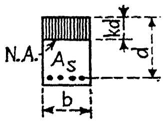
$$j = 1 - \frac{1}{3}k$$

bd^2

f_c' and n	f_c	f_s = 16,000				f_s = 18,000			
		p	k	j	K*	p	k	j	K*
2000 15	650	0.0077	0.379	0.874	108	0.0063	0.351	0.883	101
	700	0.0087	0.396	0.868	120	0.0072	0.368	0.877	113
	750	0.0097	0.413	0.862	133	0.0080	0.385	0.872	126
	800	0.0107	0.429	0.857	147	0.0089	0.400	0.867	139
	900	0.0129	0.458	0.847	175	0.0107	0.429	0.857	165
	1000	0.0151	0.484	0.839	203	0.0126	0.455	0.848	193
2500 12	700	0.0075	0.344	0.885	107	0.0062	0.318	0.894	100
	750	0.0084	0.360	0.880	119	0.0070	0.333	0.889	111
	800	0.0094	0.375	0.875	131	0.0077	0.348	0.884	123
	875	0.0108	0.396	0.868	150	0.0089	0.368	0.877	141
	950	0.0124	0.416	0.861	170	0.0102	0.388	0.871	161
	1000	0.0134	0.429	0.857	184	0.0111	0.400	0.867	173
	1125	0.0161	0.458	0.847	218	0.0134	0.429	0.857	207
	1250	0.0189	0.484	0.839	254	0.0158	0.455	0.848	241
3000 10	750	0.0075	0.319	0.894	107	0.0061	0.294	0.902	99
	800	0.0083	0.333	0.889	118	0.0068	0.308	0.897	111
	900	0.0101	0.360	0.880	143	0.0083	0.333	0.889	133
	975	0.0115	0.379	0.874	161	0.0095	0.351	0.883	151
	1050	0.0130	0.396	0.868	180	0.0107	0.368	0.877	169
	1125	0.0145	0.413	0.862	200	0.0120	0.385	0.872	189
	1200	0.0161	0.429	0.857	221	0.0133	0.400	0.867	208
	1350	0.0193	0.458	0.847	262	0.0161	0.429	0.857	248
	1500	0.0227	0.484	0.839	305	0.0190	0.455	0.848	289
3750 8	900	0.0087	0.310	0.897	125	0.0072	0.286	0.905	116
	1000	0.0104	0.333	0.889	148	0.0086	0.308	0.897	138
	1100	0.0122	0.355	0.882	172	0.0100	0.328	0.891	161
	1200	0.0141	0.375	0.875	197	0.0116	0.348	0.884	185
	1300	0.0160	0.394	0.869	223	0.0132	0.366	0.878	209
	1400	0.0180	0.412	0.863	249	0.0149	0.384	0.872	234
	1500	0.0201	0.429	0.857	276	0.0167	0.400	0.867	260
	1700	0.0244	0.460	0.847	331	0.0203	0.430	0.857	313
	1875	0.0284	0.484	0.839	381	0.0237	0.455	0.848	362

* Sometimes denoted by R.

TABLE 6.—COEFFICIENTS p, k, j, AND K FOR RECTANGULAR SECTIONS FOR BALANCED DESIGNS



$$k = \qquad +$$

$$p = \frac{}{bd} = \frac{}{2f_s}$$

f_c' and n	f_c	$f_s = 20{,}000$				$f_s = 22{,}000$			
		p	k	j	K*	p	k	j	K*
	650	0.0053	0.328	0.891	95	0.0045	0.307	0.898	90
	700	0.0060	0.344	0.885	107	0.0051	0.323	0.892	101
2000	750	0.0068	0.360	0.880	119	0.0058	0.338	0.887	112
15	800	0.0075	0.375	0.875	131	0.0064	0.353	0.882	125
	900	0.0091	0.403	0.866	157	0.0078	0.380	0.873	149
	1000	0.0107	0.429	0.857	184	0.0092	0.405	0.865	175
	700	0.0052	0.296	0.901	93	0.0044	0.276	0.908	88
	750	0.0058	0.310	0.897	104	0.0049	0.290	0.903	98
	800	0.0065	0.324	0.892	116	0.0055	0.304	0.899	109
2500	875	0.0075	0.344	0.885	133	0.0064	0.323	0.892	126
12	950	0.0086	0.363	0.879	152	0.0074	0.341	0.886	144
	1000	0.0094	0.375	0.875	164	0.0080	0.353	0.882	156
	1125	0.0113	0.403	0.866	196	0.0097	0.380	0.873	187
	1250	0.0134	0.429	0.857	230	0.0115	0.405	0.865	219
	750	0.0051	0.273	0.909	93	0.0043	0.254	0.915	87
	800	0.0057	0.286	0.905	104	0.0049	0.267	0.911	97
	900	0.0070	0.310	0.897	125	0.0059	0.290	0.903	118
3000	975	0.0080	0.328	0.891	142	0.0068	0.307	0.898	134
10	1050	0.0090	0.344	0.885	160	0.0077	0.323	0.892	151
	1125	0.0101	0.360	0.880	178	0.0086	0.338	0.887	169
	1200	0.0113	0.375	0.875	197	0.0096	0.353	0.882	187
	1350	0.0136	0.403	0.866	236	0.0117	0.380	0.873	224
	1500	0.0161	0.429	0.857	276	0.0138	0.405	0.865	263
	900	0.0060	0.265	0.912	109	0.0051	0.247	0.918	102
	1000	0.0072	0.286	0.905	129	0.0061	0.267	0.911	122
	1100	0.0084	0.306	0.898	151	0.0072	0.286	0.905	142
3750	1200	0.0097	0.324	0.892	173	0.0083	0.304	0.899	164
8	1300	0.0111	0.342	0.886	197	0.0095	0.321	0.893	186
	1400	0.0126	0.359	0.880	221	0.0107	0.337	0.888	209
	1500	0.0141	0.375	0.875	246	0.0120	0.353	0.882	234
	1700	0.0172	0.405	0.865	298	0.0148	0.382	0.873	283
	1875	0.0201	0.429	0.857	345	0.0173	0.405	0.865	328

* Sometimes denoted by R.

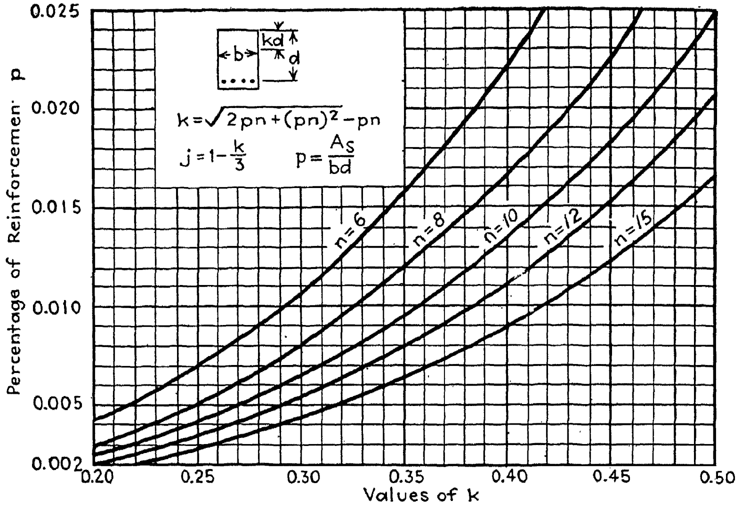
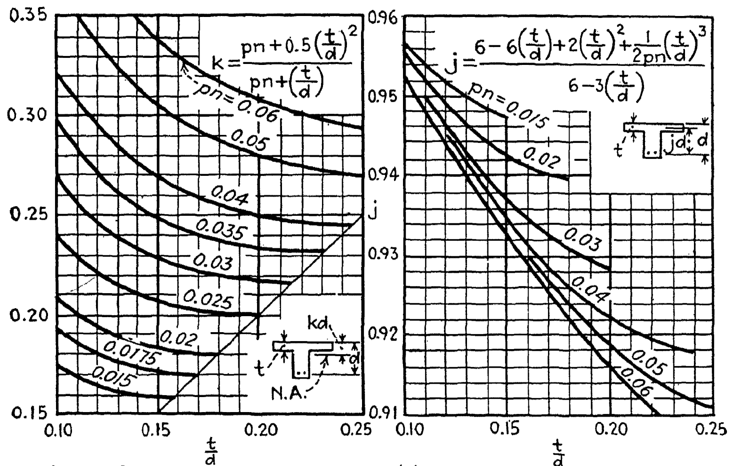


FIG. 10.—Location of neutral axis of rectangular beams with tensile steel only.



| Curves for location of neutral axis (b) Curves for location of centroid of compression

FIG. 11.—Magnitudes of k and j for T-beams.

Figure 12 *and Table* 7. Knowing the longitudinal shear to be withstood by the stirrups per inch of length of the beam, varying combinations of sizes and spacings of stirrups can be secured from Fig. 12. It is then important

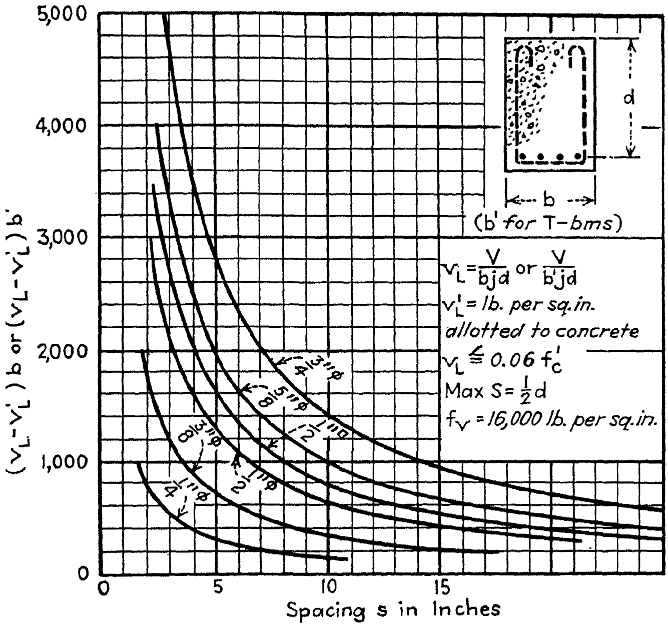


FIG. 12.—Maximum spacing of vertical U-shaped stirrups.

TABLE 7.—RECOMMENDED MINIMUM EFFECTIVE DEPTH OF BEAM IN INCHES TO DEVELOP SINGLE VERTICAL U-SHAPED STIRRUPS WITH STANDARD HOOKS

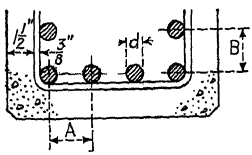
($u = 0.05f_c'$ for deformed rods)

Size of rod	f_c', lb. per sq. in.				
	2000	2500	3000	3500	4000
$\frac{1}{4}''\phi$	19	15	13	11	10
$\frac{3}{8}''\phi$	26	20	16	14	12
$\frac{1}{2}''\phi$	34	26	21	17	14
$\frac{1}{2}''\square$	33	26	21	17	13
$\frac{5}{8}''\phi$	41	31	24	20	15
$\frac{3}{4}''\phi$	48	34	26	22	18

to check the chosen size of stirrup in Table 7 to make sure that the beam is deep enough to provide adequate bond to develop the stirrups in the upper (or compression) half of the beam.

Table 8. Although the Code permits the use of narrower beams than those shown in Table 8, the data given therein have been prepared to provide generous space for thorough encasement of the steel.

TABLE 8.—MINIMUM WIDTHS OF BEAMS



Min. $A = 3d$ or $d + 1\frac{1}{2} \times$ aggregate size.
Dimensions are in inches and are increased to nearest half inch.
Allow extra for splices.

	d, in.	Number of longitudinal rods 2	3	4	5	6	7	8	9	10	Min. A or B	Preferred B
$\frac{3}{4}$ in. aggregate	$\frac{1}{2}$	6	8	$9\frac{1}{2}$	11	$12\frac{1}{2}$	14	16	$17\frac{1}{2}$	19	$1\frac{5}{8}$	2
	$\frac{5}{8}$	$6\frac{1}{2}$	$8\frac{1}{2}$	10	12	14	16	$17\frac{1}{2}$	$19\frac{1}{2}$	$21\frac{1}{2}$	$1\frac{7}{8}$	$2\frac{1}{2}$
	$\frac{3}{4}$	7	9	$11\frac{1}{2}$	$13\frac{1}{2}$	16	18	$20\frac{1}{2}$	$22\frac{1}{2}$	25	$2\frac{1}{4}$	$2\frac{1}{2}$
	$\frac{7}{8}$	$7\frac{1}{2}$	10	$12\frac{1}{2}$	$15\frac{1}{2}$	18	$20\frac{1}{2}$	23	26	$28\frac{1}{2}$	$2\frac{5}{8}$	3
	1	8	11	14	17	20	23	26	29	32	3	3
	$1\frac{1}{8}$	$8\frac{1}{2}$	12	15	$18\frac{1}{2}$	22	$25\frac{1}{2}$	$28\frac{1}{2}$	32	$35\frac{1}{2}$	$3\frac{3}{8}$	$3\frac{1}{2}$
	$1\frac{1}{4}$	9	$12\frac{1}{2}$	$16\frac{1}{2}$	20	24	$27\frac{1}{2}$	$31\frac{1}{2}$	35	39	$3\frac{3}{4}$	4
$1\frac{1}{2}$ in. aggregate	$\frac{1}{2}$	7	10	$12\frac{1}{2}$	$15\frac{1}{2}$	18	21	$23\frac{1}{2}$	$26\frac{1}{2}$	29	$2\frac{3}{4}$	3
	$\frac{5}{8}$	$7\frac{1}{2}$	$10\frac{1}{2}$	13	16	19	22	$24\frac{1}{2}$	$27\frac{1}{2}$	$30\frac{1}{2}$	$2\frac{7}{8}$	3
	$\frac{3}{4}$	$7\frac{1}{2}$	$10\frac{1}{2}$	$13\frac{1}{2}$	$16\frac{1}{2}$	$19\frac{1}{2}$	$22\frac{1}{2}$	$25\frac{1}{2}$	$28\frac{1}{2}$	$31\frac{1}{2}$	3	3
	$\frac{7}{8}$	8	11	14	$17\frac{1}{2}$	$20\frac{1}{2}$	$23\frac{1}{2}$	$26\frac{1}{2}$	30	33	$3\frac{1}{8}$	$3\frac{1}{2}$
	1	8	$11\frac{1}{2}$	$14\frac{1}{2}$	18	21	$24\frac{1}{2}$	$27\frac{1}{2}$	31	34	$3\frac{1}{4}$	$3\frac{1}{2}$
	$1\frac{1}{8}$	$8\frac{1}{2}$	12	15	$18\frac{1}{2}$	22	$25\frac{1}{2}$	$28\frac{1}{2}$	32	$35\frac{1}{2}$	$3\frac{3}{8}$	4
	$1\frac{1}{4}$	9	$12\frac{1}{2}$	$16\frac{1}{2}$	20	24	$27\frac{1}{2}$	$31\frac{1}{2}$	35	39	$3\frac{3}{4}$	4

Figure 13. These diagrams have been prepared in accordance with the 1940 report of the Joint Committee on Standard Specifications for Concrete and Reinforced Concrete.

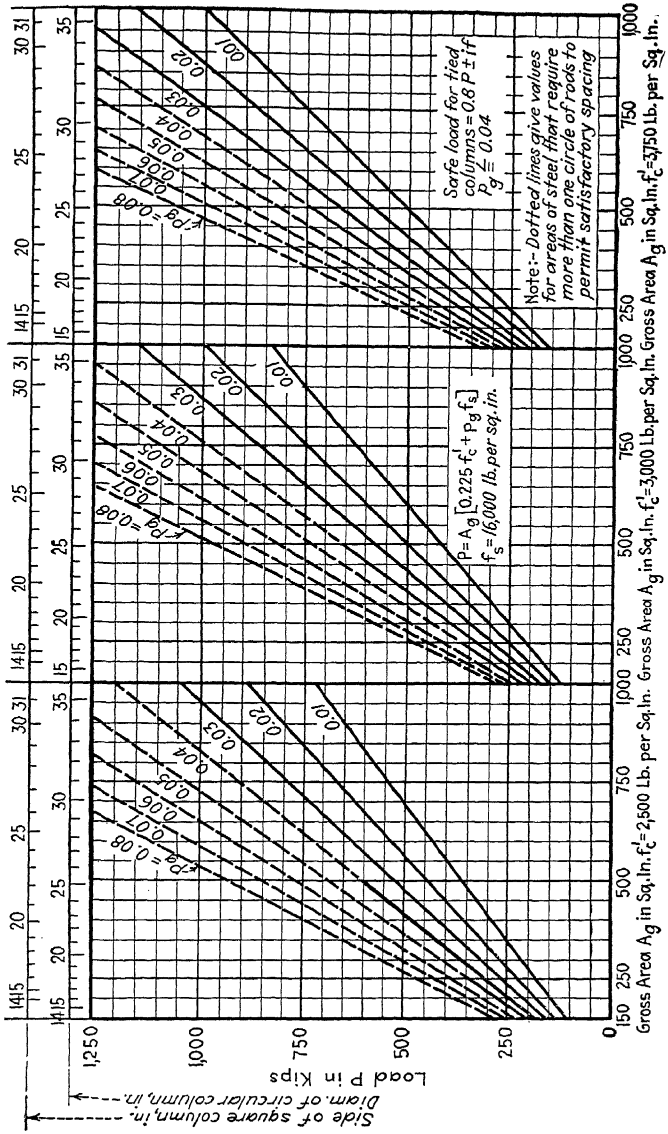


FIG. 13.—Safe loads P on spirally reinforced columns.

Table 9 and *Figure* 14. Table 9 gives a specific size and pitch for the spiral in a square or round column, the data being the same as those given in the American Concrete Institute's "Reinforced Concrete Handbook." On the other hand, Fig. 14 gives the permissible size and pitch of spirals for any value of p' (volume of spiral $\div$ volume of core).

Table 9.—Size and Pitch of Spirals*
$p' \geqq 0.0112$; hot-rolled round rods; cover = $1\frac{1}{2}$ in.

Column size, in.	Core size, in.	Square columns f_c'			Round columns f_c'		
		2500	3000	3750	2500	3000	3750
14	11				$\frac{3}{8}$–$1\frac{3}{4}$	$\frac{3}{8}$–$1\frac{3}{4}$	
15	12	$\frac{1}{2}$–2			$\frac{3}{8}$–2	$\frac{3}{8}$–$1\frac{3}{4}$	$\frac{1}{2}$–2
16	13	$\frac{1}{2}$–2			$\frac{3}{8}$–2	$\frac{3}{8}$–$1\frac{3}{4}$	$\frac{1}{2}$–2
17	14	$\frac{1}{2}$–$2\frac{1}{4}$			$\frac{3}{8}$–$2\frac{1}{4}$	$\frac{3}{8}$–$1\frac{3}{4}$	$\frac{1}{2}$–$2\frac{1}{4}$
18	15	$\frac{1}{2}$–$2\frac{1}{4}$	$\frac{5}{8}$–$2\frac{1}{2}$	$\frac{5}{8}$–$2\frac{1}{4}$	$\frac{3}{8}$–$2\frac{1}{4}$	$\frac{3}{8}$–$1\frac{3}{4}$	$\frac{1}{2}$–$2\frac{1}{2}$
19	16	$\frac{1}{2}$–$2\frac{1}{4}$	$\frac{5}{8}$–$2\frac{1}{4}$	$\frac{5}{8}$–$2\frac{1}{4}$	$\frac{3}{8}$–$2\frac{1}{4}$	$\frac{3}{8}$–$1\frac{3}{4}$	$\frac{1}{2}$–$2\frac{1}{2}$
20	17	$\frac{1}{2}$–2	$\frac{5}{8}$–$2\frac{3}{4}$	$\frac{5}{8}$–$2\frac{1}{4}$	$\frac{3}{8}$–$2\frac{1}{4}$	$\frac{3}{8}$–2	$\frac{1}{2}$–$2\frac{1}{4}$
21	18	$\frac{1}{2}$–2	$\frac{5}{8}$–$2\frac{3}{4}$	$\frac{5}{8}$–2	$\frac{3}{8}$–2	$\frac{3}{8}$–2	$\frac{1}{2}$–$2\frac{3}{4}$
22	19	$\frac{1}{2}$–2	$\frac{5}{8}$–$2\frac{3}{4}$	$\frac{5}{8}$–2	$\frac{3}{8}$–2	$\frac{3}{8}$–2	$\frac{1}{2}$–$2\frac{3}{4}$
23	20	$\frac{1}{2}$–2	$\frac{5}{8}$–$2\frac{1}{2}$	$\frac{5}{8}$–2	$\frac{3}{8}$–$1\frac{3}{4}$	$\frac{3}{8}$–$1\frac{3}{4}$	$\frac{1}{2}$–$2\frac{3}{4}$
24	21	$\frac{1}{2}$–2	$\frac{5}{8}$–$2\frac{1}{2}$	$\frac{5}{8}$–2	$\frac{1}{2}$–$3\frac{1}{4}$	$\frac{1}{2}$–$3\frac{1}{4}$	$\frac{1}{2}$–$2\frac{1}{4}$
25	22	$\frac{1}{2}$–2	$\frac{5}{8}$–$2\frac{1}{2}$	$\frac{5}{8}$–2	$\frac{1}{2}$–$3\frac{1}{4}$	$\frac{1}{2}$–$3\frac{1}{4}$	$\frac{1}{2}$–$2\frac{3}{4}$
26	23	$\frac{5}{8}$–3	$\frac{5}{8}$–$2\frac{1}{2}$	$\frac{5}{8}$–2	$\frac{1}{2}$–3	$\frac{1}{2}$–3	$\frac{1}{2}$–$2\frac{3}{4}$
27	24	$\frac{5}{8}$–3	$\frac{5}{8}$–$2\frac{1}{2}$	$\frac{5}{8}$–2	$\frac{1}{2}$–$2\frac{3}{4}$	$\frac{1}{2}$–$2\frac{3}{4}$	$\frac{1}{2}$–$2\frac{3}{4}$
28	25	$\frac{5}{8}$–$2\frac{3}{4}$	$\frac{5}{8}$–$2\frac{1}{4}$		$\frac{1}{2}$–$2\frac{3}{4}$	$\frac{1}{2}$–$2\frac{3}{4}$	$\frac{1}{2}$–$2\frac{3}{4}$
29	26	$\frac{5}{8}$–$2\frac{3}{4}$	$\frac{5}{8}$–$2\frac{1}{4}$		$\frac{1}{2}$–$2\frac{3}{4}$	$\frac{1}{2}$–$2\frac{1}{4}$	$\frac{1}{2}$–$2\frac{3}{4}$
30	27	$\frac{5}{8}$–$2\frac{1}{4}$	$\frac{5}{8}$–$2\frac{1}{4}$		$\frac{1}{2}$–$2\frac{1}{2}$	$\frac{1}{2}$–$2\frac{1}{2}$	$\frac{1}{2}$–$2\frac{1}{2}$
31	28	$\frac{5}{8}$–$2\frac{3}{4}$	$\frac{5}{8}$–$2\frac{1}{4}$		$\frac{1}{2}$–$2\frac{1}{2}$	$\frac{1}{2}$–$2\frac{1}{2}$	$\frac{1}{2}$–$2\frac{1}{2}$
32	29	$\frac{5}{8}$–$2\frac{3}{4}$	$\frac{5}{8}$–$2\frac{1}{4}$		$\frac{1}{2}$–$2\frac{1}{4}$	$\frac{1}{2}$–$2\frac{1}{4}$	$\frac{1}{2}$–$2\frac{1}{4}$
33	30	$\frac{5}{8}$–$2\frac{1}{2}$	$\frac{5}{8}$–$2\frac{1}{4}$		$\frac{1}{2}$–$2\frac{1}{4}$	$\frac{1}{2}$–$2\frac{1}{4}$	$\frac{1}{2}$–$2\frac{1}{4}$

* "A.C.I. Reinforced Concrete Design Handbook."

Tables 10 *and* 11. Table 10 is for convenience in detailing reinforcement. Table 11 is prepared as a guide in selecting reinforcement for columns to make sure that the rods can be adequately encased at lapped splices. Approximate allowance has been made for the thickness of spiral rods and ties. Spirally reinforced columns should have a minimum cover of $1\frac{1}{2}$ in. of concrete outside the spirals; tied columns should have at least 2 in. of cover.

Figures 15, 16, *and* 17. These figures have been prepared to illustrate the detailing of reinforcement. There are various procedures in use but one must remember that their purpose is the presentation of sufficient data to enable the workmen to assemble the required reinforcement, to cut and bend the bars correctly, and to place the rods properly in the forms. Ordinarily,

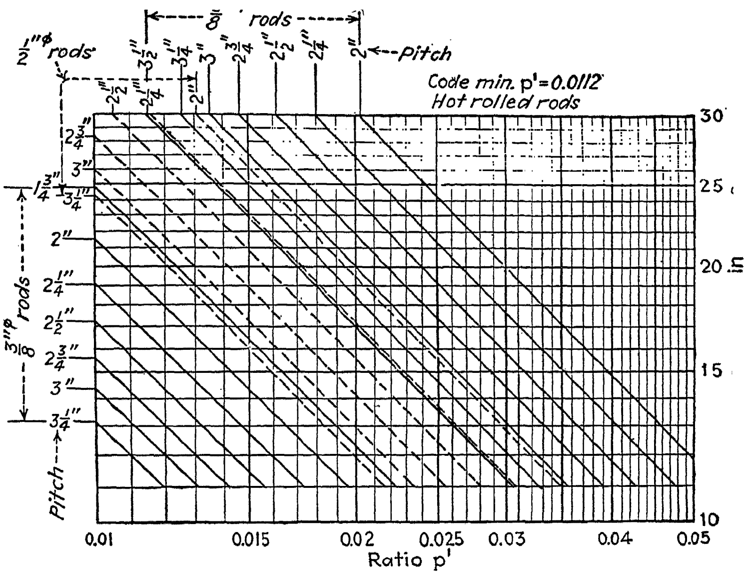
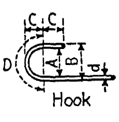
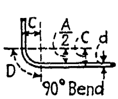


FIG. 14.—Size and pitch of spirals for circular columns.

TABLE 10.—BENDING DETAILS FOR RODS

	Dim. \ d (Size of rod)	$\frac{1}{4}''$	$\frac{3}{8}''$	$\frac{1}{2}''$	$\frac{5}{8}''$	$\frac{3}{4}''$	$\frac{7}{8}''$	$1''$	$1\frac{1}{8}''$	$1\frac{1}{4}''$
 Hook	A	$1\frac{1}{2}$	$2\frac{1}{4}$	3	$3\frac{3}{4}$	$4\frac{1}{2}$	$5\frac{1}{4}$	6	$6\frac{3}{4}$	$7\frac{1}{2}$
	B	2	3	4	5	6	7	8	9	10
	C	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5
	D	$2\frac{3}{4}$	$4\frac{1}{8}$	$5\frac{1}{2}$	$6\frac{7}{8}$	$8\frac{1}{4}$	$9\frac{5}{8}$	11	$12\frac{3}{8}$	$13\frac{3}{4}$
	$C+D$	$3\frac{3}{4}$	$5\frac{5}{8}$	$7\frac{1}{2}$	$9\frac{3}{8}$	$11\frac{1}{4}$	$13\frac{1}{8}$	15	$16\frac{7}{8}$	$18\frac{3}{4}$
 90° Bend	C	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5
	D	$1\frac{3}{8}$	$2\frac{1}{16}$	$2\frac{3}{4}$	$3\frac{7}{16}$	$4\frac{1}{8}$	$4\frac{13}{16}$	$5\frac{1}{2}$	$6\frac{3}{16}$	$6\frac{7}{8}$

$A = 6 \times$ diameter of rod
D is measured along C. L. of rod.

Radius of bend refers to inner face of rod.
Length of bend refers to C. L. of rod.
In computing lengths of rods with bends = or < 45°, neglect curvature.
All dimensions are in inches.

the bar schedule and summary of weights are on the drawing containing the details of the members, but, owing to lack of space, they have been assembled in Fig. 17.

TABLE 11.—MAXIMUM NUMBER OF RODS IN ONE CIRCLE OR SQUARE IN COLUMNS

Core diam. or side, in.	Circular and square spiral columns†							Square tied columns						
	Min. p_g = 0.01; max. p_g = 0.08							Min. p_g = 0.01; max. p_g = 0.04						
	Size of rod							Size of rod						
	$\frac{5}{8}''\phi$	$\frac{3}{4}''\phi$	$\frac{7}{8}''\phi$	$1''\phi$	1″ □	$1\frac{1}{8}''$ □	$1\frac{1}{4}''$ □	$\frac{5}{8}''\phi$	$\frac{3}{4}''\phi$	$\frac{7}{8}''\phi$	$1''\phi$	1″ □	$1\frac{1}{8}''$ □	$1\frac{1}{4}''$ □
10	9	8	7	7	6	..	..	12	8	6*	4	4		
11	10	9	8	8	7	6	..	12	10*	8	6*	4		
12	11	10	9	9	7	6	6	12	12	8	6*	4	4	
13	13	11	10	9	8	7	6	16	12	10*	8	6*	4	4
14	14	12	11	10	9	8	7	16	16	12	10*	8	6*	4
15	15	13	12	11	10	9	8	16	16	14*	10*	8	6*	4
16	16	14	13	12	11	9	8	20	16	16	12	10*	8	6*
17	17	16	14	13	11	10	9	20	20	16	14*	10*	8	6*
18	18	17	15	14	12	11	9	20	20	20	16	12	10*	8
19	19	18	16	15	13	11	10	24	20	20	16	14*	10*	8
20	21	19	17	16	14	12	11	24	24	20	20	16	12	10*
21	22	20	18	17	14	13	11	28	24	20	20	16	14*	10*
22	23	21	19	18	15	13	12	28	24	24	20	16	14*	12
23	24	22	20	18	16	14	13	28	28	24	24	20	16	12
24	25	23	21	19	17	15	13	32	28	24	24	20	16	14*
25	26	24	22	20	18	16	14	32	28	28	24	20	18*	16
26	27	25	23	21	18	16	14	32	32	28	24	24	20	16
27	29	26	24	22	19	17	15	36	32	28	28	24	20	18*
28	30	27	25	23	20	18	16	36	32	32	28	24	20	20
29	31	28	26	24	21	18	16	36	36	32	28	24	24	20
30	32	29	27	25	22	19	17	40	36	32	32	28	24	20
Min. cc of rods	$2\frac{3}{4}$	3	$3\frac{1}{4}$	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	←Recommended minimum spacing of rods in columns to permit adequate bond at splices.						

* Requires unequal arrangement of rods.
† Taken from "A.C.I. Reinforced Concrete Design Handbook," circular spirals.

The following points are worthy of note as a guide in making such details:

1. All rods are numbered differently if they vary in size, length or detail.
2. All bars must be supported adequately. Such rods as *a*11 in Fig. 15, Section *D-D*, appear to have no supporting ties but they are actually tied to rods which project from the wall below (already poured).

3. Rods should be of such size and length that they can be handled and placed with reasonable facility. Ordinary lengths for ordering are 30 ft. to 60 ft. The last are often inconveniently long and heavy.

4. The thickness of the rods must be allowed for in the details. In Fig. 16, the top reinforcement in Section *A-A* is placed 4 in. below the top of the slab so that it will pass under the rods shown 3 in. below the top of the slab in Section *B-B*. Furthermore, the depth of the beam *B*1 is such that its lower rods will not hit those of beam *B*2. As an aid in visualizing such matters, it is enlightening for one to lay a few pencils on his desk to simulate the proposed arrangement of the bars.

5. Such figures as "3×13" shown for bars *a*3 in the upper left corner of Fig. 15 indicate that there are 3 similar sets of 13 bars each.

6. In detailing walls and slabs, as in Fig. 15, it is frequently advisable to make separate plans for top and bottom mats so as to clarify the details.

7. Bending diagrams for bars can be shown ordinarily by single lines, as in Fig. 17(*c*), Type 1, but they are sometimes shown by the use of double lines, as in Type 4, when the over-all dimensions are important for clearance. However, the latter arrangement often leads to errors in computing the required length of the rod; e.g., for *b*9, $\frac{1}{2}$ in. must be deducted from *B* and $\frac{1}{4}$ in. from *C* to allow for the half-diameter of the rod because the length is computed along the center line of the bar. When using single lines, dimensions such as *A* for Type 1, Fig. 17(*c*), must allow for the thickness of the bar. Incidentally, bending of rods to form closed figures is usually inadvisable.

8. Avoid using detail dimensions with fractions less than $\frac{1}{8}$ in. because greater accuracy in cutting and bending is not practicable.

9. In bending stirrups, it is sometimes advisable to use sharp bends—radius to inner surface of rod = $\frac{1}{2}$ diameter of the main bar—when the main rods come in the corners, as in Fig. 17(*c*), Type 4, because large curves tend to cause the main rods to crowd together.

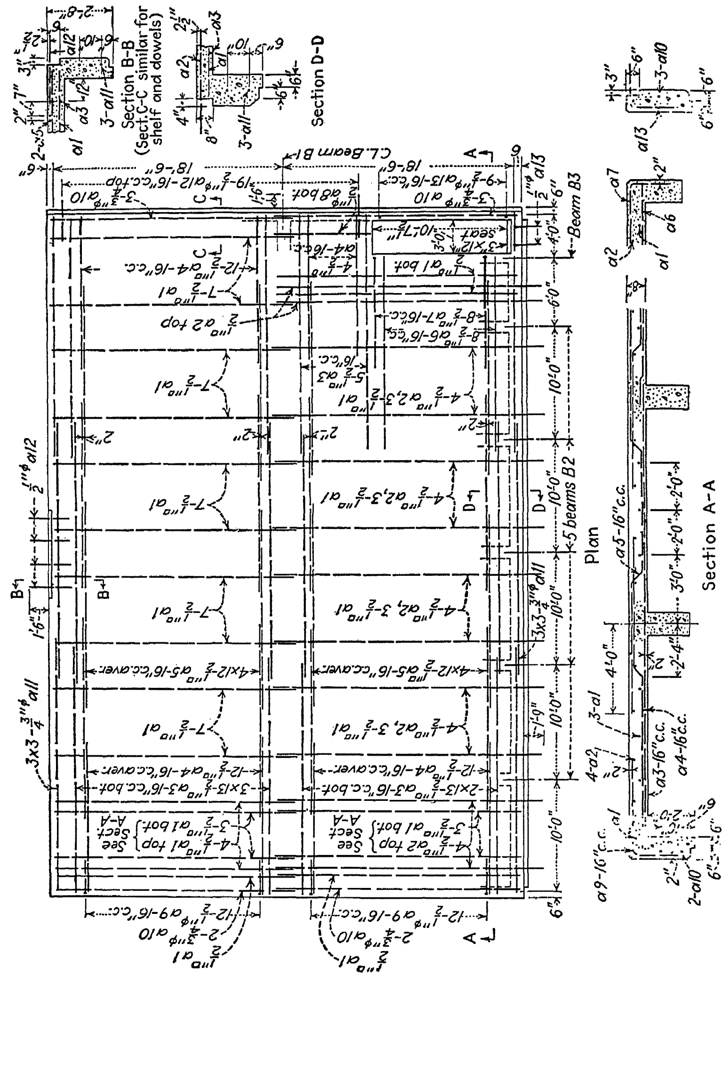


FIG. 15.—Details of floor slab.

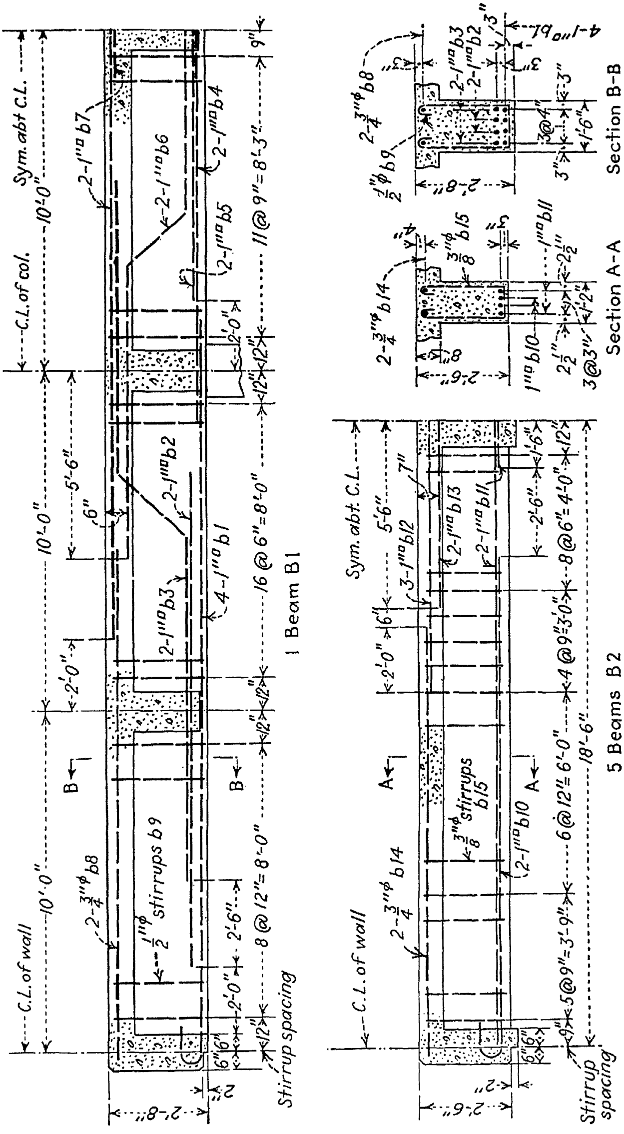
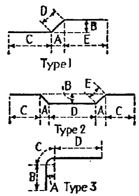


FIG. 16.—Details of beams.

(a) BAR SCHEDULE FOR FIG. 15

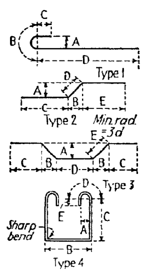


Mark	Size	No. reqd.	Type	A	B	C	D	E	Length	Total, lin. ft.	Remarks
a1	½" □	61	Str.						19'-9	1204.75	Lap at B1
a2	½" □	22	Str.						21'-9	478.50	Lap at B1
a3	½" □	70	Str.						21'-0	1470.0	
a4	½" □	40	1	4"	4"	7'-10½	5½"	6'-0	14'-4	573.33	
a5	½" □	96	2	4"	4"	6'-0	5'-4	5½"	18'-3	1752.0	
a6	½" □	8	Str.						17'-0	136.0	
a7	½" □	8	3	3¼"	1'-6	5"	10'-1		12'-0	96.0	
a8	½" φ	1	Str.						8'-0	8.0	
a9	½" φ	24	3	3¼"	1'-7	5"	4'-6		6'-6	156.0	
a10	¾" φ	10	Str.						19'-9	197.50	Wire to wall bars
a11	¾" φ	18	Str.						21'-0	378.0	Wire to wall bars
a12	½" φ	23	Str.						3'-6	80.5	
a13	½" φ	11	3	3¼"	1'-7	5"	2'-0		4'-0	44.0	

(b) SUMMARY OF WEIGHTS FOR FIG. 15

Size	Lin. ft.	Wt. per ft.	Weight, lb.
¾" φ	575.50	1.52	875
½" □	5710.58	0.86	4911
½" φ	288.5	0.68	196
Total for dwg.......			5982

(c) BAR SCHEDULE FOR FIG. 16



Mark	Size	No. reqd.	Type	A	B	C	D	E	Length	Total, lin. ft.	Remarks
b1	1" □	8	1	7"	11"	4"	21'-6		22'-9	182.0	
b2	1" □	4	Str.						14'-6	58.0	
b3	1" □	4	2	1'-11	1'-11	10'-0	2'-8½	8'-6½	21'-3	85.0	
b4	1" □	2	Str.						23'-0	46.0	Outer
b5	1" □	2	Str.						16'-0	32.0	Inner
b6	1" □	2	3	1'-8	1'-8	8'-6	10'-8	2'-4	32'-4	64.67	
b7	1" □	4	Str.						19'-6	78.0	Lap at C. L.
b8	¾" φ	4	Str.						15'-0	60.0	
b9	½" φ	76	4	3"	1'-2	2'-2	5½"	2"	6'-8	506.67	
b10	1" □	20	1	7"	11"	4"	14'-6		15'-9	315.0	
b11	1" □	20	Str.						20'-0	400.0	Lap in vert. plane
b12	1" □	15	Str.						16'-0	240.0	One in center
b13	1" □	10	Str.						11'-0	110.0	
b14	¾" φ	20	Str.						12'-9	255.0	
b15	⅜" φ	240	4	2¼"	10½"	2'-0	4⅛"	1½"	5'-9	1380.0	

(d) SUMMARY OF WEIGHTS FOR FIG. 16

Size	Lin. ft.	Wt. per ft.	Weight, lb.
1" □	1610.67	3.44	5541
¾" φ	315.0	1.52	479
½" φ	506.67	0.68	345
⅜" φ	1380.0	0.38	524
Total for dwg......			6889

FIG. 17.—Illustrative bar schedules and summaries.

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